Integrability and AdS / CFT correspondence

Integrability ~ Solvable models

classical mechanics

classical & quantum field theories

AdS/CFT : duality in string theory

Unified high-energy theory of

gauge & gravity interactions

"equivalence"

Disclaimer :

Just Introduction Today

To explain concepts, ideas,
terminology, backgrounds

Not many equations

Assumes broad audience

CFT : conformal field theory

quantum field theory (QFT)

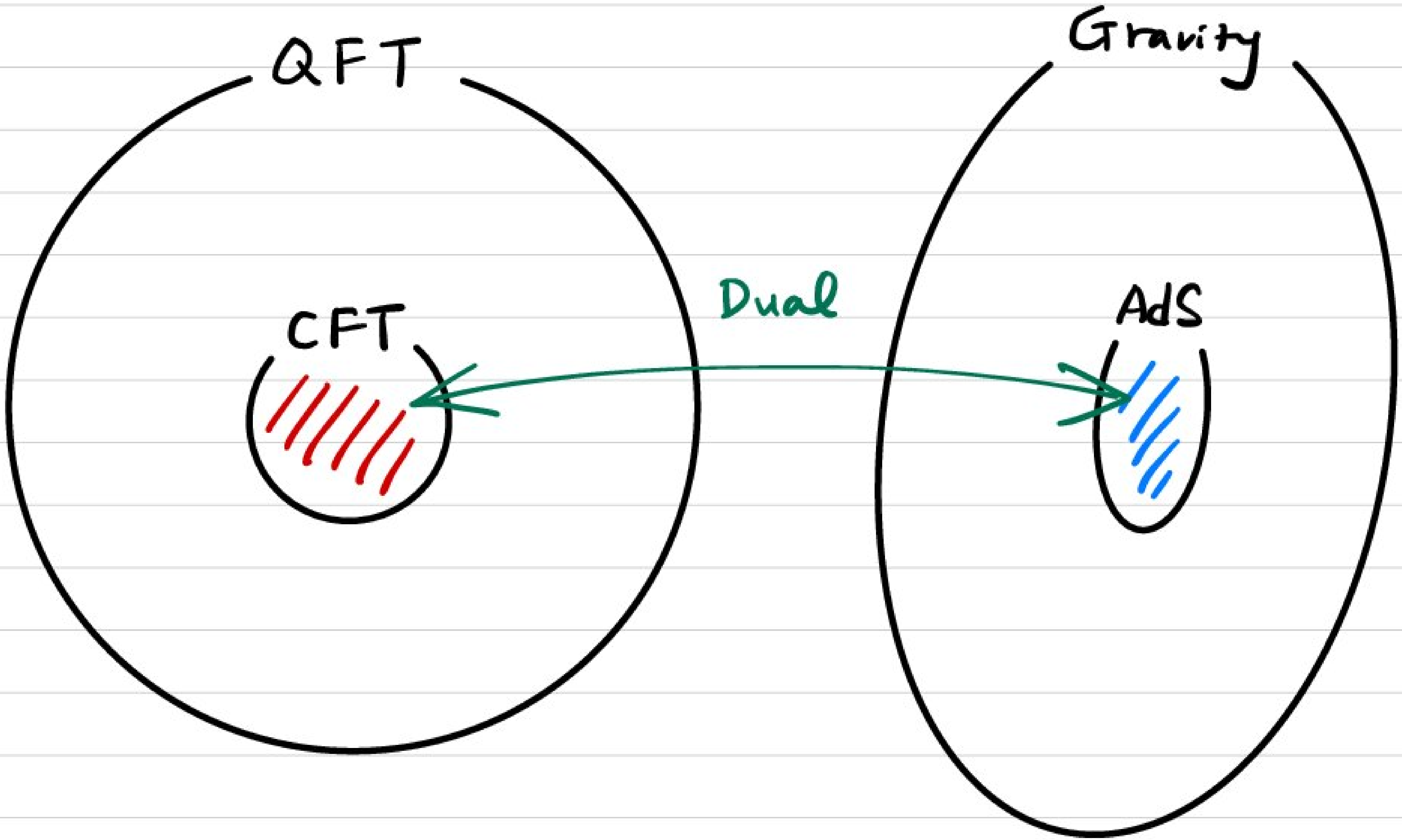
invariant under conformal transformation

generalization of scaling transform

AdS : gravity theory on Anti de-Sitter spacetime

analytic continuation of hyperbolic space

Conjecture : (something in CFT) = (something in AdS)


QFT
CFT
Dual
Gravity
AdS

AdS / CFT correspondence is a conjecture

- Why does this relation exist ?
- What is the simplest example ?

one CFT $\longleftrightarrow$ one gravity on AdS

- What is the quantitative prediction ?

$$\langle \mathcal{O} \rangle_{CFT} \longleftrightarrow \psi_{AdS}$$

- How to check these predictions ?

study the simplest example

What is the simplest example of AdS / CFT ?

Simple CFT :

massless free theory , $\mathcal{L} = (\partial\phi)^2$ or $\psi \not{\partial} \psi$

(mass breaks scaling invariance)

Simple AdS :

Einstein action with negative cosmological const.

$$\mathcal{L} = \frac{1}{16\pi G}\left(R - 2\Lambda\right) \Rightarrow \text{AdS solution}$$

→ difficult examples of the correspondence !

AdS / CFT is a strong-weak duality w.r.t. 't Hooft coupling λ

- λ is a usual coupling in CFT (free if $\lambda \to 0$)
- $\frac{1}{\lambda}$ controls higher derivative terms in gravity

$$\mathcal{L} = \frac{1}{16\pi G}\left\{ (R - 2\Lambda) + \sum_{n \geq 2} c_n \left(\frac{L^2}{\sqrt{\lambda}} \right)^{n-1} R^n \right\}$$

N.B. not derivation. Need to go through Maldacena's paper to understand the identification of parameters

How to control higher derivative terms ?

→ study string theory !

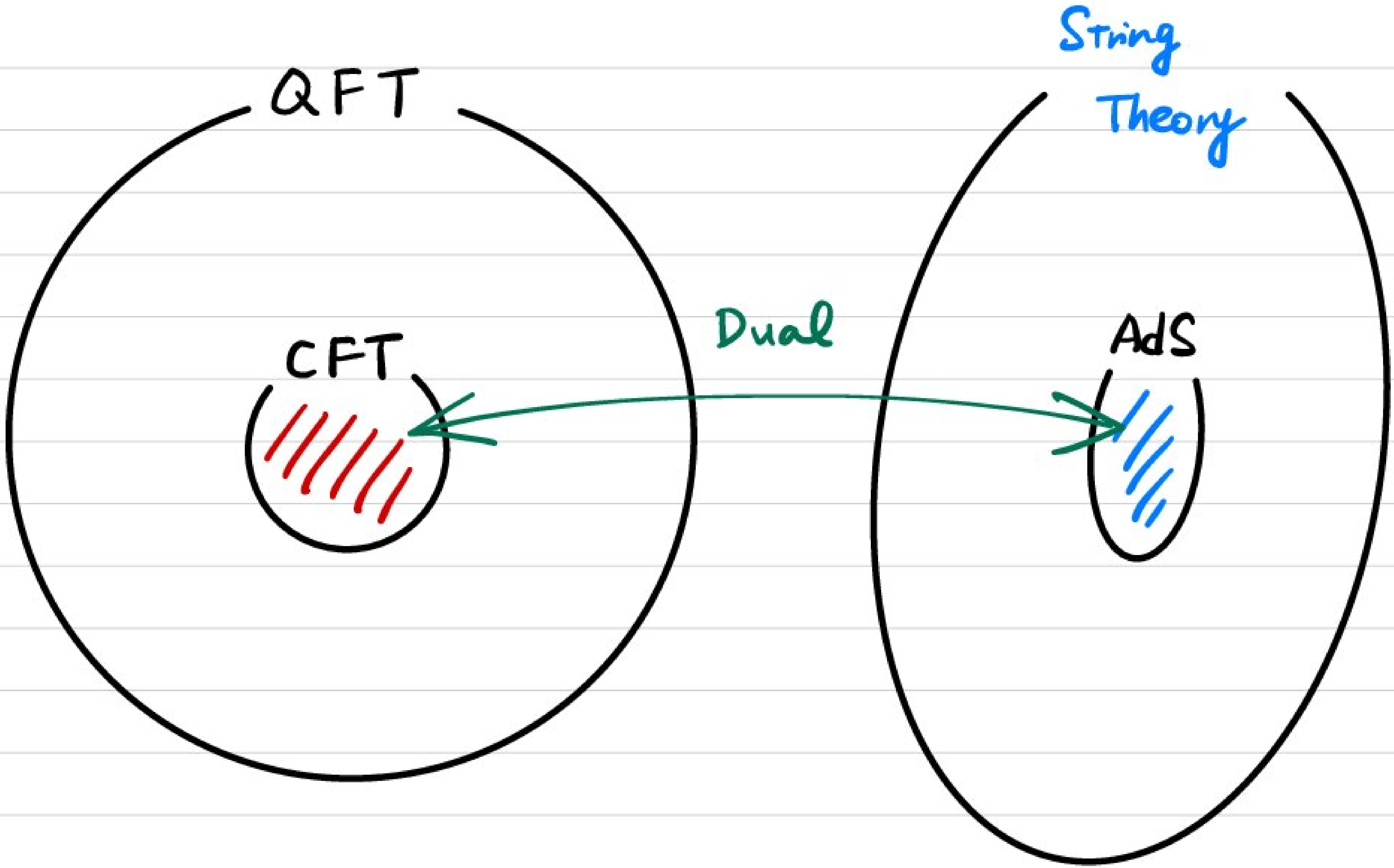
gravity $\neq$ rigid string

↓ quantize

↑ infinite tension ($\alpha' \to 0$)

graviton (particle) ⟶ extend ⟶ string

string tension $\sim \frac{1}{\alpha'}$


String Theory
QFT
Dual
CFT
AdS

Q : How to check the AdS / CFT correspondence?

- Both sides must be interacting theories

$$\begin{cases} \lambda \to 0 & : \text{free CFT} \\ \lambda \to \infty & : \text{rigid string theory on AdS} \end{cases}$$

- Both sides must be exactly soluble in λ

~ integrability, but connection not straight forward

Integrability (Integrable models / systems)

Liouville's classical integrability :

(constants of motion) =

(degrees of freedom)

- free theory
- Coulomb potential , $V(r) = 1/r$
- many 2D field theories

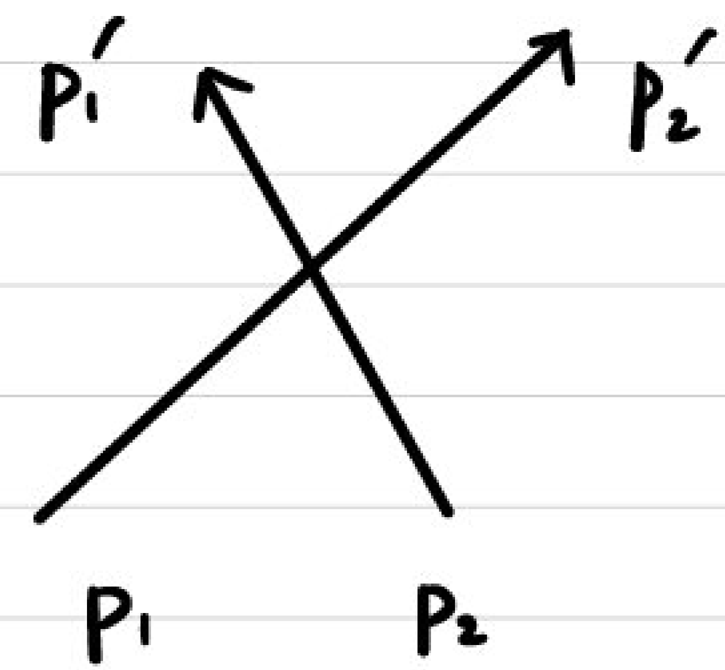
Why two dimensions ?

Integrable system has many conserved charges

$$\left\{ \sum p_i \, , \, \sum p_i^2 \, , \, \sum p_i^3 \, , \, \cdots \right\}$$

Consider S-matrix :

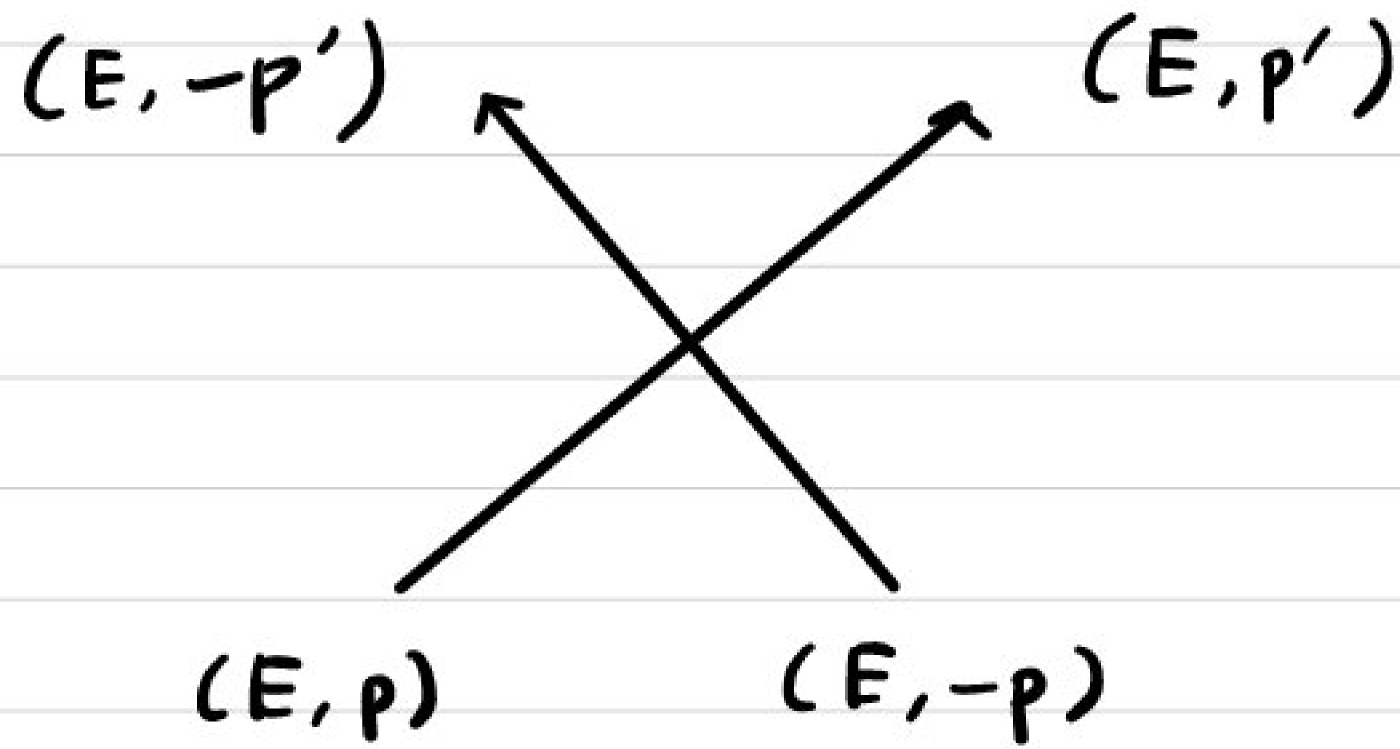
$$S(p_1, p_2, p_1', p_2') =$$



If $\sum_i p_i^n = \sum_i (p_i')^n$ for all n, then

$\{p_i'\}$ should be a permutation of $\{p_i\}$

Take the center of mass frame,



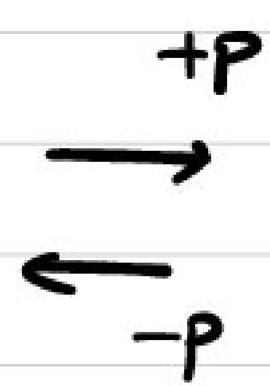
integrability

$\Downarrow$

S-matrix $\sim \delta(p-p')$ ← not analytic in $D>2$

In $D=2$, $\delta(x) \sim \frac{1}{x}$

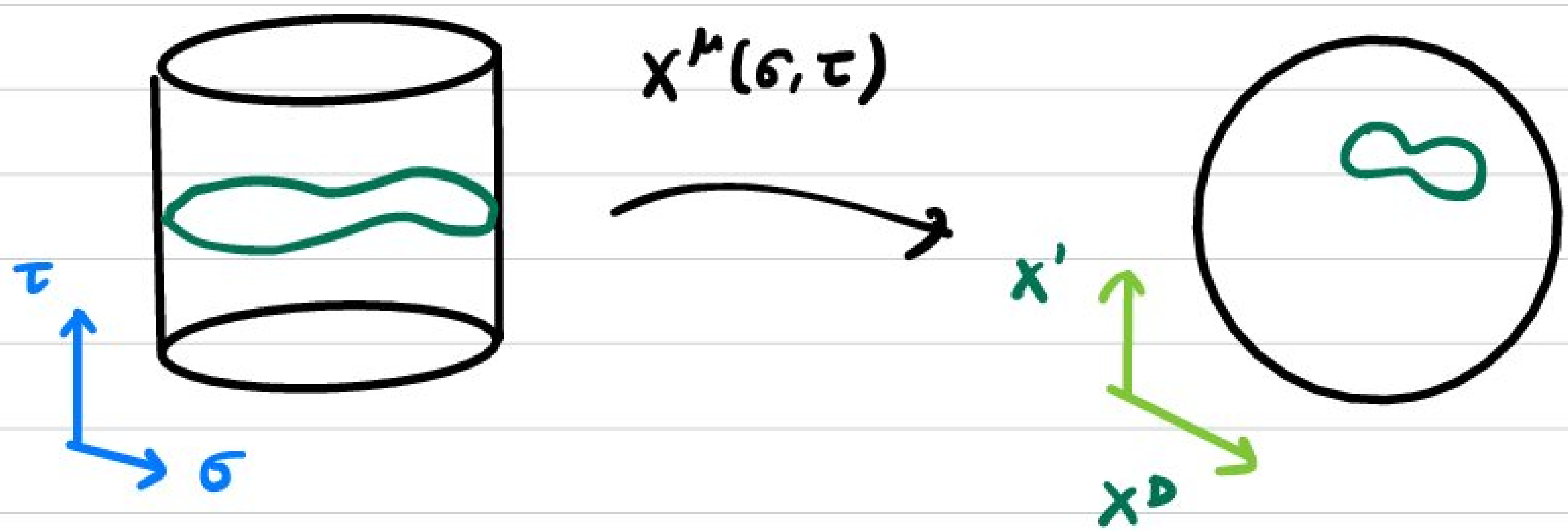
or there's no "angle" in 1 space dimension,



Where is the integrability in AdS / CFT ?

AdS (string theory) side :

integrability ⇝ string worldsheet theory (2D)



Worldsheet action (Nambu-Goto)

$$\begin{cases} S = -\frac{1}{2\pi\alpha'} \int d\sigma d\tau \sqrt{-\det h} \\ h_{ab} = G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \end{cases}$$

N.B. More complicated in $AdS_5 \times S^5$

we need to make it supersymmetric

can be made integrable if carefully choose S

Where is integrability in CFT side?

This question is hard & non-trivial

How to find "string worldsheet"?

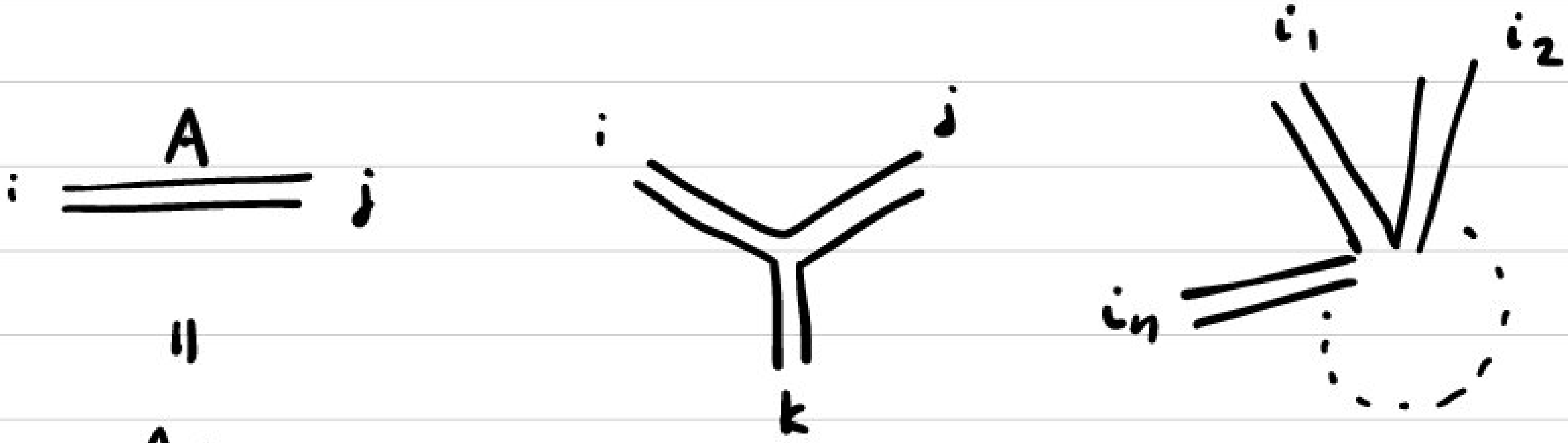
This question is easier ; large N limit

Consider $N \times N$ matrix A_{ij} $(i,j = 1, 2, \cdots, N)$

& draw a graph for the space of indices

effective worldsheet

't Hooft's double-line notation



A_{ij} $\qquad C_{ijk}$ $\qquad C_{i_1 i_2 \cdots i_n}$

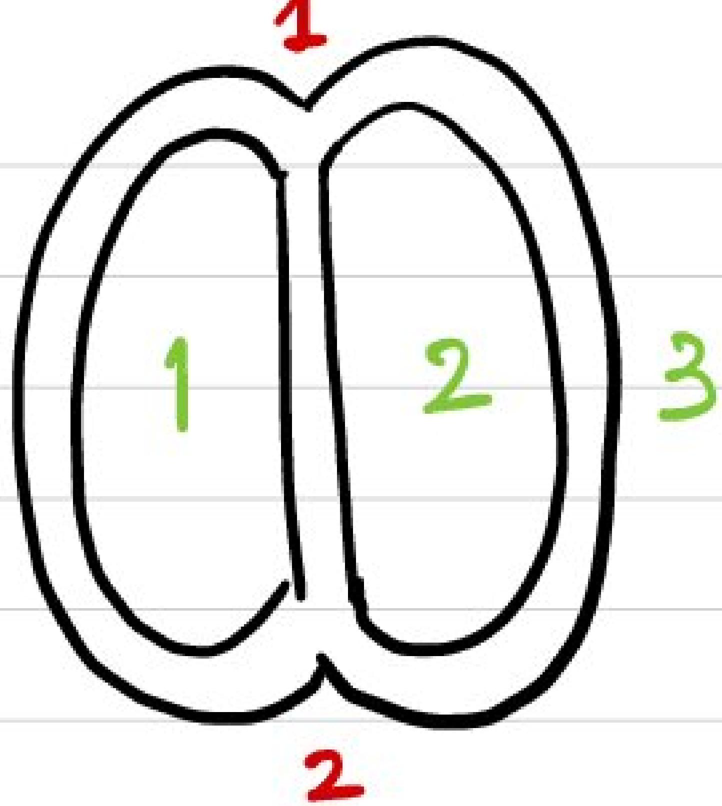
Consider triangulation of a Riemann surface

& assign a factor of N

Coming back to the basic questions:

- What is the AdS / CFT correspondence
- Why does such a relation exist

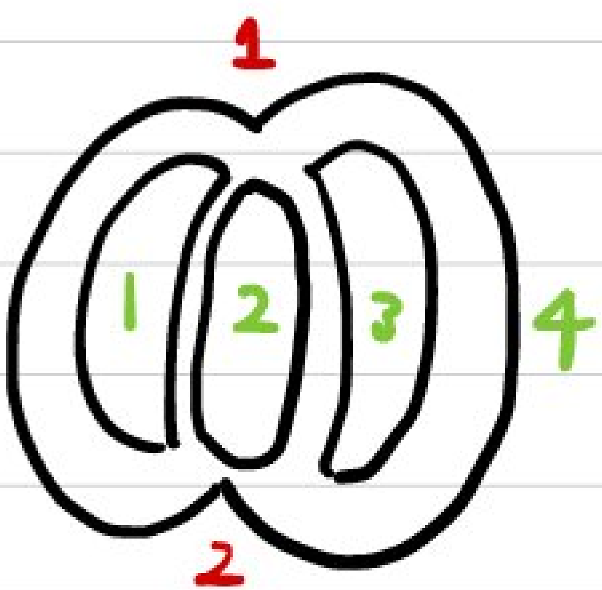
{ Blackhole entropy
  Holography
  Open-closed duality in string theory



three faces $\longrightarrow N^3$

two 3pt vertices $\longrightarrow N^{-1}$

total $\longrightarrow N^2$



N^4

N^{-2}, n-pt vertex : $N^{-(n-2)/2}$

$N^2 = N^{\chi}$, $\chi =$ Euler number of S^2

total factor is invariant for any triangulations

Pitfall : Why gravity is NOT QFT

Weinberg - Witten theorem (1980)

$$\begin{cases} |p\rangle & \text{massless one-particle state with spin } j \\ T^{\mu\nu} & \text{energy-momentum tensor} \end{cases}$$

Lorentz transformation on

$$\langle -p | T^{\mu\nu} | p \rangle = \langle -p | U^{\dagger} \cdot U T^{\mu\nu} U^{\dagger} \cdot U | p \rangle$$

rep. theory constraint, non-zero solution at $j = 0, \frac{1}{2}, 1$

For massless $j=2$ (graviton), $\langle -p| T^{\mu\nu} |p\rangle = 0$

$\Rightarrow$ $|p\rangle$ has zero charge (energy)

$\Rightarrow$ gravity cannot be a composite particle of QFT

N.B. Massive spin-2 particle may have $E \gtrsim 0$

WW theorem also applies to AdS/CFT ($\mathcal{N}=4$ SYM)

But CFT creates bulk states, charged under $T^{\mu\nu}_{bulk}$

also $T_{\mu\nu}$ in gravity is pseudo tensor, not covariant.

1) Blackhole entropy

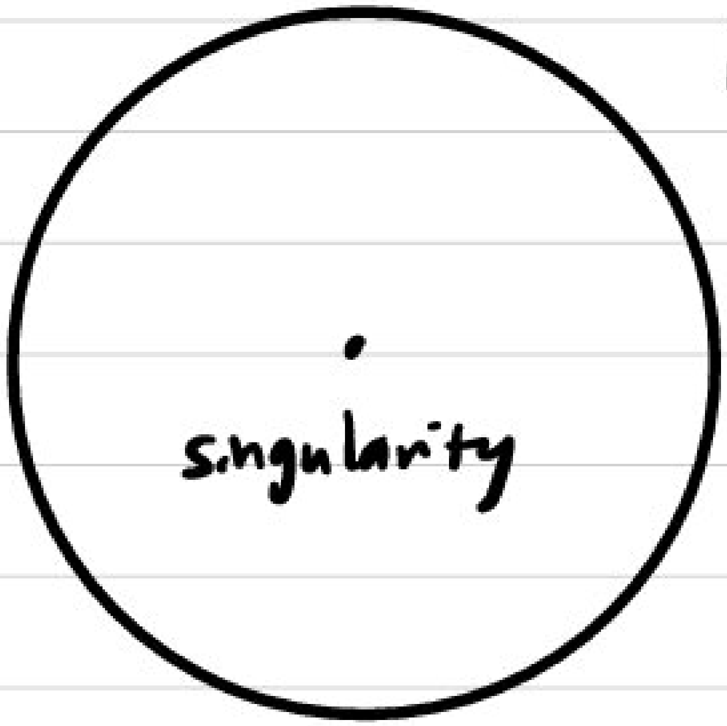
According to BH thermodynamics,

$$\underline{dM} = \underline{T} \quad \underline{dS} + (\text{charges})$$

mass — surface gravity — horizon area

= Hawking temperature — = Bekenstein-Hawking entropy

if you throw a particle into BH, it increases mass & entropy according to $dM = TdS + \cdots$



horizon : $ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \cdots$

$$\Rightarrow \quad r_H = 2GM$$

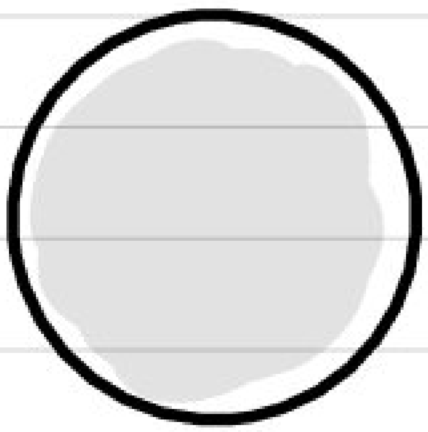
In QFT , entropy S is proportional to the volume

$$S = \left(1 - \beta \frac{\partial}{\partial \beta}\right) \log Z$$

$$\log Z = -\sum_k \ln\left(1 - e^{-\beta \varepsilon_k}\right) \rightarrow V \int d^3k \, \ln(\cdots)$$

However, BH entropy is given by area

⇝ BH physics is effectively described by QFT defined on horizon ('t Hooft)

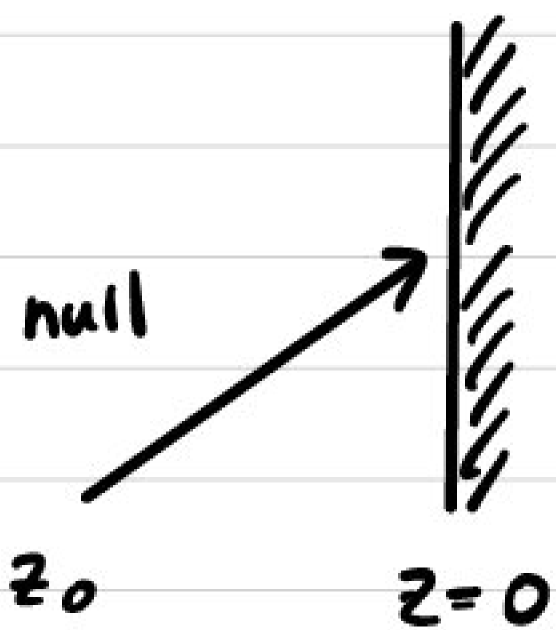


effective description of BH interior

but we can't see BH interior anyway

2) Holography

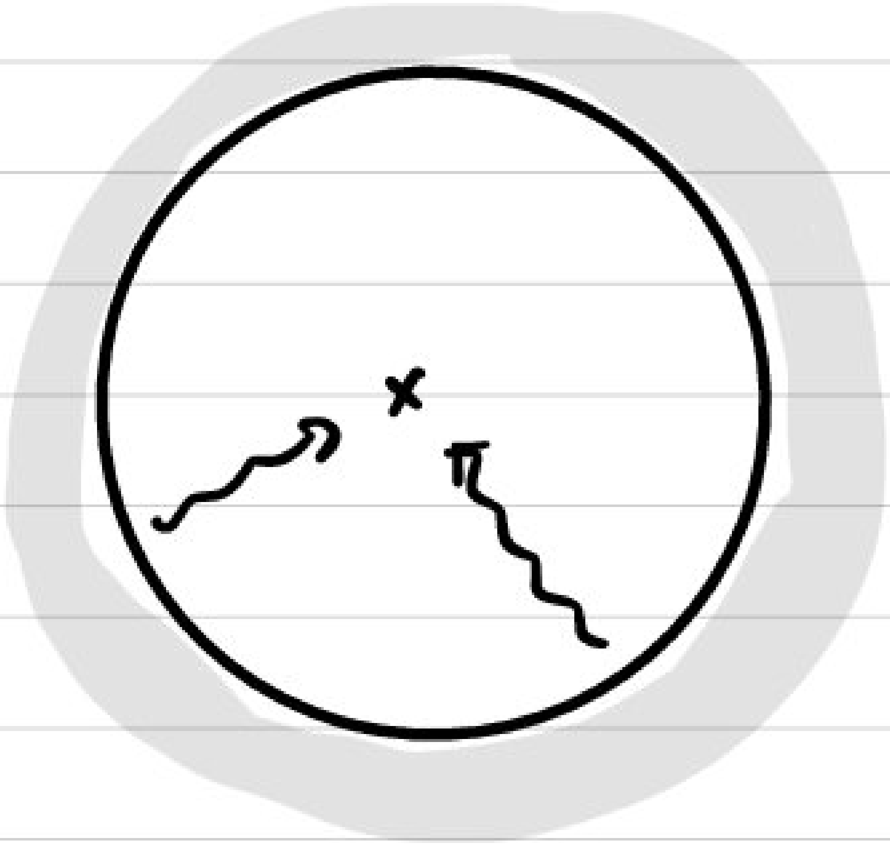
AdS is a "finite box" in the sense that it takes a "finite time" to send a signal from bulk to boundary



$$ds^2 = \frac{dz^2 - dt^2 + d\vec{x}^2}{z^2} \quad \text{(Poincaré patch)}$$

$$dt \sim dz \Rightarrow t \sim z_0$$

(geodesical distance = ∞)



For a finite 'box',
we "should" be able to
reconstruct bulk data from
boundary

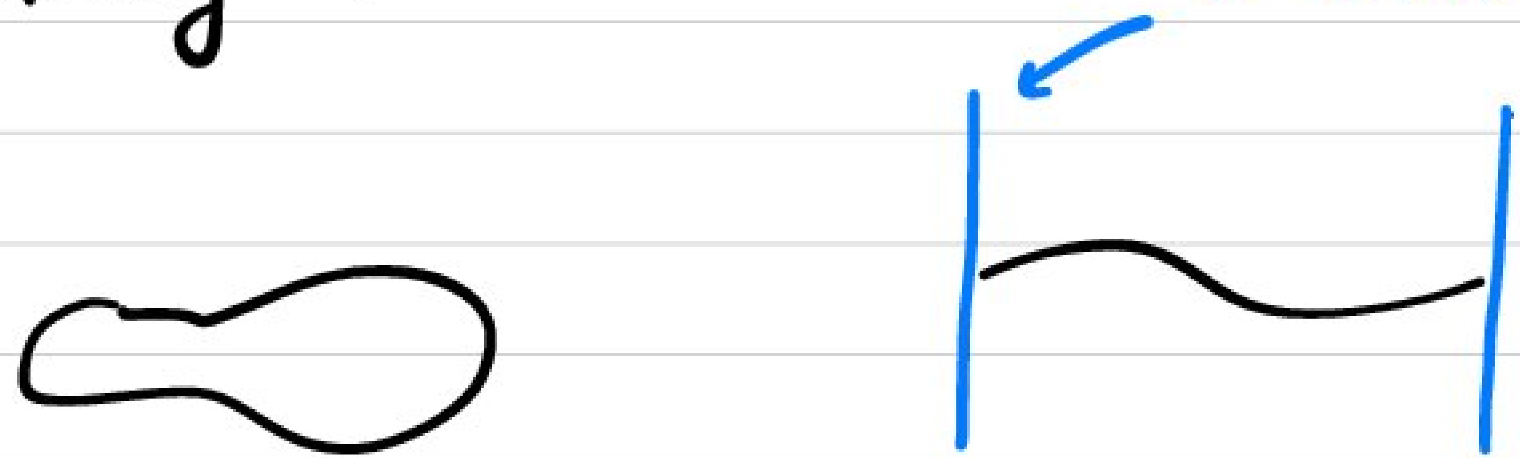
- assuming no black hole

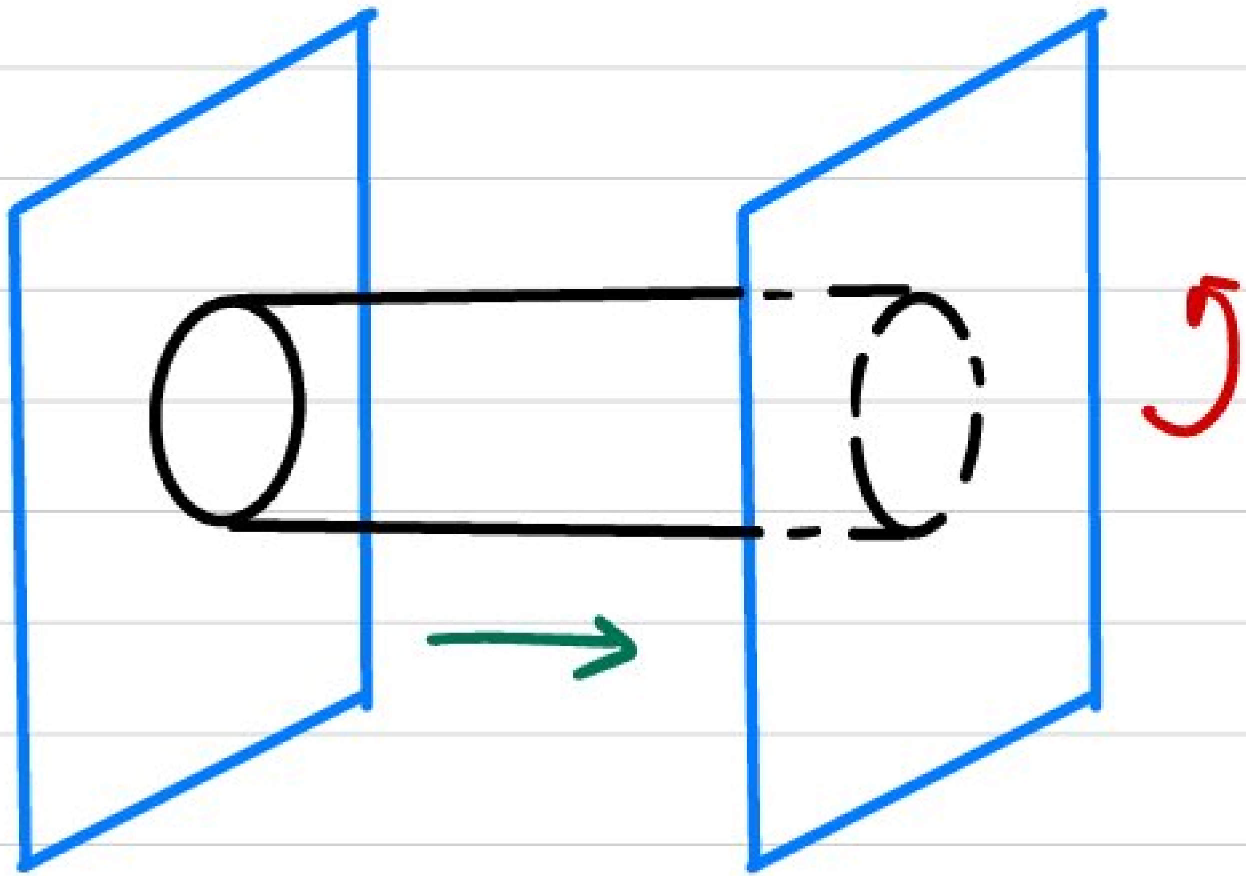
N.B. This argument relates bulk gravity & boundary gravity. AdS / CFT is a different duality

3) Open-closed duality in string theory

- Standard 'top-down' argument for AdS / CFT
- 'bottom-up' approach is possible, less rigorous

String theory :





① Open string 1-loop partition function

|| duality

② Closed string propagation between D-branes

long cylinder limit

⇒ only massless particles contribute to ②

supergravity (no stringy modes)

Compute ① by flat-space superstring

Compute ② by 10 d supergravity + D-brane action

Dirac-Born-Infeld (DBI) action

$$S_{D_p} = -T_p \int \sqrt{-\det(G+B)} + \mu_p \int (C_p + \cdots)$$

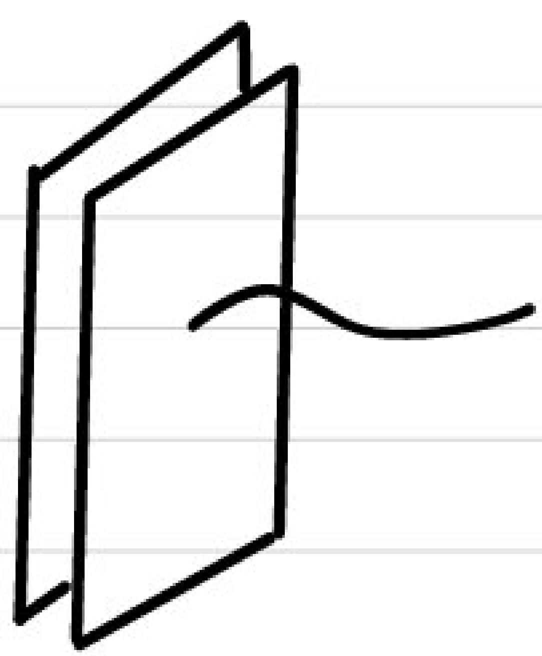
D-brane tension   Ramond-Ramond charge

① = ② fixes these parameters

Dp-brane : (p+1)-dimensional worldvolume $\mathcal{M}_{p+1}$

$$\begin{cases} n^a \partial_a X^\nu = 0 & (\nu = 0, 1, \cdots, p) \\ X^\nu = 0 & (\nu = p+1, \cdots, 10) \end{cases}$$

D3-brane ~ like 'our 4D world'



open string ending on N coincident D3-branes

→ SU(N) (susy) gauge theory in 4 dimensions

Closed string viewpoint :

D3- brane has a tension $\Rightarrow$ non-zero energy

$\Rightarrow$ backreaction to the flat spacetime

$\Rightarrow$ black 3-brane solution in 10d supergravity

Action of type IIB sugra

$$S_{10} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left[R - \frac{(\partial\phi)^2}{2} - \frac{F_5^2}{4\cdot 5!} \right] + \cdots$$

Total action $S = S_{10} + N\, S_{D3}$

Ansatz (4+6) $$ds^2 = \sqrt{f(\rho)}\, \eta_{\mu\nu} dx^\mu dx^\nu + \frac{d\rho^2}{f(\rho)^2} + \rho^2 d\Omega_5^2$$

$$F_5 = Q\left(d\mathrm{vol}_{S_5} + * d\mathrm{vol}_{S_5} \right)$$

EoM for $g_{\mu\nu}$ $\Rightarrow$ $$\begin{cases} f(\rho) = 1 - \dfrac{L^4}{\rho^4}, & L: \text{horizon radius} \\ Q = 2L^4 \end{cases}$$

Horowitz, Strominger NPB 360 (1991)

EoM for C_4 $(F_5 = dC_4)$:

$$S = -\frac{1}{2 g_s \kappa_{10}^2} \int d^{10}x \, (F_5)^2 + N \mu_3 \int F_5$$

$$\Rightarrow \quad d * F_5 = 2 \kappa_{10}^2 \mu_3 \, g_s N \, \delta^6(x)$$

use brane RR charge

$$= 16 \pi^4 \alpha'^2 g_s N \, \delta^6(x)$$

if we integrate ansatz, $\int F_5 = 2\pi^3 Q = 4\pi^3 L^4$

$$\Rightarrow \quad L^4 = 4\pi \alpha'^2 g_s N$$

In 4d $U(N)$ gauge theory, we define

't Hooft coupling $\lambda \equiv N g_{YM}^2$

- gauge boson ~ open string
- graviton ~ closed string ~ $(\text{open})^2$
- g_s = closed string coupling

one identifies $\lambda = N g_{YM}^2 = 4\pi N g_s = \dfrac{R^4}{\alpha'^2}$

α' is 'typical size of string'

($1/\alpha'$ is string tension)

string $\longrightarrow$ { particles in the limit $\alpha' \to 0$
rigid string

$\sqrt{\lambda} = \frac{R^2}{\alpha'} \longrightarrow \infty$ in this limit (not $\lambda \to 0$)

$\Rightarrow$ Higher derivative terms in gravity disappear