Week 9 : Centrally Extended $PSU(2|2) \ltimes \mathbb{R}^3$

Final Week : Dressing phase

AFS phase

α' corrections , HL phase

Crossing symmetry

DHM representation

§ AFS phase (hep-th/0406256)

KMMZ paper : String finite-gap solution

$\parallel$

continuum limit of <u>one-loop</u> BAE

BDS, RSS : One-loop BAE

$\longrightarrow$ <u>Hubbard BAE</u> at higher loops

Higher order corrections disagree with KMMZ

Rescue : Introduce dressing phase to cancel extra term

All-loop BAE in su(2) sector

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{M} \frac{u_k - u_j + 2i/g}{u_k - u_j - 2i/g}\, e^{2i\theta(u_k, u_j)}$$

$$\theta_{AFS}(u_k, u_j) = \sum_{r=0}^{\infty} \left(\frac{\lambda}{16\pi^2}\right)^{r+2} \left\{ q_{r+2}(p_k)\, q_{r+3}(p_j) - (k \leftrightarrow j) \right\}$$

Take log :

$$iLp_k + 2\pi n_k i = \sum_{j\neq k} \ln\left(\frac{u_k - u_j + 2i/g}{u_k - u_j - 2i/g} \cdot e^{2i\theta_{kj}}\right)$$

$(n_k \in \mathbb{Z})$

Continuum limit : $u_k \to \infty$ (or $u_k = O(1)$, $g \to \infty$)

$$x_k = \frac{1}{2}\left(u_k \pm \sqrt{u_k^2 - 4}\right) \longrightarrow u_k - \frac{1}{u_k} + O(u_k^{-3})$$

Introduce the density :

$$\sum_j \ln\left(\frac{u_k - u_j + 2i/g}{u_k - u_j - 2i/g}\right) \longrightarrow \frac{4i}{g}\ \underline{\text{P.V.}} \int_C dv\ \frac{L\rho(v)}{u_k - v}$$

(remove $v = u_k$ part)

Momentum :

$$ip_k = \ln \frac{x(u_k + i/g)}{x(u_k - i/g)} \longrightarrow \frac{2i}{g} \frac{1}{\sqrt{u_k^2 - 4}} + O(g^{-3})$$

BAE:

$$\frac{2i}{g}\frac{1}{\sqrt{u_k^2-4}}+\frac{2\pi i\, n_k}{L}=\fint_c dv\,\rho(v)\left\{\frac{4i}{g}\frac{1}{u_k-v}+2i\,\theta_{AFS}\right\}$$

removing "k":

$$\frac{1}{\sqrt{u^2-4}}+\frac{g\pi n}{L}=\fint_c dv\,\rho(v)\left\{\underline{\frac{2}{u-v}}+g\,\theta_{AFS}\right\}$$

introduce $x_v \equiv x(v)$, $x_u \equiv x(u)$, $dv\,\rho(v)=dx\,\sigma(x)$

$$\fint dv\,\frac{\rho(v)}{u-v}=\fint dx\,\sigma(x)\left\{\frac{1}{x_u-x_v}+\cdots\right\}$$

Apply $\frac{d}{dv}\log(\cdots)$ on $(u-v) = (x_u - x_v)\left(1 - \frac{1}{x_u x_v}\right)$

$$-\frac{1}{u-v} = -\frac{dx_v}{dv}\left\{\frac{1}{x_u - x_v} - \frac{1}{x_u x_v^2}\frac{1}{\left(1-\frac{1}{x_u x_v}\right)}\right\}$$

$$\Rightarrow \quad \frac{1}{u-v} = \frac{dx_v}{dv}\left(\frac{1}{x_u - x_v} - \frac{1}{x_v(x_u x_v - 1)}\right)$$

$$\fint_C dv\, \frac{\rho(v)}{u-v} = \fint_C dx_v\, \frac{\sigma(x_v)}{x_u - x_v} - \underline{\int_C dx_v}\, \frac{\sigma(x_v)}{x_v(x_u x_v - 1)}$$

regular kernel, PV not needed

BAE becomes :

$$\frac{1}{\sqrt{u^2-4}} + \frac{g\pi n}{L} = \fint_c dx' \sigma(x') \left\{ \frac{2}{x-x'} - \underline{\frac{2}{x'(xx'-1)}} + g\,\theta_{AFS} \right\}$$

compare with finite-gap

$$\frac{1}{x'(xx'-1)} \quad \text{must be} \quad \frac{1}{(x')^2\,(x-1/x)}$$

$$\text{difference} = \frac{dx}{du} \frac{-(x-x')}{x^2(x')^2(xx'-1)}$$

$$= \frac{dx}{du} \sum_{r=0} \left\{ \frac{1}{x^{r+3}\,(x')^{r+2}} - \frac{1}{x^{r+2}\,(x')^{r+3}} \right\}$$

Higher conserved charges at $g \gg 1$:

$$q_r = \frac{i}{r-1}\left\{\frac{1}{(x^+)^{r-1}} - \frac{1}{(x^-)^{r-1}}\right\} \longrightarrow \frac{2}{g}\frac{dx}{du}\frac{1}{x^r}$$

$$(\text{difference}) = \sum_{r=0}\left\{q_{r+3}\frac{1}{(x')^{r+2}} - q_{r+2}\frac{1}{(x')^{r+3}}\right\}$$

Define $Q_r \equiv \int dx'\,\sigma(x')\,\frac{1}{(x')^r}$

$$(\text{BAE difference}) = -2\sum_r\left(q_{r+3}Q_{r+2} - q_{r+2}Q_{r+3}\right)$$

$$+\ g \fint dx'\,\sigma(x')\,\theta_{AFS}$$

If we write

$$\theta_{AFS}(u_k, u_j) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{r,s} \left\{ q_r(u_k)\, q_s(u_j) - (k \leftrightarrow j) \right\}$$

and evaluate RHS at strong coupling,

$$c_{r,s} = g\, \delta_{r+1,s}$$

$\Rightarrow$ Asymptotic BAE + AFS phase

agrees with strong finite-gap

§ α' corrections [McLoughlin 1012.3987]

Methods :

- WKB / soliton quantization, Daschen - Hasslacher - Neveu
  Chen - Dorey - Lima Matos
- Solving Virasoro constraint, Roiban - Tseytlin
- Adding a pole in finite-gap, Gromov - Schafer Nameki - Vieira
- Heuristic (below)

Sum of non-absolutely convergent series

$$\delta E \sim \sum_n \sqrt{n^2 + m_B^2} - \sqrt{n^2 + m_F^2} \rightsquigarrow \text{ambiguous}$$

2d sigma model :

$$\mathcal{L} = G_{MN}\, \partial X^M \bar{\partial} X^N = \pi^a \bar{\pi}^a \,, \quad \pi^a = \underline{E_M{}^a}\, \partial X^M$$

vierbein

- don't use integrability
- expand around any classical sol : $X^M = X_0^M + x^M$

$$\delta \pi^a = E_M{}^a \partial x^M + x^N \partial_N E_M{}^a \partial X^M$$

use $\xi^a \equiv E_M{}^a x^M$, then

$$\partial \xi^a = E_M{}^a \partial x^M + \partial X^N \partial_N E_M{}^a x^M$$

$$\delta\Pi^a = \partial\xi^a - \partial X_0^N \partial_N E_M^{\ a} E^{bM} \xi^b + \xi^b E^{bN} \partial_N E_M^{\ a} \partial X_0^M$$

$$= \partial\xi^a + \underbrace{\partial X_0^M \left(E^{bN} \partial_N E_M^{\ a} - E^{bN} \partial_M E_N^{\ a} \right)}_{= \Omega^{ab}(X_0)} \xi^b$$

Lagrangian becomes :

$$\delta\mathcal{L} = \left(\partial\xi^a + \Omega^{ab}\xi^b \right)^2 + \mathcal{O}(\xi^3)$$

$\longrightarrow$ free particles

$\longrightarrow$ one-loop corrections to $Z = \int DX\, e^{-S[X]}$

for supervierbein , "a" runs for bosons & fermions

Consider Hamiltonian formalism

$$Z \equiv \mathrm{tr}\left(e^{-\beta H}\right) = \int D\phi \; e^{-S[\phi]} \Big|_{\phi(\tau)=\phi(\tau+\beta)}$$

$\because$) $U(t) = e^{itH}$ is time translation

Compute the 2d (worldsheet) energy

$$\langle H \rangle = -\frac{\partial}{\partial\beta} \ln Z = -\frac{\partial}{\partial\beta} \ln\left(Z_0 \, Z_2 \, Z_{\text{higher}}\right)$$

- $H_{2d} = E - J$ in lightcone gauge
- $H_{2d} = 0$ in conformal gauge $\rightsquigarrow$ use Virasoro

One-loop correction to Z :

If Lagrangian is $\delta \mathcal{L} = -(\partial_\tau \phi)^2 + m^2 \phi^2$,

then $\phi = a_\omega e^{i\omega\tau}$, $\omega = \frac{2\pi n}{\beta}$ for $n \in \mathbb{Z}$

$$\frac{1}{\sqrt{\det(-\partial^2+m^2)}} = \exp\left(-\frac{1}{2} \operatorname{tr} \log(-\partial^2+m^2)\right)$$

$$= \exp\left(-\frac{1}{2} \log \prod_n \left\{ \left(\frac{2\pi n}{\beta}\right)^2 + m^2 \right\}\right)$$

$$= \exp\left(-\frac{m\beta}{4\pi} + O(1)\right) \quad \text{as } \beta \to \infty$$

Use : $\displaystyle\prod_{n=1}^{\infty} \frac{n^2+\mu^2}{n^2} = \frac{\sinh \mu\pi}{\mu\pi}$ $\left(\mu = \frac{m\beta}{2\pi}\right)$

Thus

$$\langle H_{1\text{-loop}} \rangle = -\frac{\partial}{\partial\beta} \ln Z_1 = \frac{1}{2}\left(\partial_\beta \mu\right) + O(1/\beta)$$

In string sigma model,

$$\xi(\tau,\sigma) = \xi(\tau+\beta,\sigma)$$

$$\delta\mathcal{L} = -\partial\xi\bar{\partial}\xi + m^2\xi^2, \qquad = \xi(\tau,\sigma+2\pi)$$

$$\xi(\tau,\sigma) = \sum_{n_1,n_2} a_{n_1,n_2} \exp\left(i\frac{2\pi n_1}{\beta}\tau + i n_2\sigma\right)$$

For complicated sols, $m = m(\sigma)$ $\rightarrow$ non-diagonal Fourier modes

Consider simple cases,

$$\mu(n_1,n_2)^2 = \left(\frac{\beta}{2\pi}\right)^2\left(m^2 + n_2^2\right)$$

Then

$$\langle H_{1\text{-loop}} \rangle = -\frac{\partial}{\partial\beta} \log Z_{1\text{-loop}} \Big|_{\beta\to\infty}$$

$$= \frac{1}{2} \sum_r (-1)^{\mathrm{sig}(r)} \sum_{n_2} \partial_\beta \, \mu(m_r, n_2)$$

$$= \frac{1}{4\pi} \sum_{r, n_2} (-1)^{\mathrm{sig}(r)} \sqrt{m_r^2 + n_2^2}$$

- mass depends on the parameters of X_0 ($t = \kappa\tau$, $\phi = \nu\tau$)
- should add ghosts for conformal gauge (total d.o.f. $= 8_B + 8_F$)

§ HL phase [Beisert - Tseytlin, 0509084] [HL, 0603204]

Circular string "$k = 2m$" solution on $\mathbb{R}_t \times S^3$

$$X_1 = \frac{1}{\sqrt{2}} \exp(i w \tau + i m \sigma), \quad X_2 = \frac{1}{\sqrt{2}} \exp(i w \tau - i m \sigma)$$

$$J_1 = J_2 \equiv \frac{\sqrt{\lambda}}{2} \mathcal{J}, \quad E_{cl} = \sqrt{\lambda (\mathcal{J}^2 + m^2)}$$

Corresponding resolvent (1 cut between $x = \pm i / 4\pi m$)

$$G(x) = 2\pi m + \frac{\sqrt{1 + (4\pi m g)^2} - \sqrt{1 + (4\pi m x)^2}}{2(x - g^2/x)}$$

regular at $x = \pm g$, $g \equiv 1/4\pi \mathcal{J} = \sqrt{\lambda}/4\pi J$

After regularization of 1-loop sum:

$$E_{1\text{-loop}} = \frac{1}{\sqrt{J^2+m^2}}\left(m^2 + 2J^2 \log\frac{J^2}{J^2+m^2} - \frac{J^2-m^2}{2}\log\frac{J^2-m^2}{J^2+m^2}\right)$$

$$\doteqdot -\frac{m^6}{3J^5} + \frac{m^8}{3J^7} - \frac{49}{120}\frac{m^9}{J^{10}} + \cdots$$

→ want to reproduce this from adding phase in BAE

as in AFS case, we look for equations like

$$2\int dy\, \frac{\rho(y)}{x-y} + g\,\theta_{HL} = \cdots$$

For checking the claim, start from resolvent identity:

$$G(z) = \int dy \, \frac{\rho(y)}{z-y} \longleftarrow \frac{1}{N} \sum_{j=1}^{N} \frac{1}{z-z_j}$$

Use $\displaystyle \frac{1}{(z-z_j)(z-z_k)} = \frac{1}{(z-z_j)(z_j-z_k)} + (j \leftrightarrow k)$

RHS is meaningful only if $z_j \neq z_k$; subtract P.V. in LHS

$$\frac{1}{N^2} \sum_{j,k} \frac{1}{(z-z_j)(z-z_k)} \longrightarrow G^2(z)$$

$$\frac{1}{N^2} \sum_{j=k} \frac{1}{(z-z_j)(z-z_k)} \longrightarrow \frac{1}{N} G'(z)$$

The resolvent satisfies (cf. (4.53) of KMMZ)

$$2 \fint dy \frac{\rho(y)}{x-y} = 2\pi k + \frac{1}{x - g^2/x}\Bigg[1 + g^2 \underbrace{\int dy \frac{\rho(y)}{y^2}}_{Q_2 \,=\, \text{energy}} \Bigg] + (\mathrm{HL})$$

HL proposal :

$$(\mathrm{HL}) = -\frac{2}{\sqrt{\lambda}}\, a_{r,s}\, g^{r+s-1} \int dy\, \rho(y) \left\{ \frac{1}{x^{r-1} y^{s-1}} - \frac{1}{x^{s-1} y^{r-1}} \right\}$$

$$a_{r,s} = 8 \frac{(r-1)(s-1)}{(r+s-2)(r-s)} \quad \text{for odd } r+s$$

$(a_{r,s} = 0$ for even $r+s)$

$$G^2(z) + \frac{1}{N} G'(z) = \frac{1}{N^2} \sum_{j \neq k} \frac{1}{(z - z_j)(z_j - z_k)}$$

Continuum limit & normalize density:

$$G^2(z) + G'(z) = \int dy_1 \, \frac{\rho(y_1)}{z - y_1} \underbrace{⨍ dy_2 \, \frac{\rho(y_2)}{y_1 - y_2}}_{\text{substitute BAE}}$$

$$⨍ dy_2 \, \frac{\rho(y_2)}{y_1 - y_2} = 2\pi k + \underbrace{\frac{1}{y_1 - g^2/y_1}}_{\text{approximate as } 1/y_1} \left[1 + 2g^2 \int dy_2 \, \frac{\rho(y_2)}{y^2} + \ldots \right]$$

Final 1-loop correction is a series of

$$\frac{m^2}{J^2} = (mg)^2 \quad ; \qquad g \ll 1 \quad \text{but} \quad m \gg 1$$

if you expand
$$\begin{cases} G(x) = -\sum_{n=0} Q_{n+1}\, x^n \\ G^{(r)}(x) = -\sum_{n=r} Q_{n+1}\, x^{n-r} \end{cases}$$

$$Q_n \sim O(g^{-n}) \gg 1$$

also
$$\begin{cases} \text{Virasoro} \Rightarrow G(0) = Q_1 = -2\pi m \\ \text{FG} \Rightarrow E = J(1 + 2g^2 Q_2) \end{cases}$$

Equation for resolvent including HL term

$$G^2 - 2\pi k G - G^{(1)} = g^2\left(G^{(1)2} - \underline{2\pi k\, G^{(2)}}\right)$$

$$+ 2g^2\left(G^{(1)}Q_2 - \underline{G^{(2)}Q_1}\right)$$

cancel by $k=2m$

$$- \frac{2}{\sqrt{\lambda}}\, a_{r,s}\, g^{r+s-1}\left(G^{(r)}Q_s - G^{(s)}Q_r\right)$$

Expand $G = G_0 + \delta G$, set $x=0$, use $G^{(1)}(0) = -Q_2$

$$\rightsquigarrow \quad (1+2g^2 Q_2)\,\delta Q_2 = \frac{2}{\sqrt{\lambda}}\, a_{r,s}\, g^{r+s-1}\left(Q_{r+1}Q_s - Q_{s+1}Q_r\right)$$

We know classical resolvent $G(x)$

$\Rightarrow$ Compute $$\delta E = a_{r,s}\, g^{r+s} \frac{Q_{r+1}Q_s - Q_{s+1}Q_r}{\pi(1+2g^2 Q_2)}$$

$\Rightarrow$ matches string theory computation

Remark :

- HL phase is "odd crossing", somewhat universal

- Also agrees with semiclassical scattering of GM

§ Crossing Relations

Beisert considered the singlet state

$$|1\rangle = \frac{1}{\gamma_1 \gamma_2}\left(\frac{x_1^+}{x_1^-} - 1\right) \varepsilon_{ab} |\phi_1^a \phi_2^b Z^+\rangle + \varepsilon_{\alpha\beta} |\psi_1^\alpha \psi_2^\beta\rangle$$

with $x_2^\pm = 1/x_1^\pm$

$\Rightarrow$ The total momentum, central charges $= 0$

No representation label of $su(2|2)$

Scattering between $1, \mathcal{X}$ for any $\mathcal{X}$ is diagonal

$$S_{13} S_{12} | \mathcal{X}_3 1_{12} \rangle = \left(\frac{x_3^+ - x_1^+}{x_3^+ - x_1^-} \frac{1 - 1/x_3^- x_1^+}{1 - 1/x_3^- x_1^-} \right)^2 \times$$

$$S_0(x_3, x_1) \, S_0(x_3, 1/x_1) \, | 1_{12} \, \mathcal{X}_3 \rangle$$

Basically the same eq. in Janik (0603038)

Normalization ambiguity : AF 0901.4937 → "S^{-1}"

$$S_{su(2)}(z_1, z_2) = \frac{x_1^+}{x_1^-} \frac{x_2^-}{x_2^+} \sigma(x_1, x_2)^2 \frac{u_1 - u_2 - 2i/g}{u_1 - u_2 + 2i/g}$$

$$S_{su(2)}(z_1, z_2) \, S_{su(2)}(z_1 + \omega_1, z_2) = \left(\frac{x_1^- - x_2^-}{x_1^+ - x_2^-} \frac{1 - 1/x_1^- x_2^+}{1 - 1/x_1^+ x_2^+} \right)^2$$

Beisert - Hernández - López 0609044

- solved crossing eq. to all orders in $1/\lambda$

- solution living on ∞-genus Riemann surface

$$y^{(n)} + \frac{1}{y^{(n)}} - x - \frac{1}{x} = \frac{in}{g}$$

- Ambiguity in homogeneous sol

$$f(x_1^\pm, x_2^\pm)\, f(1/x_1^\pm, x_2^\pm) = 1$$

Beisert - Eden - Staudacher 0610251

- Analytic continuation at $\lambda \ll 1$, compute $f(g)$

$$\theta(u_k, u_j) = \sum_{r=2} \sum_{s=r+1} c_{r,s} \left(q_r(u_k)\, q_s(u_j) - (r \leftrightarrow s) \right)$$

$$c_{r,s}(g) = \underbrace{c_{r,s}^{(0)}}_{\text{AF}} + g^{-1} \underbrace{c_{r,s}^{(1)}}_{\text{HL}} + g^{-2} \underbrace{c_{r,s}^{(2)}}_{\text{BHL}} + \cdots$$

$c_{r,s}^{(n)} \longrightarrow c_{r,s}^{(-n)}$ to get $c_{2,3}^{(-2)} = -4\zeta(3)$: dressing phase at 3-loop

<u>Check</u>

BFKL seems to be the unique weak-coupling data to justify dressing without wrapping

How to express BES phase ?

[Dorey Hofman Maldacena, 0703104] [Vieira, Volin, 1012.3992]

$$\sigma(u,v)\,\sigma(u^c,v) = \frac{1-\frac{1}{x^+y^+}}{1-\frac{x^-}{y^-}}\;\frac{1-\frac{x^-}{y^+}}{1-\frac{1}{x^+y^-}}$$

Higher charge decomposition :

$$i\log\sigma = \chi(x^+,y^+) - \chi(x^+,y^-) - \chi(x^-,y^+) + \chi(x^-,y^-)$$

$$\chi(x,y) = -\chi(y,x)$$

introduce $\sigma_1 = \exp\left(i\,\chi(x,y^-) - i\,\chi(x,y^+)\right)$

$$\sigma(u,v) = \frac{\sigma_1(x^+,v)}{\sigma_1(x^-,v)} \quad , \quad \sigma(u^c,v) = \frac{\sigma_1(1/x^+,v)}{\sigma_1(1/x^-,v)}$$

γ^+ : flip x^+ only (similarly for $\gamma-$)

$$\sigma(u^{\gamma-},v) = \frac{\sigma_1(x^+,v)}{\sigma_1(1/x^-,v)} \quad , \quad \sigma(u^{\gamma+},v) = \frac{\sigma_1(1/x^+,v)}{\sigma_1(x^-,v)}$$

$$\frac{\sigma_1(x^+,v)\,\sigma_1(1/x^+,v)}{\sigma_1(x^-,v)\,\sigma_1(1/x^-,v)} = \frac{1-\frac{1}{x^+y^+}}{1-\frac{1}{x^+y^-}} \cdot \frac{1-\frac{1}{x^-y^+}}{1-\frac{1}{x^-y^-}}$$

if we define shift operator $Df(u) = f(u+\frac{i}{g})$

$$\left(\sigma_1(x,v)\, \sigma_1\!\left(\frac{1}{x}, v\right) \right)^{D-D^{-1}} = \left(\frac{x - \frac{1}{y^+}}{x - \frac{1}{y^-}} \right)^{D+D^{-1}}$$

e.g. $$\alpha^{D+D^-} = \exp\left((D^+ + D^-) \log \alpha(u) \right)$$

$$= \exp\left(\log \alpha(u+\tfrac{i}{g}) + \log \alpha(u-\tfrac{i}{g}) \right)$$

$$= \alpha\left(u+\tfrac{i}{g}\right) \alpha\left(u-\tfrac{i}{g}\right)$$

formally

$$\sigma_1(x,v)\,\sigma_1\left(\frac{1}{x},v\right) = \left(\frac{x-\frac{1}{y^+}}{x-\frac{1}{y^-}}\right)^{f(D)},$$

$$f(D) = \frac{D+D^{-1}}{D-D^{-1}} = \frac{D^{-2}}{1-D^{-2}} - \frac{D^2}{1-D^2} = \sum_{n=1}^{\infty}\left(D^{-2n} - D^{2n}\right)$$

$\longrightarrow$ $f(D)$ should be written by even powers of D, D^{-1}

using χ.

$$LHS = e^{\,i\chi(x,y^-) + i\chi(\frac{1}{x},y^-) - i\chi(x,y^+) - i\chi(\frac{1}{x},y^+)}$$

We get

$$e^{i\chi(x,y) + i\chi(1/x,\, y)} = \left(\frac{x - \frac{1}{y}}{\sqrt{x}} \right)^{-f(D)}$$

$\rightarrow$ replaced $x^{\pm}, y^{\pm}$ to x, y by inverting D-operator

Symmetric combination

$$e^{i\chi(x,y) + i\chi(1/x,\, y) + i\chi(x,\, \frac{1}{y}) + i\chi(\frac{1}{x},\, \frac{1}{y})} = (u-v)^{-f(D)}$$

one can solve

$$(u-v)^{-f(D)} = \frac{\Gamma\left(1 + \frac{ig}{2}(u-v)\right)}{\Gamma\left(1 - \frac{ig}{2}(u-v)\right)}$$

$\because$) $$G(z) = \frac{\Gamma(1 + igz/2)}{\Gamma(1 - igz/2)}$$

$$(D - D^{-1}) \log G = \log \frac{\Gamma(\frac{1}{2} + \frac{igz}{2})}{\Gamma(\frac{3}{2} - \frac{igz}{2})} \frac{\Gamma(\frac{1}{2} - \frac{igz}{2})}{\Gamma(\frac{3}{2} + \frac{igz}{2})}$$

$$= -\log \left(\frac{1}{2} + \frac{igz}{2}\right)\left(\frac{1}{2} - \frac{igz}{2}\right)$$

$$= -(D + D^{-1}) \log \left(\frac{igz}{2}\right)$$

$\therefore$ $$G(z) = \left(\frac{igz}{2}\right)^{-f(D)}$$ up to zero modes

Now

$$\chi(x,y) + \chi\left(\frac{1}{x}, y\right) + \left(y \leftrightarrow \frac{1}{y}\right) = \frac{1}{i} \log \frac{\Gamma\left(1+\frac{ig}{2}(u-v)\right)}{\Gamma\left(1-\frac{ig}{2}(u-v)\right)}$$

⇒ DHM representation :

$$\chi(x,y) = -i \oint_{|w_1|=1} \frac{dw_1}{2\pi i} \oint_{|w_2|=1} \frac{dw_2}{2\pi i} \frac{1}{x-w_1} \frac{1}{y-w_2} \times$$

$$\log \frac{\Gamma\left(1+\frac{ig}{2}\left(w_1+\frac{1}{w_1}-w_2-\frac{1}{w_2}\right)\right)}{\Gamma\left(1-\frac{ig}{2}\left(w_1+\frac{1}{w_1}-w_2-\frac{1}{w_2}\right)\right)}$$

$\because$) either x, $1/x$ is inside the unit circle, same for y

Conclusion

Studied string theory $\rightsquigarrow$ AdS / CFT

AdS / CFT $\longleftrightarrow$ Integrable systems

$psu(2|2)$ symmetry + dressing phase $\rightsquigarrow$ all-loop

Construction of AdS / CFT dictionary

$\mathcal{N}=4$ SYM $\longleftrightarrow$ (asymptotic) BAE $\longleftrightarrow$ $AdS_5 \times S^5$ superstring

- needs wrapping corrections $\rightsquigarrow$ TBA, QSC
- other observables, other theories, ...