Last week : GKP-Witten relations

Motion of a BPS particle on $AdS_5 \times S^5$

This week : One-loop mixing in $\mathcal{N}=4$ SYM

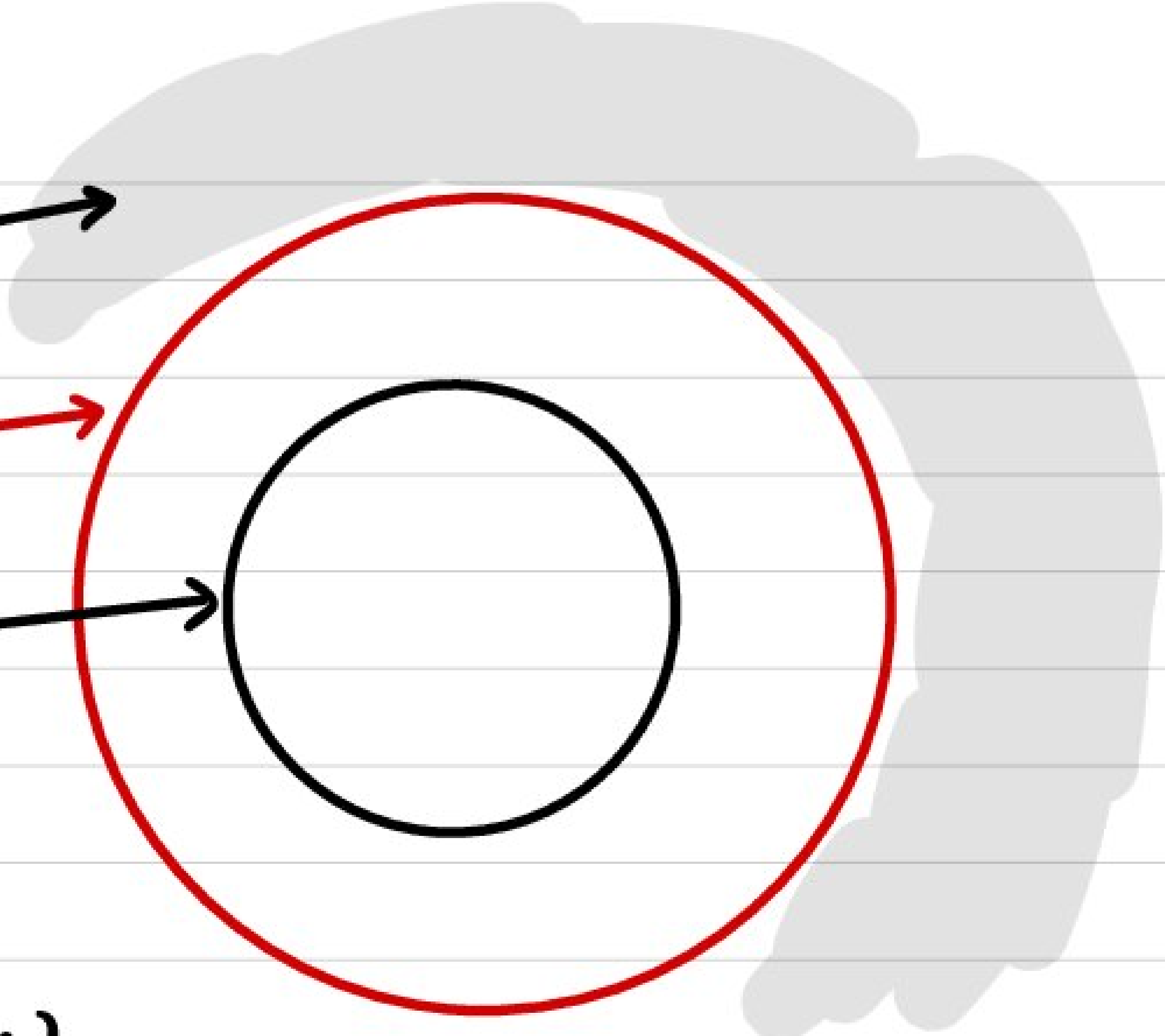
Quantum integrable models

§ Overview

Unsolvable models

solvable models

Solved models
(numerical values)



free theory

conformal theory (gapless)

e.g. (gauged) WZW model ← free-field representation

integrable models (gapped)

Three hard questions :

1) What is "quantum integrable" or "exactly solvable"

Something related to Yang-Baxter Equations

2) How to test if a quantum system is integrable?

diagnostics : level-spacing statistics, ETH, ...

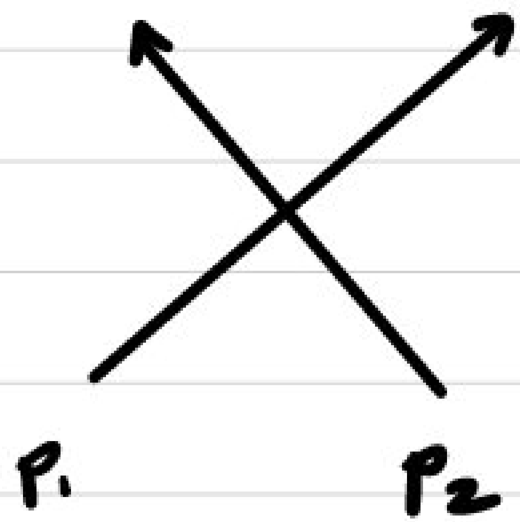
2) How is it related to QFT ?

based on S-matrix (on-shell scattering)

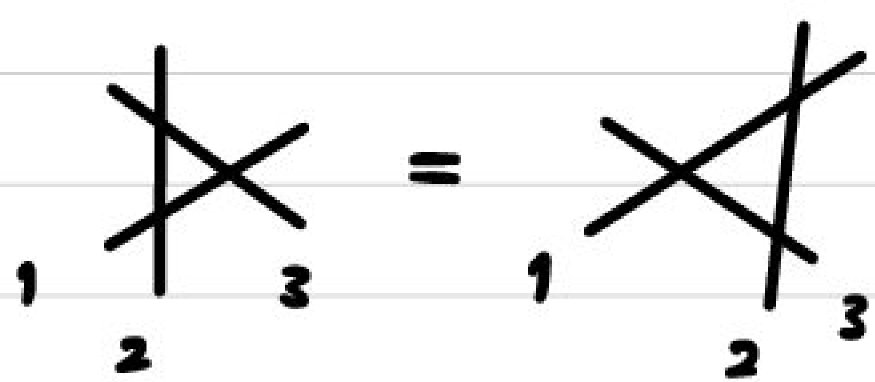
1) Definition of quantum integrability :

no "axiom-style" definition

Something related to Yang-Baxter equation



$= R_{12}(p_1, p_2)$: R-matrix



$\underline{R_{13}}\, R_{23}\, \underline{R_{12}} = \underline{R_{12}}\, R_{23}\, \underline{R_{13}}$

: YBE

N.B. YB is not so useful if

- $R_{ij}(p,q) = \delta_{ij}\, e^{i\phi(p,q)}$ $\rightsquigarrow$ trivial solution
- $R_{ij}(p,q) = R_{ij}$ (constant) $\rightsquigarrow$ needs "Baxterization"

Energy is a sum of one-particle energy $\epsilon(p)$

$$E(p_1, p_2, \cdots, p_N) = \sum_{i=1}^{N} \epsilon(p_i)$$

and the momenta are quantized :

$$\underbrace{\{p_i\} \in \mathbb{C}^N \longleftrightarrow}_{\text{nonlinear}} \underbrace{\{n_i\} \in \mathbb{Z}}_{\text{mode number (parameters)}}$$

2) Diagnostics of integrable systems

classical eom $\longrightarrow$ Lax pair

one-parameter extension

hard to find !

quantum Hamiltonian $\longrightarrow$ solution of YBE

hard to find !

Berry-Tabor conjecture : $\Delta E_n = E_{n+1} - E_n$ obeys Poisson statistics

ETH conjecture : $O_n = \langle n|O|n\rangle$ is <u>not</u> smooth function of the energy $E = E_n$

3) Often hard to compare QIM & QFT

QFT: Lagrangian $\rightarrow$ Correlator $\longrightarrow$ S-matrix

QIM: Asymptotic states & factorized S-matrix

- no off-shell formulation w. spectral parameter

  (Costello-Yamazaki-Witten for topological case)

- Correlator from form-factor expansion

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \sum_n \langle 0 | \mathcal{O}_1 | n \rangle \langle n | \mathcal{O}_2 | 0 \rangle$$

(hard to compute & sum may diverge)

§ One-loop operator mixing in $\mathcal{N}=4$ SYM

[Minahan, 1012.3983]

Lorentzian action of $\mathcal{N}=4$ SYM

$$S = \frac{-1}{2g_{YM}^2}\int d^4x \ \mathrm{tr}\left(\frac{F_{\mu\nu}F^{\mu\nu}}{2} + |D_\mu \phi^I|^2 + \sum_{I<J} [\phi^I, \phi^J]^2 + \cdots \right)$$

Euclidean action after rescaling

$$S_E = \int d^4x \ \mathrm{tr}\Big(\frac{F_{\mu\nu}^2}{4} + \frac{|D_\mu \phi_I|^2}{2} - \frac{g_{YM}^2}{2} \underbrace{\sum_{I<J} [\phi^I, \phi^J]^2}_{\text{negative definite}} + \cdots \Big)$$

Goal: planar one-loop mixing matrix in SU(2) sector

$$\Delta = \Delta_0 + \Gamma, \qquad \Gamma = \frac{N g_{YM}^2}{8\pi^2} \sum_{k=1}^{L} \left(I_{k,k+1} - P_{k,k+1} \right)$$

- Compute tree-level 2pt for real scalars
- Introduce single-trace operators in the SU(2) sector
- Planar scalar exchange at one-loop
  (using DREG, but regularization choice may be subtle)
- Fix the coefficient of $I_{k,k+1}$

If $L = \frac{1}{2}(\partial\phi)^2$ in 4D, the two-point function is

$$W[J] = \int D\phi \, e^{\int -\frac{1}{2}\partial\phi\partial\phi + J\phi} = e^{J\partial^{-2}J}$$

$$\langle \phi(x)\phi(0) \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{e^{i\vec{p}\vec{x}}}{|p|^2}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{1}{\Gamma(\alpha)} \int_0^\infty dt \, t^{\alpha-1} e^{i\vec{p}\vec{x} - t|p|^2} \qquad (\alpha = 1)$$

$$= \frac{1}{16\pi^4\Gamma(\alpha)} \int_0^\infty dt \, t^{\alpha-3} e^{-\frac{|x|^2}{4t}} \qquad u \equiv |x|^2/4t \qquad \frac{du}{u} = -\frac{dt}{t}$$

$$\langle \phi(x)\, \phi(0) \rangle = \frac{1}{16\pi^2 \Gamma(\alpha)} \left(\frac{4}{|x|^2} \right)^{2-\alpha} \underbrace{\int_0^\infty du \; u^{1-\alpha} e^{-u}}_{\Gamma(1)}$$

$$= \frac{1}{4\pi^2 |x|^2}$$

Thus, $\mathcal{L} = \frac{1}{2} \, \mathrm{tr}\left(D_\mu \phi^I \; D^\mu \Phi^I \right)$

$$\Rightarrow \left\langle \left(\Phi^I \right)_a{}^b(x) \; \left(\Phi^J \right)_c{}^d(y) \right\rangle = \frac{\delta_a^d \, \delta_c^b}{4\pi^2 \, |x-y|^2}$$

$I, J = 1, 2, \cdots 6$, $a, b, c, d = 1, 2, \cdots N$

tree-level 2pt
for real scalars

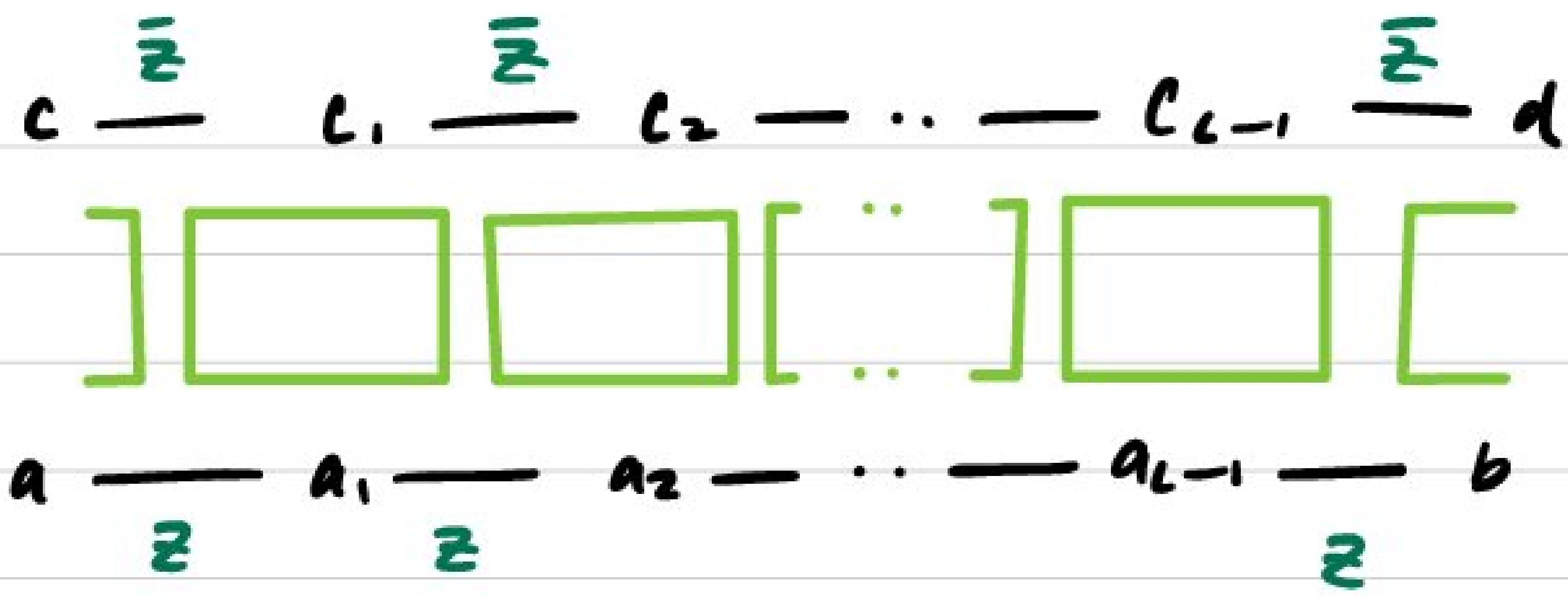
Complex scalars :

$$Z = \frac{\phi^5 + \sqrt{-1}\,\phi^6}{\sqrt{2}}, \quad W = \frac{\phi^3 + \sqrt{-1}\,\phi^4}{\sqrt{2}}, \quad \ldots$$

$$\frac{1}{2}\left| D_\mu \phi^I \right|^2 = D_\mu Z D^\mu \bar{Z} + D_\mu W D^\mu \bar{W} + \cdots$$

$$\langle Z(x)\, \bar{Z}(y) \rangle = \frac{\langle \phi^5 \phi^5 \rangle + \langle \phi^6 \phi^6 \rangle}{2} = \frac{\delta_a{}^d \delta^b{}_c}{4\pi^2 |x-y|^2}$$

Composite operator : $\displaystyle (Z^L)^a{}_b \equiv \sum_{a_1, a_2 \cdots}^{N} Z^a{}_{a_1} Z^{a_1}{}_{a_2} \cdots Z^{a_{L-1}}{}_b$



leading large N
$\shortparallel$
planar diagrams

$$\langle \left(Z^{a}{}_{a_1} Z^{a_1}{}_{a_2} \cdots Z^{a_{L-1}}{}_{b}\right)\left(\bar{Z}^{d}{}_{c_{L-1}} \cdots \bar{Z}^{c_2}{}_{c_1} \bar{Z}^{c_1}{}_{c}\right) \rangle$$

$$= \delta^{a}_{c} \underbrace{\delta^{c_1}_{a_1} \quad \delta^{a_1}_{c_1}}_{N} \underbrace{\delta^{c_2}_{a_2} \cdots}_{N} \quad \delta^{b}_{d} = \frac{N^{L-1} \delta^{a}_{c} \delta^{b}_{d}}{(4\pi^2)^{L} \, |x-y|^{2L}}$$

SU(2) sector

$$\mathcal{O} = \mathrm{tr}\,(\underbrace{ZZ\cdots Z}_{m}\,\underbrace{WW\cdots W}_{n}) + (\text{permute } Z \leftrightarrow W)$$

"Konishi descendant" $\quad \mathcal{O}_K = \mathrm{tr}\,(ZZWW - ZWZW)$

SU(2) sector ~ HWS of global PSU(2,2|4) rep.
(+ su(2) descendants)

representation & $\Delta_0 = m+n$ $\Rightarrow$ no other operators mix under (planar) perturbation of $\mathcal{N}=4$ SYM

Interaction :

$$S_E \sim \int d^4x \ \mathrm{tr} \left(\frac{1}{2} |D_\mu \phi^I|^2 + V \right)$$

$$V = -\frac{g_{YM}^2}{2} \sum_{I<J} \mathrm{tr} \left(\phi^I \Phi^J - \phi^J \phi^I \right)^2$$

$$= -g_{YM}^2 \sum_{I<J} \mathrm{tr} \left(\phi^I \phi^J \phi^I \phi^J - \underline{\phi^I \phi^J \phi^J \phi^I} \right)$$

responsible for scalar exchange

$ZW \to WZ$, or $ZW\bar{W}\bar{Z}$

complex scalar notation : $(\phi^5)^2 + (\phi^6)^2 = Z\bar{Z} + \bar{Z}Z$

$$V \sim g_{YM}^2 \sum_{I<J} \mathrm{tr}\left(\phi^I \phi^J \phi^J \phi^I\right)$$

$$= g_{YM}^2 \, \mathrm{tr}\big(ZW\bar{W}\bar{Z}$$: leading at large N

$$+ \bar{Z}\bar{W}WZ$$: wrong order

$$\underline{+ Z\bar{W}W\bar{Z} + \bar{Z}W\bar{W}Z \big)}$$

clearly not planar

$$\langle (Z^{a_1}{}_{a_2} W^{a_2}{}_{a_3})(\bar{W}^{b_1}{}_{b_2} \bar{Z}^{b_2}{}_{b_3})(Z^{c_1}{}_{c_2} W^{c_2}{}_{c_3} \bar{W}^{c_3}{}_{c_4} \bar{Z}^{c_4}{}_{c_1}) \rangle$$

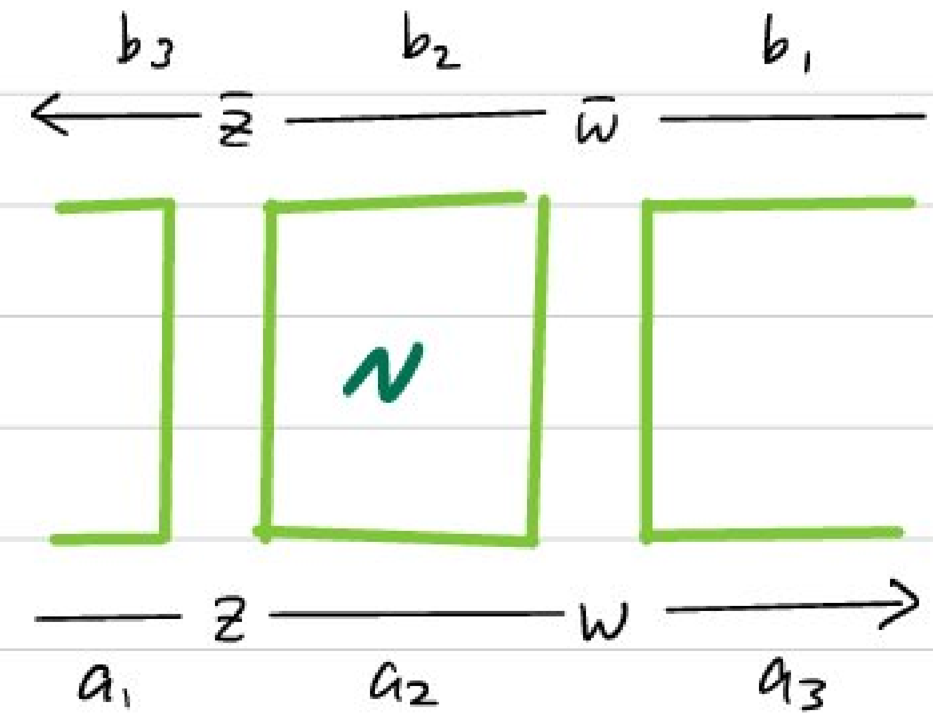
$$= \delta^{a_1}{}_{c_1} \underbrace{\delta^{c_4}{}_{a_2} \delta^{a_2}{}_{c_4}}_{N} \delta^{c_3}{}_{a_3} \delta^{b_1}{}_{c_3} \underbrace{\delta^{c_2}{}_{b_2} \delta^{b_2}{}_{c_2}}_{N} \delta^{c_1}{}_{b_3}$$

$$= \delta^{a_1}{}_{c_3} \delta^{b_1}{}_{a_3} N^2$$

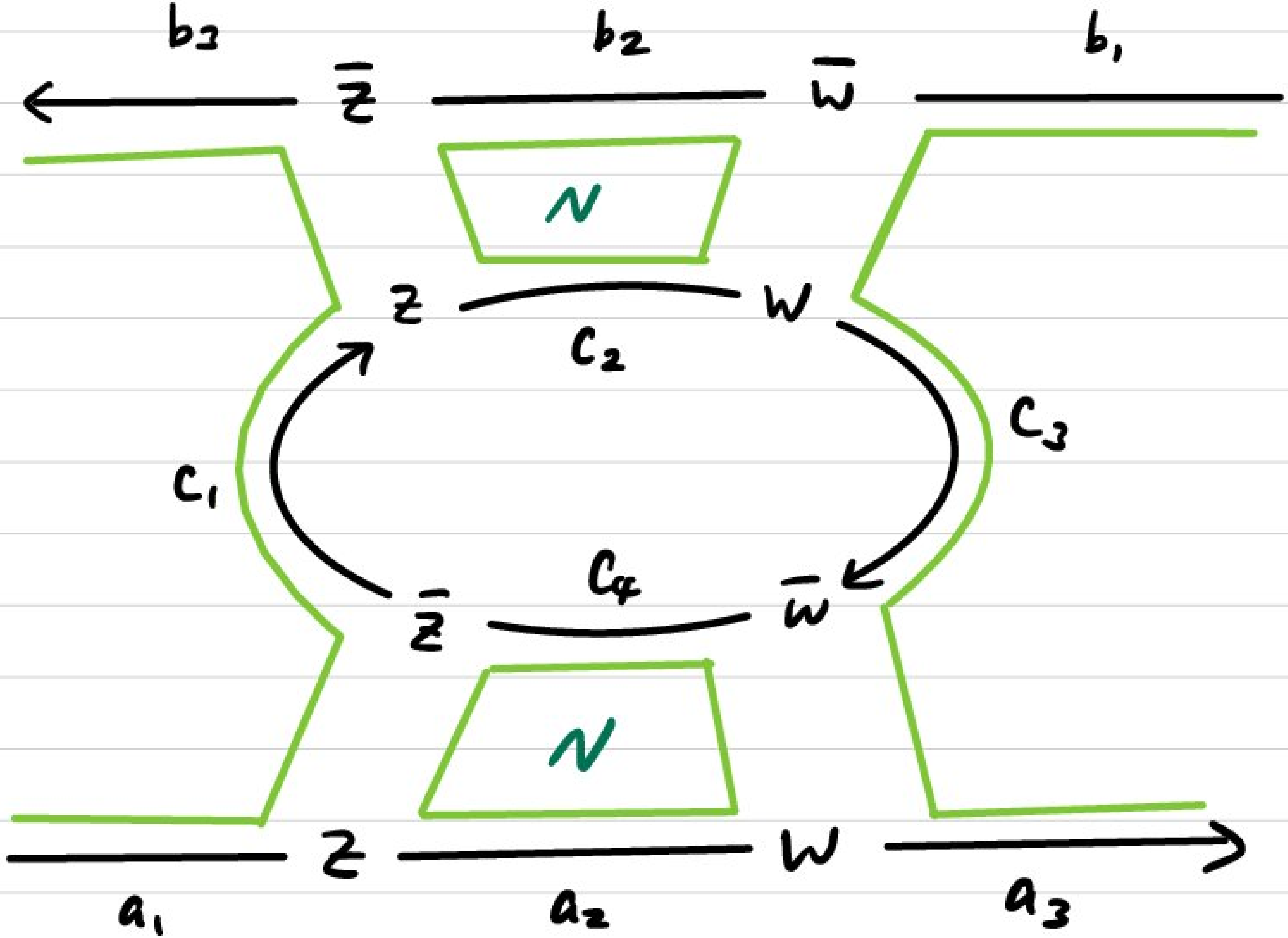
tree-level

double line

diagram



one-loop
double-line
diagram



$$\langle ZW(x) \cdot \bar{W}\bar{Z}(y) \, (1+V) \rangle$$

$$= \frac{N}{4\pi^2 |x-y|^4} \left\{ 1 + \frac{\lambda}{8\pi^2} \ln \Lambda^2 |x-y|^2 \right\}$$

Compare with general formula

$$\frac{1}{|x-y|^{2(\Delta_0+\gamma)}} \doteq \frac{1}{|x-y|^{2\Delta_0}} \left(1 - \gamma \log |x-y|^2 \right)$$

$\Rightarrow$ scalar exchange contribution $\Gamma = -\frac{\lambda}{8\pi^2} \sum_k P_{k,k+1}$

If we remember the spacetime dependence,

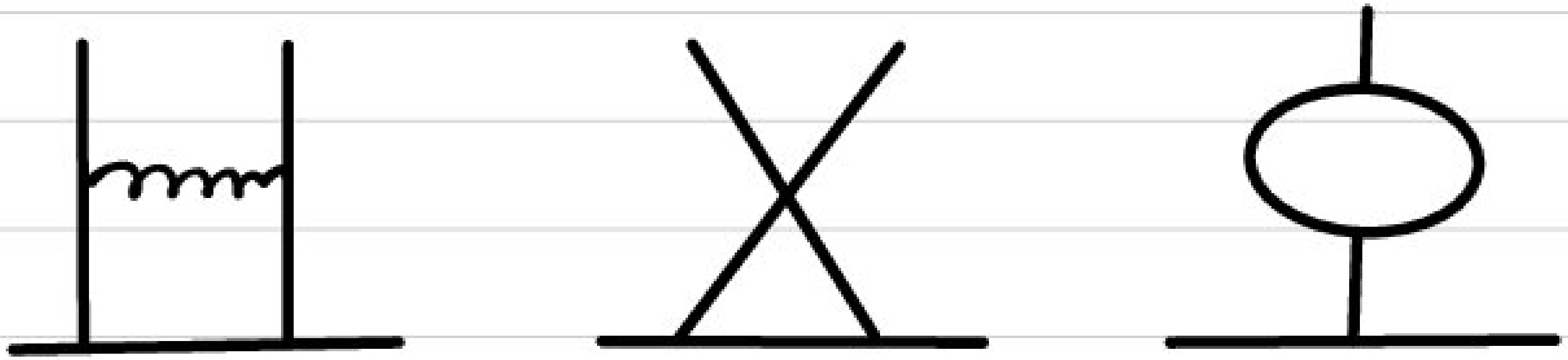
$$\langle ZW(x) \cdot \bar{W}\bar{Z}(y) \cdot g_{YM}^2 \int d^4z \; tr(ZW\bar{W}\bar{Z})(z) \rangle$$

$$= \underbrace{\frac{N}{(4\pi^2)^2}}_{\text{tree-level}} \cdot \frac{N g_{YM}^2}{16\pi^4} \cdot \underbrace{\int d^4z \frac{1}{|z-x|^4 \, |z-y|^4}}_{\text{operator mixing is UV property}}$$

$z = x + \Lambda^{-1}$ or $z = y + \Lambda^{-1}$, $\Lambda \gg 1$

$$\int \frac{d^4z}{|z-x|^4 \, |z-y|^4} = \frac{2}{|x-y|^4} \int_{1/\Lambda} \frac{2\pi^2 \, dz}{z} = \frac{-2\pi^2}{|x-y|^4} \ln\left(\frac{1}{\Lambda^2}\right)$$

The other diagrams contribute to the identity



$$\Gamma = \frac{\lambda}{8\pi^2} \sum_{k=1}^{L} \left(c\, I_{k,k+1} - P_{k,k+1} \right)$$

BPS state $\mathrm{tr}\, Z^L$ must be the zero eigenstate of Γ

$$\Gamma \cdot \mathrm{tr}\, Z^L = 0 \quad \Rightarrow \quad c = 1$$

$\Rightarrow$ this Γ is Hamiltonian of XXX spin chain

Spin-chain picture comes from the identity

$$P_{k,k+1} = \frac{1}{2}\left(I_k \otimes I_{k+1} + \vec{\sigma}_k \otimes \vec{\sigma}_{k+1} \right)$$

on the basis $\{ |\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle \}$

$$= \{ |ZZ\rangle, |ZW\rangle, |WZ\rangle, |WW\rangle \}$$

$$\vec{\sigma}_k \otimes \vec{\sigma}_{k+1} = \sigma_k^+ \otimes \sigma_{k+1}^- + \sigma_k^- \otimes \sigma_{k+1}^+ + \sigma_k^z \otimes \sigma_{k+1}^z$$

$$\sigma^{\pm} = \frac{\sigma^1 \pm i\sigma^2}{\sqrt{2}}, \qquad A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B \\ A_{21}B & A_{22}B \end{pmatrix}$$

After some algebra:

$$\frac{1}{2}\left(I_k \otimes I_{k+1} + \vec{\sigma}_k \otimes \vec{\sigma}_{k+1} \right) = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}$$

exchanges $e_2 = |\uparrow\downarrow\rangle$

$e_3 = |\downarrow\uparrow\rangle$

This "integrability" is generalized to

- higher loop order in SU(2) sector
- full sector of $\mathcal{N}=4$ SYM
- other local / nonlocal observables
- but mostly only in the planar large N limit

§ Algebraic Bethe Ansatz for XXX spin chain

[Faddeev , 9605187]

- One cannot "derive" R-matrix from spin chain Hamiltonian. We can only start from a solution of YBE , and check that it eventually reproduces :

$$H_{XXX} = \sum_{k=1}^{L} \vec{\sigma}_k \cdot \vec{\sigma}_{k+1}$$

- Faddeev promoted R-matrix to operators

  (L-operator , R-operator , fusion rules, ⋯)

R-matrix for fundamental representation of $SU(2)$:

$$R_{k,k+1}(\lambda) \equiv \lambda I_{k,k+1} + i P_{k,k+1}$$

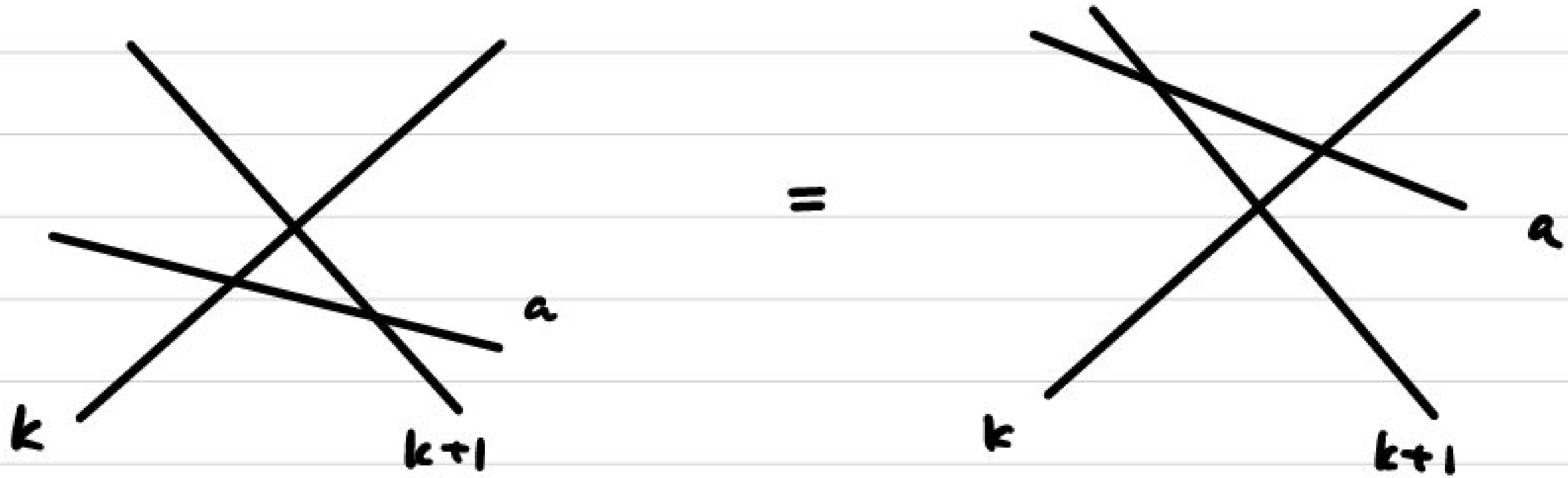
$$= \begin{pmatrix} \lambda+i & & & \\ & \lambda & i & \\ & i & \lambda & \\ & & & \lambda+i \end{pmatrix} \qquad (S^{\pm} = S^1 \pm i S^2)$$

$$= \underline{\begin{pmatrix} \lambda + i S^3_{k+1} & i S^-_{k+1} \\ i S^+_{k+1} & \lambda - i S^3_{k+1} \end{pmatrix}} + \frac{i}{2} I_{k,k+1}$$

$\equiv L_{k+1,k}(\lambda)$, L-operator

R-matrix satisfies Yang-Baxter

$\Rightarrow$ L-matrix satisfies RLL = LLR relation



$$R_{k,k+1}(\lambda-\mu)\, \underline{L_{k,a}}(\lambda)\, \underline{L_{k+1,a}}(\mu) = \underline{L_{k+1,a}}(\mu)\, \underline{L_{k,a}}(\lambda)\, R_{k,k+1}(\lambda-\mu)$$

$\because$) use identities, $P_{ab} = P_{ba}$, $P_{ab}^2 = 1$, $P_{na} P_{nb} P_{na} = P_{ab}$

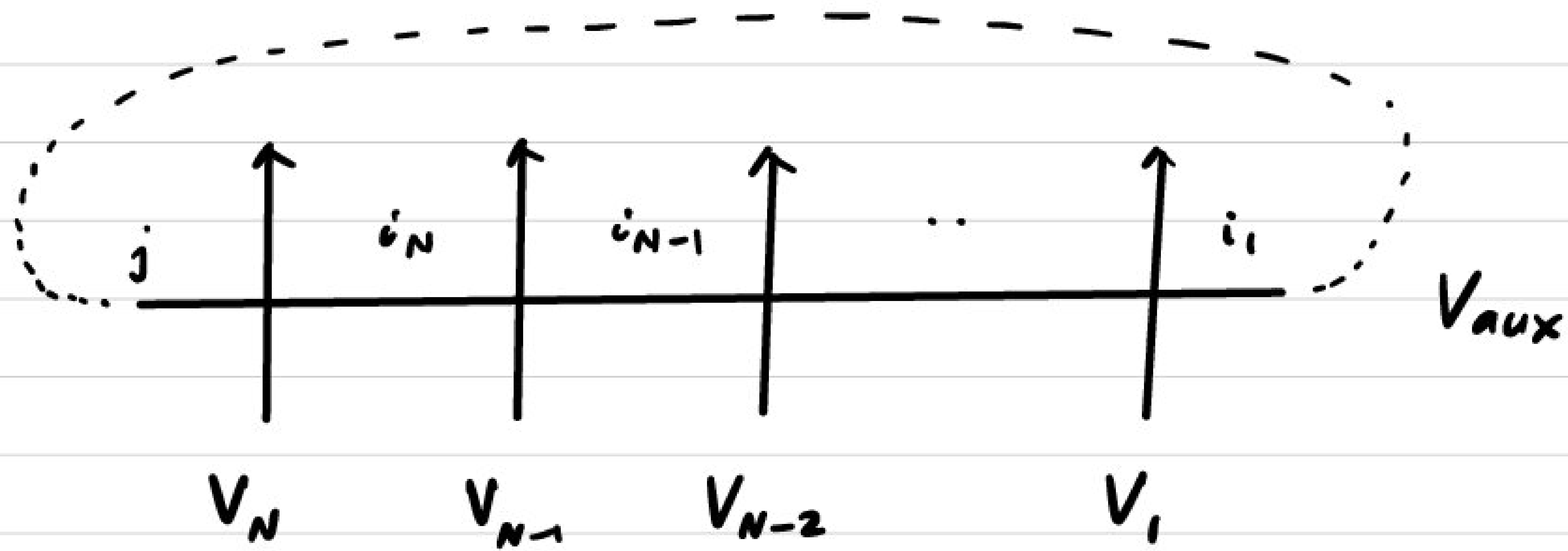
a = auxiliary space V_{aux} , $\dim V_{aux} = 2$

$$\begin{pmatrix} (L_{k,a})_{11} & (L_{k,a})_{12} \\ (L_{k,a})_{21} & (L_{k,a})_{22} \end{pmatrix} = \begin{pmatrix} \lambda + iS_k^3 & iS_k^- \\ iS_k^+ & \lambda - iS_k^3 \end{pmatrix}$$

$$\begin{cases} \text{take a tensor product over } V_N \otimes V_{N-1} \otimes \cdots \otimes V_1 \\ \text{take a matrix product over } V_{aux} \end{cases}$$

$$\sum_{i_1 i_2 \cdots i_N = 1}^{2} (L_{N,a})_{j\, i_N} \otimes (L_{N-1,a})_{i_N i_{N-1}} \otimes \cdots \otimes (L_{1,a})_{i_2 i_1}$$

$$\equiv M_a(\lambda)_{ji} \quad : \text{ monodromy matrix}$$

$M_a(\lambda)_{ji} =$ 

$$\begin{pmatrix} M_a(\lambda)_{11} & M_a(\lambda)_{12} \\ M_a(\lambda)_{21} & M_a(\lambda)_{22} \end{pmatrix} \equiv \begin{pmatrix} A_a(\lambda) & B_a(\lambda) \\ \underline{C_a(\lambda)} & \underline{D_a(\lambda)} \end{pmatrix}$$

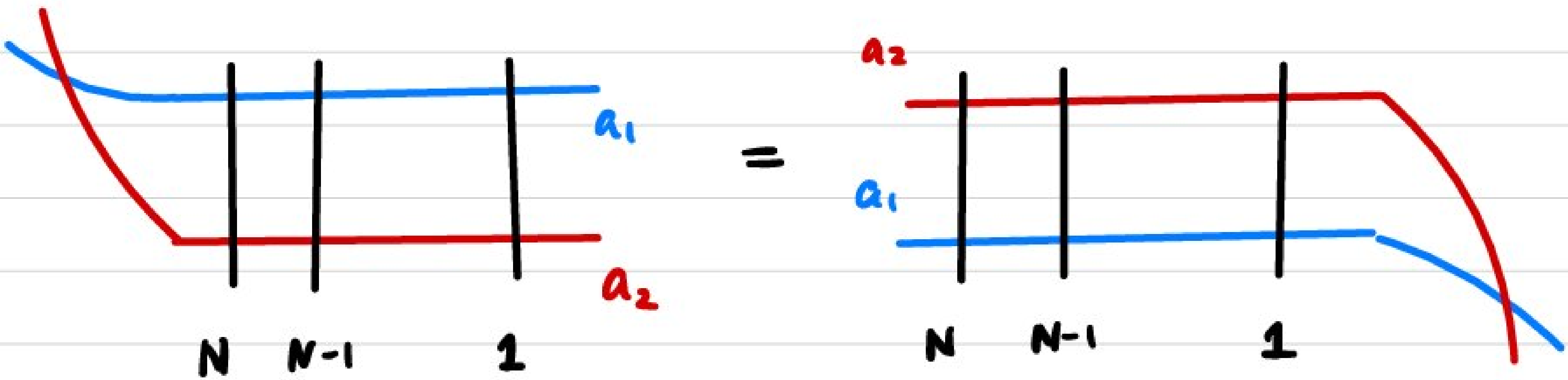
each entry is an operator on $V_{phys} \equiv V_N \otimes \cdots \otimes V_1$

$\dim V_{phys} = 2^N$

By using RLL = LLR relations, we obtain

$$R_{a_1 a_2}(\lambda-\mu)\, M_{a_1}(\lambda)\, M_{a_2}(\mu) = M_{a_2}(\mu)\, M_{a_1}(\lambda)\, R_{a_1, a_2}(\lambda-\mu)$$

as the operator on V_{aux}



Transfer matrix : $T(\lambda) \equiv tr_a M_a(\lambda) = A_a(\lambda) + D_a(\lambda)$

$$R_{12}\, M_1(\lambda)\, M_2(\mu)\, R_{12}^{-1} = M_2(\mu)\, M_1(\lambda)$$

$$\left\{ \begin{array}{l} \text{matrix indices on } V_{phys} \text{ are contracted} \\ \text{matrix indices on } V_{aux_1} \otimes V_{aux_2} \text{ are independent} \end{array} \right.$$

Take the trace over $V_{aux_1} \otimes V_{aux_2}$

$$\Rightarrow \quad T(\lambda)\, T(\mu) = T(\mu)\, T(\lambda) \quad \forall \lambda, \mu \in \mathbb{C}$$

$2^N \times 2^N$ matrices with $\lambda \neq \mu$ commute!

Expand $T(\lambda) = \mathrm{tr}\left(L_N L_{N-1} \cdots L_1\right)$

using $L_n(\lambda) = \begin{pmatrix} \lambda + iS_n^3 & iS_n^- \\ iS_n^+ & \lambda - iS_n^3 \end{pmatrix} = (\lambda - \frac{i}{2}) I_n + i P_n$

$\lambda \to \infty$: Global charge

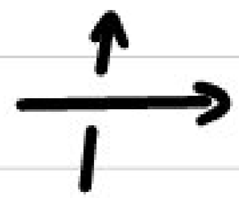
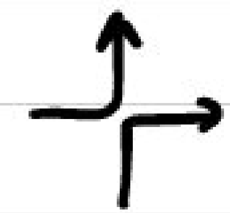
$$T(\lambda) = \lambda^N \, \mathrm{tr}_a \left[\bigotimes_n \left(\mathbb{I}_n + \frac{i}{\lambda} \mathcal{S}_n \right) \right]$$

$$= \lambda^N \, \mathrm{tr}_a \left[\underbrace{\mathbb{I}}_{2} + \frac{i}{\lambda} \underbrace{\sum_n \mathcal{S}_n}_{0} - \frac{1}{\lambda^2} \underbrace{\sum_{m<n} \mathcal{S}_m \otimes \mathcal{S}_n}_{\text{conserved charge}} + \cdots \right]$$

cf. off-diagonal $\to$ level-1 $Y(su_2)$, $Q_1 = \sum_{m<n} \epsilon^{abc} S_m^a S_n^b$

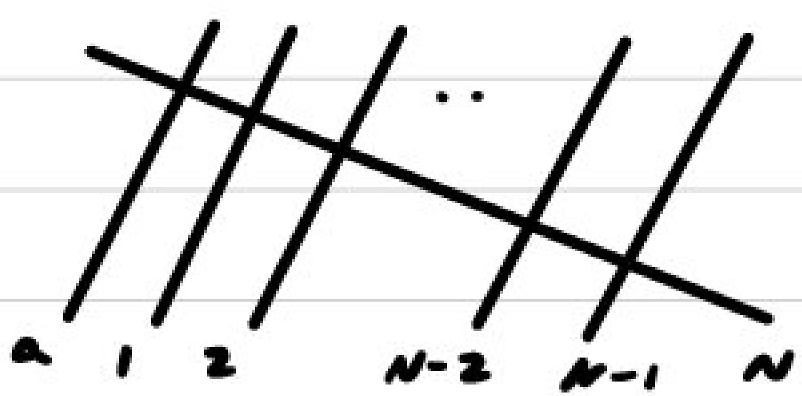
$\lambda \to \frac{i}{2}$: Local charge

$$T(\lambda) = i^N \,\mathrm{tr}_a \left[\bigotimes_n \left(P_{n,a} - i\left(\lambda - \frac{i}{2}\right) I_{n,a} \right) \right]$$

leading order, $I_{n.a} =$  , $P_{n.a} =$ 

$\hat{U} \equiv$ 

(permutation on the sites (n,a) not particle)

$=$ 

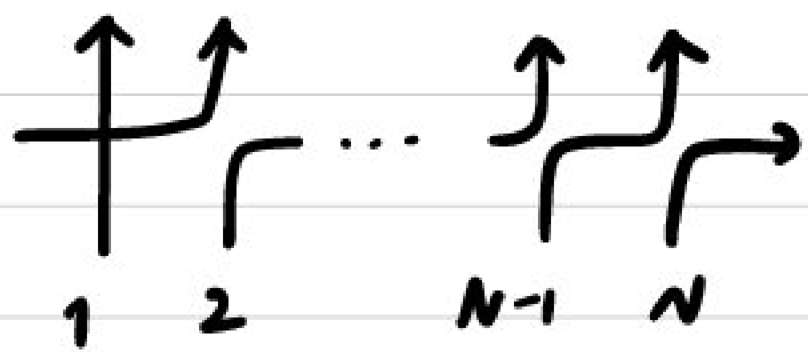
$$= P_{a,1} \underbrace{P_{1,2} P_{2,3} \cdots P_{N-1,N}}_{\equiv U}$$

$$P_{a,n} = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}, \quad \mathrm{tr}_a P_{a,n} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = 1_n$$

Expansion : $T(\lambda) = i^N U + i^{N-1}(\lambda - \frac{i}{2}) \, \mathrm{tr} \, \hat{F}_2 + \cdots$

shift operator, $U^{-1} X_n U = X_{n-1}$, $U^{-1} X_1 U = X_N$

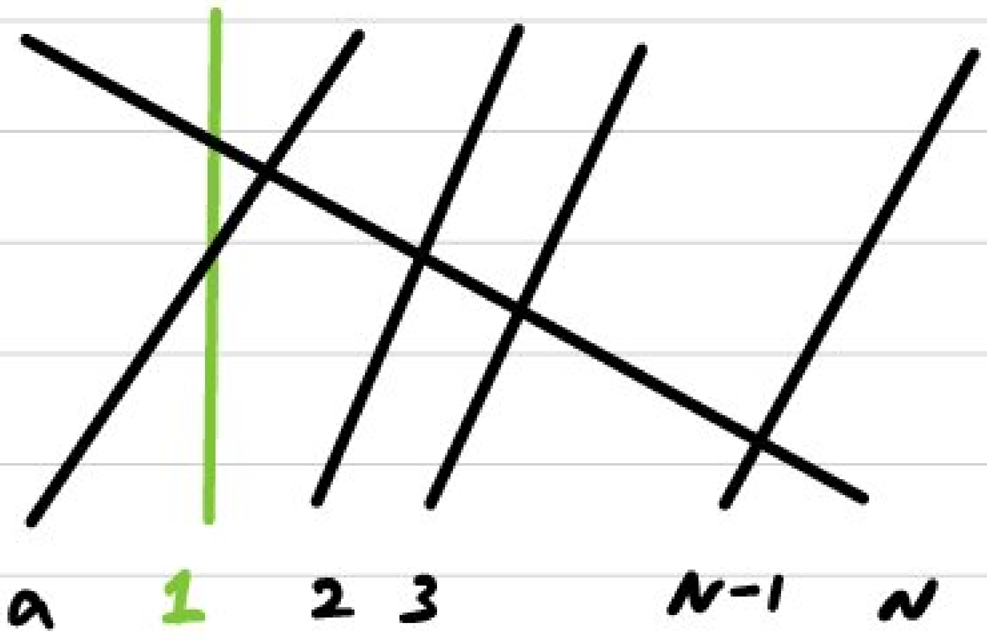
$$\hat{F}_2 = a$$



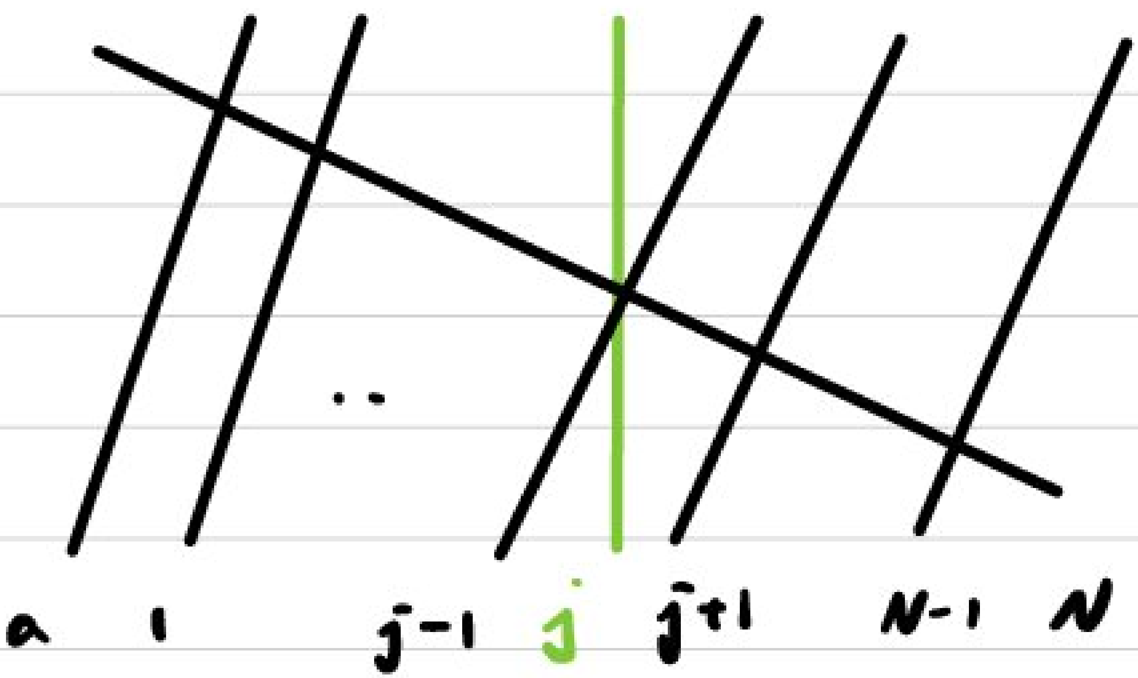
$$+ \sum_{j=2}^{N} \left(P_{a,j} \to I_{a,j} \right)$$

$P_{a,1}$ replaced by $I_{a,1}$

$$= P_{a,N} \; P_{2,3} \cdots P_{N-1,N} + \sum_j P_{a,1} P_{1,2} \cdots P_{j-1,j+1} \cdots P_{N-1,N}$$



$$\frac{P_{a,2} P_{2,3} \cdots P_{N-1,N}}{\begin{aligned} &= P_{1,2} U \\ &= U P_{N,1} \end{aligned}}$$



$$\frac{P_{a,1} P_{1,2} \; P_{j-1,j+1} \cdots P_{N-1,N}}{\begin{aligned} &= U \; P_{j,j+1} P_{j-1,j} P_{j-1,j+1} \\ &= U P_{j-1,j} \end{aligned}}$$

$$\mathrm{tr}\,\hat{F}_2 = U\left(P_{N,1} + \sum_{j=2}^{N} P_{j-1,j} \right)$$

$$i \frac{d}{d\lambda} \log T(\lambda) \Big|_{\lambda = \frac{i}{2}} = \sum_k P_{k,k+1} = \sum_k \frac{I_k \otimes I_{k+1} + \vec{\sigma}_k \otimes \vec{\sigma}_{k+1}}{2}$$

⇒ 2nd local charge = Spin $\frac{1}{2}$ XXX Hamiltonian

$$H_{XXX} = \sum_k \vec{\sigma}_k \cdot \vec{\sigma}_{k+1} = 2i \frac{d \log T}{d\lambda} \Big|_{\lambda = \frac{i}{2}} - N$$

⇒ This spin chain is integrable!

§ History (long!)

Heisenberg (1928) model

Bethe (1931) Ansatz

Lieb-Liniger (1963) model $V(x) = \sum_{i<j} \delta(x_i - x_j)$

McGuire (64) Yang (67) Baxter (72) relations

$\rightsquigarrow$ compute partition functions, $Z = \mathrm{tr}(e^{-\beta H})$

coordinate Bethe Ansatz for energy eigenstates

$$\psi = \sum_{x_1 < x_2} \left(e^{ip_1x_1 + ip_2x_2} + S(p_1,p_2)\, e^{ip_1x_2 + ip_2x_1} \right)$$

- Boussinesq (1877), Korteweg - de Vries (1895)

$$u_t - 6 u u_x + u_{xxx} = 0 \quad : \quad \text{KdV}$$

- Gelfand - Levitan (1951) : inverse scattering

$$-\psi_{xx} + V \psi = E \psi$$

scattering data for ψ at $x = \pm\infty$ $\Rightarrow$ Potential V

cf. Stokes sectors in ODE/IM

- Gardner, Greene, Kruskal, Miura (1967) : $V = u_{KdV}$

- Faddeev, Sklyanin, Takhtajan (1978) : quantum inv. scattering

R-matrix by quantizing classical QFT