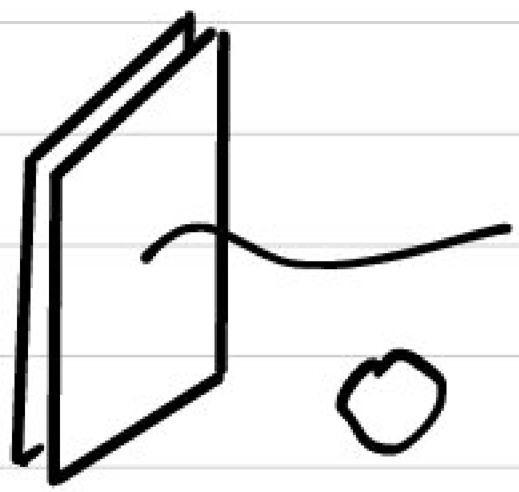
Last week : String theory with N D_p - branes



- open string in low energy limit

U(N) super Yang-Mills in $(p+1)$ dim

scale-invariant if $p=3$

CFT/AdS

- closed string in low energy limit

Backreaction of D-branes to the spacetime

Supergravity on AdS space

Black 3-brane metric

$$\begin{cases} ds^2 = \sqrt{f(\rho)}\left(-dt^2 + dx_i^2\right) + \dfrac{d\rho^2}{f(\rho)^2} + \rho^2 d\Omega_5^2 \\ f(\rho) = 1 - \dfrac{r_H^4}{\rho^4} \equiv \dfrac{1}{1 + \dfrac{r_H^4}{r^4}} \quad \text{for} \quad r^4 \equiv \rho^4 - r_H^4 \end{cases}$$

near-horizon limit : $r \ll 1$, $f \to \left(\dfrac{r}{r_H}\right)^4$, $z \equiv \dfrac{r_H^2}{r}$

$$ds^2 = r_H^2 \left(\frac{dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu}{z^2} + d\Omega_5^2 \right)$$

horizon :
$r \to 0$, $z \to \infty$

metric of Poincaré $AdS_5 \times S^5$ with radius r_H

N coincident D3 branes on flat spacetime

- Superconformal field theory (SCFT) in 4d

dual $\quad \mathcal{N}=4$ SYM w. $U(N)$ gauge group

- Supergravity (or superstring) on $AdS_5 \times S^5$

Unique Lagrangian theory with maximum SCF symm.

$\Rightarrow$ "proof" of the AdS/CFT correspondence

We identified the dual pair

What is the 'dictionary' of correspondence?

GKP - Witten relation

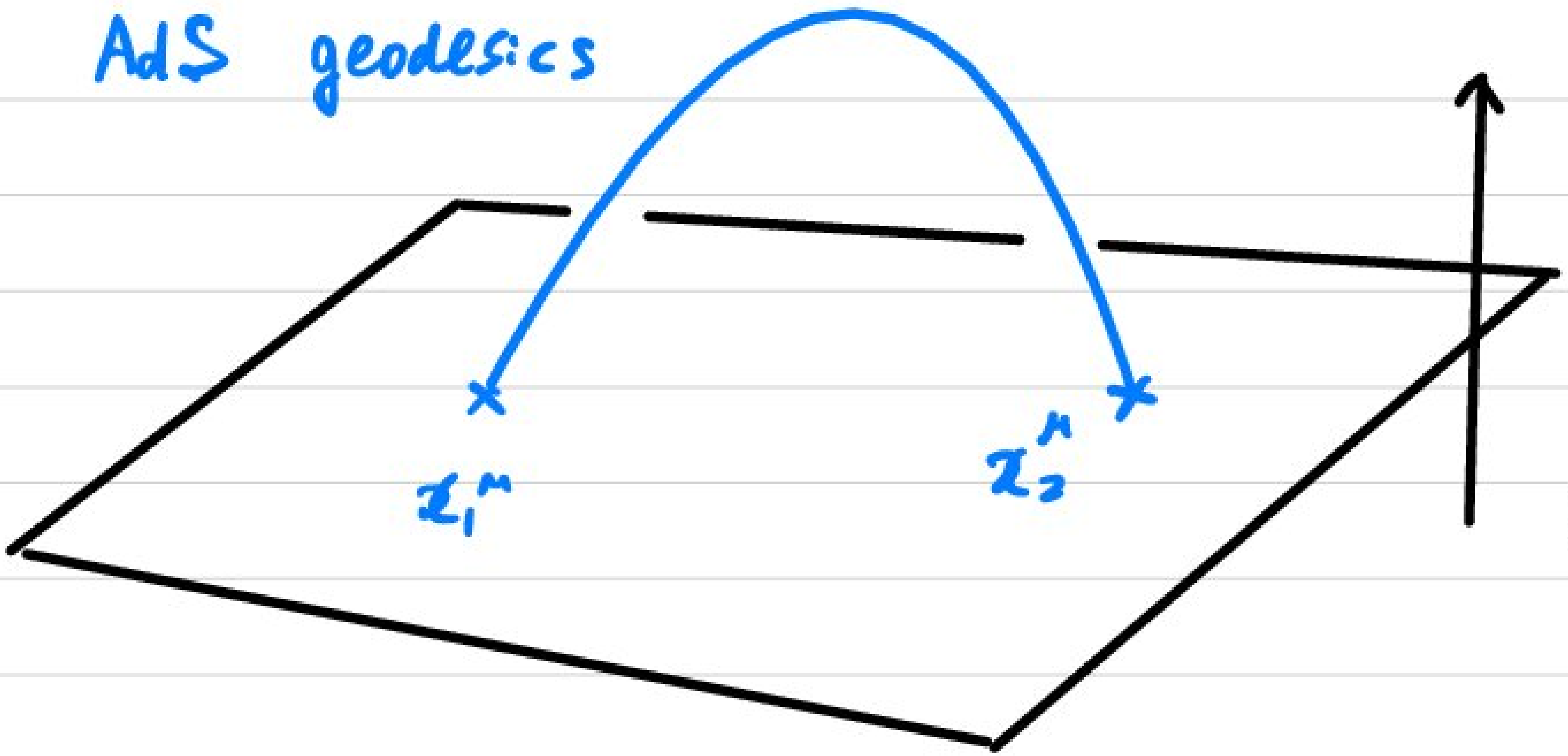
$$Z_{AdS}[\phi \to \phi_0] = \left\langle e^{i\mathcal{O}\phi_0} \right\rangle_{CFT}$$

gravity path integral with fixed boundary conditions

$\mathcal{O}$: source operator for ϕ_0

(different from topological gravity, no boundaries)

AdS geodesics



$z = \infty$: horizon (inside)

$z = 0$: AdS boundary

$$\langle \mathcal{O}(x_1)\, \mathcal{O}(x_2) \rangle \sim \frac{1}{|x_1 - x_2|^{2\Delta}} \sim \exp\left[-\Delta \underbrace{\log \frac{(x_1-x_2)^2}{\epsilon^2}}_{\sim \text{ geodesic length}} \right]$$

- Will revisit AdS calculation next week
- GKPW doesn't explain how to find $\Delta = \Delta(\lambda)$

Plan of this week (& next week)

- Superconformal symmetry

- $\mathcal{N}=4$ super Yang-Mills theory in 4D

  & superconformal invariance

- GKP-Witten relation , PP-wave limit

~ AdS / CFT setup without integrability

Maximal superconformal symmetry in 4d

$$psu(2,2|4) \supset su(2,2) \times su(4)_R$$

$$\underbrace{so(4,2)}_{\text{4d conformal group}} \times \underbrace{so(6)_R}_{\text{R-symmetry}}$$

Victor Kac : classified super Lie algebra (70's)

Werner Nahm : classified interacting SCFT (1978)

Dim	Algebra	String Theory Construction
6d :	$osp(8^*\|2\mathcal{N})$	$\mathcal{N}$ M5-branes
5d :	F_4	brane-web
4d :	$su(2,2\|\mathcal{N})$	$\mathcal{N}$ D3-branes
3d :	$osp(4\|\mathcal{N})$	$\mathcal{N}$ M2-branes
2d :	many possibilities	(N_1, N_5) D1-D5
		(uplift to M2-M5)

§ Superconformal Symmetry

$$\mathrm{sig}(\phi) = \begin{cases} 0 & \text{boson} \\ 1 & \text{fermion} \end{cases} \qquad ab = (-1)^{\mathrm{sig}(a)\cdot \mathrm{sig}(b)}\, ba$$

supervector $\quad \Phi_I = (\phi_i \mid \psi_\alpha) \qquad \begin{array}{l} i = 1 \cdots m \\ \alpha = 1 \cdots n \end{array}$

(N.B. BF dual $\check{\Phi}_I = (\check{\psi}_i \mid \check{\phi}_\alpha)$ $\quad \begin{array}{l} \phi : \text{boson} \\ \psi : \text{fermion} \end{array}$)

supermatrix

$$M_{IJ} = \begin{pmatrix} A_{ij} & B_{i\beta} \\ C_{\alpha j} & D_{\alpha\beta} \end{pmatrix} = \left(\begin{array}{c|c} B & F \\ \hline F & B \end{array}\right)$$

Supertrace : $\mathrm{Str}\, M = \mathrm{tr} A - \mathrm{tr} D$

Superdet : $$\mathrm{sdet}\, M \equiv \frac{\det(A - BD^{-1}C)}{\det D}$$

$$= \exp(\mathrm{str}(M))$$

$$\because) \quad M = \underbrace{\begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix}}_{\substack{\det = \mathrm{sdet} \\ = 1}} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \underbrace{\begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix}}_{\substack{\det = \mathrm{sdet} \\ = 1}}$$

super adjoint (supertranspose if $M = \overline{M}$)

$$\langle \Phi, \Phi' \rangle = \sum_i \bar{\phi}_i \phi_i' + \sum_\alpha \bar{\psi}_\alpha \psi_\alpha'$$

$$\langle M\Phi, \Phi' \rangle \equiv \langle \Phi, M^\dagger \Phi' \rangle$$

$$= \sum_i \overline{(A\phi + B\psi)_i}\, \phi'_i + \sum_\alpha \overline{(C\phi + D\psi)_\alpha}\, \psi'_\alpha$$

$$= \bar{A}_{ij} \bar{\phi}_j \phi'_i + \underbrace{\bar{B}_{i\alpha} \bar{\psi}_\alpha \phi_i'}_{-\bar{\psi}_\alpha \bar{B}_{i\alpha} \phi'_i} + \bar{C}_{\alpha j} \underbrace{\bar{\phi}_j}_{\text{Bos.}} \psi'_\alpha + \underbrace{\bar{D}_{\alpha\beta}}_{\text{Bos.}} \bar{\psi}_\beta \psi'_\alpha$$

$$= \bar{\phi}_j \left(\bar{A}_{ij} \phi'_i + \bar{C}_{\alpha j} \psi'_\alpha \right) + \bar{\psi}_\alpha \left(-\bar{B}_{i\alpha} \phi'_i + \bar{D}_{\alpha\beta} \psi'_\alpha \right)$$

$$M^{\dagger} = \begin{pmatrix} A^{\dagger} & C^{\dagger} \\ -B^{\dagger} & D^{\dagger} \end{pmatrix}$$

superunitary group $U(m|n)$

$$\langle U\phi , U\phi' \rangle = \langle \phi, \phi' \rangle , \quad \underbrace{U^{\dagger}}_{\text{super transpose}} U = 1$$

$$U(m|n) \supset U(m) \times U(n)$$

$$SU(m|n) \supset S(U(m) \times U(n)) \qquad \text{sdet } M = 1$$

$$PSU(m|m) \supset SU(m) \times SU(m) \qquad \text{sdet}(\lambda I) = 1 \;\; \forall \lambda \in \mathbb{R}$$

Consider infinitesimal transformation

$$M(x, \theta) = e^{ixT + i\theta Q}$$

$\rightsquigarrow$ super Lie algebra ($\mathbb{Z}_2$ - graded Lie alg)

graded commutator

$$[T, T'\} = TT' - (-1)^{\mathrm{sig}(T)\,\mathrm{sig}(T')} T'T$$

graded Jacobi id

$$[T_1, [T_2, T_3\}\} + [T_2, [T_3, T_1\}\} + [T_3, [T_1, T_2\}\} = 0$$

Poincaré algebra , $SO(1, d-1)$

[Sohnius (1985)]

[D'Hoker Freedman hep-th/0201253]

$$[P_\mu, P_\nu] = 0$$

$$[M_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho} P_\nu + \eta_{\nu\rho} P_\mu)$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho} M_{\nu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} - (\mu \leftrightarrow \nu))$$

$$\eta_{\mu\nu} = (-, +, +, \cdots, +) \quad , \quad M_{\mu\nu} = -M_{\nu\mu}$$

Conformal algebra , $SO(2, d)$

$$D = M_{d+1,d} \quad , \quad P_\mu = M_{d,\mu} + M_{d+1,\mu} \quad , \quad K_\mu = M_{d,\mu} - M_{d+1,\mu}$$

P_μ : momentum $\qquad P_\mu \phi(x) \sim i\partial_\mu \phi$

D : dilatation $\qquad D\phi(x) \sim i x\cdot\partial \phi$

K_μ : special conformal

$$K_\mu \phi(x) \sim i\{2x_\mu(x\cdot\partial) - x^2\partial_\mu\}\phi$$

$$[K_\mu, P_\nu] = 2i(\eta_{\mu\nu} D - M_{\mu\nu})$$

K and P are roughly "inverse"

Conformal primary ; $K_\mu \mathcal{O}(0) = 0$

P_μ creates descendants $\{\mathcal{O}, \partial_\mu \mathcal{O}, \partial_\mu\partial_\nu \mathcal{O}, \dots\}$

4 dim Supersymmetry

Q^a_α , $\bar{Q}_{\dot{\alpha} a}$

$\alpha = 1, 2$

$\dot{\alpha} = \dot{1}, \dot{2}$

$a = 1, 2, \cdots, N$

Dirac γ-matrix

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot{\beta}} \\ \bar{\sigma}^{\mu\dot{\alpha}\beta} & 0 \end{pmatrix} , \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

$(+, -, -, -)$

$$\sigma^\mu = (1, \sigma^i) , \quad \bar{\sigma}^\mu = (1, -\sigma^i)$$

with $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$

$$\{ Q^a_\alpha , \bar{Q}_{\dot{\beta} b} \} = 2 \delta^a_b (\sigma^\mu)_{\alpha\dot{\beta}} P_\mu$$

$$\{ Q^a_\alpha , Q^b_\beta \} = 2 \epsilon_{\alpha\beta} Z^{ab}$$

$Z^{ab} = - Z^{ba}$: central charge

$[Q, P_\mu] = [\bar{Q}, P_\mu] = 0$ and $\{Q, Q\} \sim Z$

apply the graded Jacobi for $[\{Q, Q\}, \bar{Q}]$

$$\rightsquigarrow [Q, Z^{ab}] = [Z^{ab}, Z^{cd}] = 0$$

Z^{ab} generates abelian symmetry

$$Q^a Q^b + Q^b Q^a = 2\epsilon \underline{Z^{ab}}$$

complex antisymmetric

$U(\mathcal{N})_R$ R-symmetry rotation:

$$Q^a \to U^{aa'} Q^{a'} \quad \text{for} \quad U\underline{U^\dagger} = 1$$

$$U^{aa'} U^{bb'} \{Q^{a'}, Q^{b'}\} = 2\epsilon \left(U Z \underline{U^T}\right)^{ab}$$

canonical form

$$\Rightarrow Z \sim \begin{pmatrix} 0 & z_1 & & & \\ -z_1 & 0 & & & \\ & & 0 & z_2 & \\ & & -z_2 & 0 & \\ & & & & \ddots \end{pmatrix} \qquad \underline{z_i \in \mathbb{R}_{\geq 0}}$$

Superconformal symmetry

Poincaré P_μ, $M_{\mu\nu}$

Conformal D, K_μ

SUSY Q, $\bar{Q}$, R $\quad \begin{cases} [Q,R] \sim Q \\ [\bar{Q},R] \sim -\bar{Q} \end{cases}$

SCF S, $\bar{S}$

$K_\mu \sim$ inverse of P_μ, $S \sim$ inverse of Q

4d SCF symmetry with $\mathcal{N}=4$ $\rightsquigarrow$ $\mathcal{N}=4$ SYM

$\{Q, \bar{Q}\} = \not{P}$, $\{Q, Q\} = Z$, $\{\bar{Q}, \bar{Q}\} = Z$

eigenvalues $\pm i Z_k$

$$\{Q + \bar{Q}, Q + \bar{Q}\} = 2(\not{P} + Z)$$

For massive particles, $p^\mu = (m, 0, 0, 0)$

$$\not{P} = \begin{pmatrix} 2m & 0 \\ 0 & 2m \end{pmatrix}$$

Assuming $\bar{Q} = Q^\dagger$, $(\not{P} + Z) \longrightarrow (m \mp Z_k) \geq 0$

unitarity bound

But $m = 0$ for gauge bosons; $Z_{ab} = 0$ for massless mult.

Commutation Relations :

$$\{Q, Q\} = \{S, S\} = \{Q, \bar{S}\} = \{\bar{Q}, S\} = 0$$

$$\{Q^a_\alpha, \bar{Q}_{\dot{\beta} b}\} = 2 \not{P}_{\alpha\dot{\beta}} \, \delta^a_b$$

$$\{S_{\alpha a}, \bar{S}^b_{\dot{\beta}}\} = 2 \not{K}_{\alpha\dot{\beta}} \, \delta^b_a$$

$$\{Q^a_\alpha, S_{\beta b}\} = \epsilon_{\alpha\beta}\left(\delta^a_b D + \underbrace{R^a_b}_{SU(4)_R}\right) + \frac{\delta^a_b}{2} L_{\mu\nu} \underbrace{\sigma^{\mu\nu}_{\alpha\beta}}_{\text{Pauli matrix}}$$

$$\sigma^\mu_{\alpha\dot{\gamma}} \, \bar{\sigma}^\nu_{\dot{\delta}\beta} \, \epsilon^{\dot{\delta}\dot{\gamma}} \equiv \eta^{\mu\nu} \epsilon_{\alpha\beta} - i \, \sigma^{\mu\nu}_{\alpha\beta}$$

We also have $[Q, K_\mu] = \gamma_\mu S$, $[S, K_\mu] = 0$

Conformal inversion : $I : x_\mu \to \dfrac{x_\mu}{x^2}$, $I^2 = 1$

$$IQI = S \ , \quad IP_\mu I = K_\mu$$

Radial quantization :

$$\langle \mathcal{O}_1 | \mathcal{O}_2 \rangle = \lim_{x \to 0} \langle I \cdot \mathcal{O}_1(x) \ \mathcal{O}_2(x) \rangle$$

$Q = S^\dagger$ in this inner product

unitarity bound in SCFT

$$\Rightarrow \{Q, S\} \sim (D + R + L) \geq 0$$

Conformal primary : $K_\mu |\mathcal{O}\rangle = 0$

Superconformal primary : $K_\mu |\mathcal{O}\rangle = S_{\alpha a} |\mathcal{O}\rangle = 0$

BPS states (short mult.) : $Q^a_\alpha |\mathcal{O}\rangle = 0$

BPS & superconformal primary

$$0 = \{Q, S\} |\mathcal{O}\rangle \sim (D + R + L) |\mathcal{O}\rangle$$

$\Rightarrow$ BPS states saturate the unitarity bound

$\mathcal{N}=4$ super Yang-Mills Lagrangian

- Dimensional reduction of 10D $\mathcal{N}=1$ SYM

$$\mathcal{L} = \mathrm{tr}\left(-\frac{1}{4} F_{MN} F^{MN} + \frac{i}{2} \bar{\lambda} \not{D} \lambda \right)$$

λ : Majorana - Weyl

- In $\mathcal{N}=1$ language,

vector multiplet + 3 massless chiral multiplet

adjoint representation of $G = SU(N_c)$

all interactions are commutators

Vector mult: $(\lambda, \quad F_{\mu\nu}, \quad \underline{D})$ ← auxiliary

Chiral mult: $(X_i, \ \psi_i, \ \underline{F_i})$ $\quad i = 1, 2, 3$

define $\quad X_i = \Phi_{2I-1} + i\,\Phi_{2I}, \quad \{\Phi_I, \ I = 1 \cdots 6\}$

$$\mathcal{L} = \frac{-1}{2 g_{YM}^2} \operatorname{tr} \left\{ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \Phi_I D^\mu \Phi_I + \frac{[\Phi_I, \Phi_J]^2}{2} \right.$$

$$\left. + i \bar{\lambda}_A \not{D} \lambda^A + [\bar{\Phi}_{AB}, \lambda^A] \lambda^B - (c.c.) + \theta\text{-term} \right\}$$

$\lambda^A = (\psi_1, \psi_2, \psi_3, \lambda)$: fund. rep of $SU(4)_R$

$$\Phi_I = \frac{1}{2}(T_I)^{AB}\bar{\Phi}_{AB} = \frac{1}{2}(\bar{T}_I)_{AB}\,\Phi^{AB}$$ $A, B = 1, 2, 3, 4$

$I = 1 \sim 6$

$(X_i, \bar{X}_i)$ $3 \oplus \bar{3}$ of $SU(3)_R$

$\Phi^{AB} = -\Phi^{BA}$ 6 of $SO(6)_R \simeq SU(4)_R$

$T_I{}^{AB}, (\bar{T}_I)_{AB}$: Γ-matrix of $SO(6)_R$

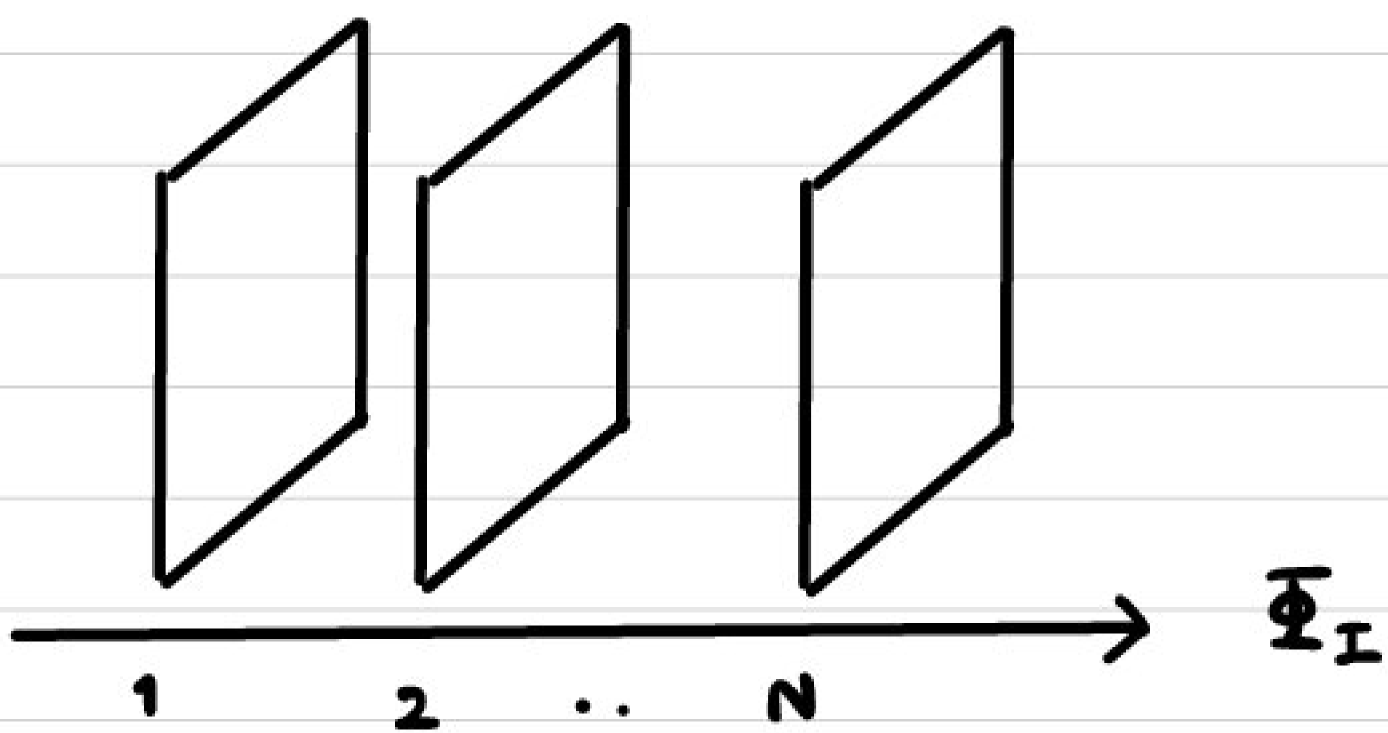
Φ_I 6 real scalars

Lagrangian has $SU(4)_R$ invariance.

Bosonic potential $\quad \mathrm{tr}\left([\Phi_I, \Phi_J]^2\right)$

Classical vacuum $\Rightarrow$ $[\Phi_I, \Phi_J] = 0$

Simultaneous eigenvalues $(\Phi_I)_{ii} \qquad i = 1, \cdots, N$



Superconformal phase :

$\langle \Phi_I \rangle = 0$, $U(N)$ or $SU(N)$ symmetry

- $U(1)$ controls the shift of origin

- $U(1)$ always decouple from commutators

Coulomb phase ; $\langle \Phi_I \rangle \sim \text{diag}\,(x_{I,1} \cdots x_{I,N})$

- $U(N) \longrightarrow U(1)^N$

- mass scale $\delta x \sim (x_{I,i} - x_{I,i+1})$

S-duality (conjecture)

$$\tau \equiv \frac{4\pi i}{g_{YM}^2} + \frac{\theta}{2\pi}, \qquad \mathcal{N}=4 \text{ SYM is invariant under}$$

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

Central charge for BPS dyons

$$Z \sim \sum_i \left(\underline{n_e^i} + \tau\, \underline{n_m^i} \right) \qquad \vec{n}_e \cdot \vec{n}_m \in \mathbb{Z}$$

change $n_e \leftrightarrow n_m$, $\tau \leftrightarrow \frac{1}{\tau}$

Usually we don't study S-duality in AdS / CFT

- Need to define $\lambda = N g_{YM}^2$, $\begin{cases} N \to \infty \\ g_{YM} \to 0 \end{cases}$

But $\dfrac{\tau}{N} = \dfrac{4\pi i}{\lambda} + \underline{\dfrac{\theta}{2\pi N}}$ disappear at $N \gg 1$

But AdS / CFT supports S-duality of $\mathcal{N}=4$ SYM

$\because$) IIB supergravity has $SL(2,\mathbb{Z})$ symmetry on axio-dilaton

[Minahan, 1012.3983]

β- function :

$\mathcal{N}=4$ SYM has only one coupling g_{YM}^2 (at $\theta=0$)

protected by SUSY $\Rightarrow$ β- function = 0.

$\because$) $\mathcal{N}=4$ SYM Lagrangian $\subset$ BPS multiplet

$$\mathrm{tr}\left(\phi^{(I}\phi^{J)}\right) - \frac{\delta^{IJ}}{6}\mathrm{tr}\left(\phi^k\phi^k\right)$$: $20'$ of $SO(6)_R$ symmetric traceless

OR

$$\mathrm{tr}\left(\phi^{ab}\phi^{cd}\right) - \frac{\epsilon^{abcd}}{4!}\epsilon_{a'b'c'd'}\,\mathrm{tr}\left(\phi^{a'b'}\phi^{c'd'}\right)$$

$\mathcal{N}=4$ on-shell multiplet (superspace)

$$\Phi = \underline{F_+} + \eta^A \chi_A + \frac{\eta^A \eta^B}{2} \underline{\phi_{AB}}$$

$$+ \frac{\eta^A \eta^B \eta^C}{3!} \epsilon_{ABCD} \bar{\chi}^D + \eta^A \eta^B \eta^C \eta^D \underline{F_-}$$

Boson : 2+6 , Fermions 4+4

on-shell condition : $\not{p}_{\alpha\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}$

$F_{\mu\nu} (\sigma^{\mu\nu})_{\alpha\beta} \sim F_+ \lambda_\alpha \lambda_\beta$, $F_{\mu\nu} (\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}} \sim F_- \bar{\lambda}_{\dot{\alpha}} \bar{\lambda}_{\dot{\beta}}$

Apply 4 supercharges to $20'$ multiplet :

$$\varepsilon^{\alpha_1\alpha_2}\varepsilon^{\alpha_3\alpha_4}\, Q^{a_1}_{\alpha_1} Q^{a_2}_{\alpha_2} Q^{a_3}_{\alpha_3} Q^{a_4}_{\alpha_4}\, \mathcal{O}^{a_1a_2a_3a_4}$$

$$\sim \varepsilon^{\alpha_1\alpha_2}\varepsilon^{\alpha_3\alpha_4}\, \mathrm{tr}\left(F_{+\,\alpha_1\alpha_2} F_{+\,\alpha_3\alpha_4} \right)$$

$$\varepsilon^{\dot\alpha_1\dot\alpha_2}\varepsilon^{\dot\alpha_3\dot\alpha_4}\, \bar{Q}^{a_1}_{\dot\alpha_1} \cdots \bar{Q}^{a_4}_{\dot\alpha_4}\, \mathcal{O}^{a_1\cdots a_4}$$

$$\sim \varepsilon^{\dot\alpha_1\dot\alpha_2}\varepsilon^{\dot\alpha_3\dot\alpha_4}\, \mathrm{tr}\left(F_{-\,\dot\alpha_1\dot\alpha_2} F_{-\,\dot\alpha_3\dot\alpha_4} \right)$$

Lagrangian $\mathcal{L} \sim -i\tau \, \underline{\mathrm{tr} F_+^2} + i\bar{\tau} \, \underline{\mathrm{tr} F_-^2}$

mass dimension = 4

protected by SUSY

coupling τ cannot run under RG flow $\longrightarrow \beta = 0$

$$\frac{\delta S}{\delta g_{\mu\nu}} = T_{\mu\nu}$$

conformal invariance $\langle T_\mu{}^\mu \rangle = 0$

(diffeo invariance $\langle T_{\mu\nu} \rangle = 0$)

$$\langle T_\mu{}^\mu \rangle = \underbrace{\beta(g)}_{=0} \, \mathrm{tr}(F^2) \underbrace{- a E + c W^2}_{\text{curvature corrections}}$$

$$\sim c_1 R_{\mu\nu\rho\sigma}^2 + c_2 R_{\mu\nu}^2 + c_3 R$$

vanishes in the (conformally) flat spacetime ; $\mathbb{R}^{1,3}$, $\mathbb{R} \times S^3$, ...

$\Rightarrow$ $\langle T_\mu{}^\mu \rangle = 0$, superconformal invariance of $\mathcal{N}=4$ SYM on $\mathbb{R}^{1,3}$

[Jack, Osborn 1312.0428] [Komargodski, Schwimmer 1107.3978]

In 2d $\langle T_\mu{}^\mu \rangle = \beta \mathcal{O} + \frac{c}{24} R$

- SCFT with exactly marginal deformation (like λ) is rare

- In critical string, we impose $\langle T_\mu{}^\mu \rangle = 0$ for any 2d Riemann surface $\Sigma_g \rightsquigarrow c = 0$

- In AdS/CFT, $c \sim N \to \infty$ is possible because the curvature term is 0

Question :

Why $ab = q\,ba$ not studied by Kac ?

Answer :

Suppose $[a, b\}_q \equiv a\,b - q^{\deg(a) + \deg(b)}\,b\,a$

Check generalized Jacobi identity

$$[a_1, [a_2, a_3\}\} + (\text{permutations}) \propto (q^2 - 1)$$

which is 0 only for $q = \pm 1$