Last week: One-loop planar operator mixing in the SU(2) sector of $\mathcal{N}=4$ SYM

$$\Delta = L + \frac{\lambda}{8\pi^2} \underbrace{\sum_{k=1}^{L} \left(I_{k,k+1} - P_{k,k+1} \right)}_{\text{Hamiltonian of } XXX_{1/2}}$$

$$H_{XXX} \sim \frac{d}{d\lambda} \log \underbrace{\mathrm{tr}\left(\mathcal{L}_{La}(\lambda) \cdots \mathcal{L}_{1a}(\lambda) \right)}_{= T(\lambda), \text{ transfer matrix}} \Big|_{\lambda = \frac{i}{2}}$$

This week : How to find (all) eigenvalues of H_{XXX}

by solving Bethe Ansatz Equations

Singular solutions of BAE

Separation of Variables

cf. XXX spin chain is related to many other models :

$$\left\{ \begin{array}{l} \text{6-vertex, sine-Gordon, massive Thirring} \\ c=1 \text{ CFT at self-dual point, } \ldots \end{array} \right\}$$

§ Transfer matrix eigenvalues [Faddeev, 9605187]

$$\mathcal{L}_{n,a}(\lambda) = \begin{pmatrix} \lambda + iS_n^3 & iS_n^- \\ iS_n^+ & \lambda - iS_n^3 \end{pmatrix}$$

$S^+|\uparrow\rangle = 0 \Rightarrow$ Define Bethe vacuum $|0\rangle = |\uparrow\uparrow\cdots\uparrow\rangle$

In terms of $\mathcal{N}=4$ SYM: $\{\uparrow, \downarrow\} \longleftrightarrow \{Z, W\}$

$|0\rangle \longleftrightarrow \mathrm{tr}\, Z^L$

$$\mathcal{L}_{n,a}(\lambda)\,|0\rangle = \begin{pmatrix} \lambda + iS_n^3 & * \\ 0 & \lambda - iS_n^3 \end{pmatrix}$$

Monodromy matrix :

$$M_a(\lambda)_{ij} = \left(\mathcal{L}_{L,a}(\lambda) \cdots \mathcal{L}_{1,a}(\lambda) \right)_{ij} \equiv \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$$

$i,j = 1,2$, A, B, C, D : $2^L \times 2^L$ matrices

$$M_a(\lambda)_{ij} \,|0\rangle = \begin{pmatrix} \lambda + \frac{i}{2} & * \\ 0 & \lambda - \frac{i}{2} \end{pmatrix}^L |0\rangle$$

$$\Rightarrow \quad A(\lambda)\,|0\rangle = \left(\lambda + \frac{i}{2}\right)^L |0\rangle$$

$$D(\lambda)\,|0\rangle = \left(\lambda - \frac{i}{2}\right)^L |0\rangle$$

$$\frac{d}{d\lambda} \log \left(A(\lambda) + D(\lambda) \right) \Big|_{\lambda = \frac{i}{2}} = (\text{const})$$ vacuum energy

For excited states, consider RTT relations

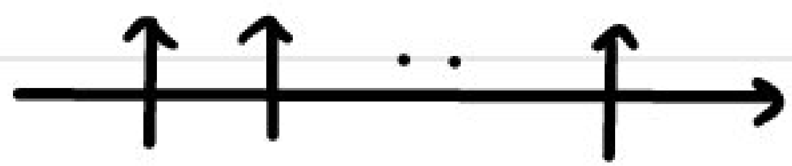
(In my notation, RMM = MMR)

$$R_{a_1 a_2}(\lambda-\mu)\, M_{a_1}(\lambda) M_{a_2}(\mu) = M_{a_2}(\mu) M_{a_1}(\lambda)\, R_{a_1 a_2}(\lambda-\mu)$$

$$R_{ab}(\lambda-\mu) = (\lambda-\mu) I_{ab} + i P_{ab}$$

$$= \begin{array}{cc} \begin{pmatrix} \lambda-\mu+i & 0 & 0 & 0 \\ 0 & \lambda-\mu & i & 0 \\ 0 & i & \lambda-\mu & 0 \\ 0 & 0 & 0 & \lambda-\mu+i \end{pmatrix} & \begin{array}{c} \uparrow\uparrow \\ \uparrow\downarrow \\ \downarrow\uparrow \\ \downarrow\downarrow \end{array} \end{array}$$

$\{\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow\}$ is natural basis of $V_{a_1} \otimes V_{a_2}$

$M_a(\lambda)_{ij} = i$  $j = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$

$V_L \otimes V_{L-1} \cdots \otimes V_1$

$$M_{a_1}(\lambda) \otimes M_{a_2}(\mu) = \begin{pmatrix} A(\lambda)A(\mu) & A(\lambda)B(\mu) & B(\lambda)A(\mu) & B(\lambda)B(\mu) \\ A(\lambda)C(\mu) & A(\lambda)D(\mu) & \cdot & \cdot \\ C(\lambda)A(\mu) & \cdot & \cdot & D(\lambda)B(\mu) \\ C(\lambda)C(\mu) & \cdot & \cdot & D(\lambda)D(\mu) \end{pmatrix}$$

$$M_{a_2}(\mu) \otimes M_{a_1}(\lambda) = \begin{pmatrix} A(\mu)\,A(\lambda) & B(\mu)\,A(\lambda) & A(\mu)\,B(\lambda) & B(\mu)\,B(\lambda) \\ C(\mu)\,A(\lambda) & D(\mu)\,A(\lambda) & C(\mu)\,B(\lambda) & D(\mu)\,B(\lambda) \\ A(\mu)\,C(\lambda) & \cdot & \cdot & B(\mu)\,D(\lambda) \\ A(\mu)\,C(\lambda) & \cdot & \cdot & D(\mu)\,D(\lambda) \end{pmatrix}$$

note that tensor structure is different, because we need to keep track of (a_1, a_2) indices

$RMM = MMR \Rightarrow$ 16 equations for $2^L \otimes 2^L$ matrices

$$\begin{pmatrix} \lambda-\mu+i & & \\ & \ddots & \\ & & \end{pmatrix}\begin{pmatrix} A(\lambda)A(\mu) & & \\ & \ddots & \\ & & \end{pmatrix} = \begin{pmatrix} A(\mu)A(\lambda) & & \\ & \ddots & \\ & & \end{pmatrix}\begin{pmatrix} \lambda-\mu+i & & \\ & \ddots & \\ & & \end{pmatrix}$$

(1,1) component $\Rightarrow$ $A(\lambda)A(\mu) = A(\mu)A(\lambda)$

similarly $[B(\lambda), B(\mu)] = [C(\lambda), C(\mu)] = [D(\lambda), D(\mu)] = 0$

$$\begin{cases} A+D \longrightarrow \text{eigenvalues of } T(\lambda) \\ B \longrightarrow \text{creation op.} \\ C \longrightarrow \text{annihilation op.} \end{cases}$$

We need $XB = * BX + * BX$ for $X = A, D$

Non-trivial components : (1,3) and (2,4)

$$\begin{cases} (\lambda-\mu+i)\, B(\lambda) A(\mu) = i\, B(\mu) A(\lambda) + (\lambda-\mu)\, \underline{A(\mu) B(\lambda)} \\ (\lambda-\mu)\, \underline{D(\lambda) B(\mu)} + i\, B(\lambda)\, D(\mu) = (\lambda-\mu+i)\, B(\mu)\, D(\lambda) \end{cases}$$

Ansatz for excited states (Bethe states) :

$$|\vec{u}\rangle = B(u_1)\, B(u_2) \cdots B(u_M)\, |0\rangle$$

Evaluate transfer matrix using commutation relations :

$$\big(A(v) + D(v) \big)\, B(u_1) \cdots B(u_M)\, |0\rangle$$

$$A(v)\,B(u_1) = \frac{v-u_1-i}{v-u_1}\,\underline{B(u_1)\,A(v)} + \frac{i}{v-u_1}\,\underline{B(v)\,A(u_1)}$$

$A(v)\,B(u_2) = \cdots$ $\qquad$ $A(u_1)\,B(u_2) = \cdots$

Commutation relations generate 2^M terms, summarized as

$$A(v)\,|\vec{u}\rangle = \left(\prod_{j=1}^{M}\frac{v-u_j-i}{v-u_j}\right) B(u_1)\cdots B(u_M)\cdot \underbrace{A(v)\,|0\rangle}_{=\left(v+\frac{i}{2}\right)^L|0\rangle}$$

$$+\sum_{k=1}^{M} M_k(v)\cdot \underbrace{B(v)\,B(u_1)\cdots(\text{no } B(u_k))\cdots B(u_M)}_{\text{unwanted term, not } |\vec{u}\rangle}\,\underbrace{A(u_k)\,|0\rangle}_{=\left(u_k+\frac{i}{2}\right)^L|0\rangle}$$

The coefficient M_k :

The $k=1$ term comes from

$$\begin{cases} A(v)\, B(u_1) = f(v-u_1)\, B(u_1)\, A(v) + \underline{g(v-u_1)\, B(v)\, A(u_1)} \\ A(u_1)\, B(u_j) = \underline{f(u_1-u_j)\, B(u_j)\, A(u_1)} + g(u_1-u_j)\, B(u_1)\, A(u_j) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } j \geq 2 \end{cases}$$

$$\Rightarrow \quad M_1(v) = \frac{i}{v-u_1} \prod_{j=2}^{M} \frac{u_1-u_j-i}{u_1-u_j}$$

$k \geq 2$ are complicated, but $B(u_1) \cdots B(u_M)|0\rangle$ is

invariant under permutation $u_j \to u_{\sigma(j)}$, $\sigma \in S_M$

Commutation relations for DB:

$$D(v)\,B(u_1) = \frac{v-u_1+i}{v-u_1}\,B(v)\,D(u_1) - \frac{i}{v-u_1}\,B(u_1)\,D(v)$$

$$D(v)\,|\vec{u}\rangle = \left(\prod_{j=1}^{M} \frac{v-u_j+i}{v-u_j}\right) B(u_1)\cdots B(u_M)\,\underbrace{D(v)\,|0\rangle}_{=\left(v-\frac{i}{2}\right)^L|0\rangle}$$

$$+ \sum_k \tilde{M}_k(v)\,\underbrace{B(v)\,B(u_1)\cdots(\text{no } B(u_k))\cdots B(u_M)}_{\text{unwanted term}}\,\underbrace{D(u_k)\,|0\rangle}_{\left(u_k-\frac{i}{2}\right)^L|0\rangle}$$

Therefore

$$(A(v) + D(v)) |\vec{u}\rangle$$

$$= \left\{ \left(v+\frac{i}{2}\right)^L \prod_{j=1}^{M} \left(\frac{v-u_j-i}{v-u_j} \right) + \left(v-\frac{i}{2}\right)^L \prod_{j=1}^{M} \left(\frac{v-u_j+i}{v-u_j} \right) \right\} |\vec{u}\rangle$$

$$+ \sum_{k=1}^{M} \left\{ M_k(v) \left(u_k+\frac{i}{2}\right)^L + \tilde{M}_k(v) \left(u_k-\frac{i}{2}\right)^L \right\}$$

$$B(v)\, B(u_1) \cdots (\text{no } B(u_k)) \cdots B(u_M) |0\rangle$$

$$\tilde{M}_1(v) = \frac{-i}{v-u_1} \prod_{j=2}^{M} \frac{u_1-u_j+i}{u_1-u_j} \quad \propto \quad \frac{1}{v-u_1}$$

Can derive Bethe Ansatz Equations (BAE) in two ways

1) Unwanted term = 0.

$$\left(u_1+\frac{i}{2}\right)^L \prod_{j=2} \frac{u_1-u_j-i}{u_1-u_j} - \left(u_1-\frac{i}{2}\right)^L \prod_{j=2} \frac{u_1-u_j+i}{u_1-u_j} = 0$$

$$\Longrightarrow \quad \left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^L = \prod_{j\neq k}^{M} \frac{u_k-u_j+i}{u_k-u_j-i}$$

for $k=1,2,\cdots,M$

2) Transfer matrix $T(v) = A(v) + D(v)$

must be a polynomial of v in degree L

$$T(v) = \left(v + \frac{i}{2}\right)^L \prod_{j=1}^{M} \frac{v - u_j - i}{v - u_j} + \left(v - \frac{i}{2}\right)^L \prod_{j=1}^{M} \frac{v - u_j + i}{v - u_j}$$

potentially singular at $v = u_k$; $\operatorname{Res} T(v) = 0$

$$0 = -i \left(u_k + \frac{i}{2}\right)^L \prod_{j \neq k} \frac{u_k - u_j - i}{u_k - u_j} + i \left(u_k - \frac{i}{2}\right)^L \prod_{j \neq k} \frac{u_k - u_j - i}{u_k - u_j}$$

$\Rightarrow$ same equation as before

Conserved charges :

1) $$U \sim T\left(\lambda = \frac{i}{2}\right) \sim \prod_{j=1}^{M} \left(\frac{u_j + i/2}{u_j - i/2} \right) \equiv \prod_{j=1}^{M} e^{ip_j}$$

shift operator $U \longleftrightarrow$ total momentum P

2) $$E \sim \frac{d}{d\lambda} \log T(\lambda) \Big|_{\lambda = \frac{i}{2}} \sim \sum_{j=1}^{M} \frac{1}{(u_j + \frac{i}{2})(u_j - \frac{i}{2})}$$

3) Higher charges : $$Q_r \equiv \sum_j \left\{ \frac{1}{(u_j + i/2)^{r-1}} - \frac{1}{(u_j - i/2)^{r-1}} \right\}$$

SU(2) symmetry :

Taking $\lambda \to \infty$ in RTT relations ,

Using $T(\lambda) = \begin{pmatrix} \lambda^N + i\lambda^{N-1} S^3 + \cdots & i\lambda^{N-1} S^- + \cdots \\ i\lambda^{N-1} S^+ + \cdots & \lambda^N + i\lambda^{N-1} S^3 + \cdots \end{pmatrix}$

$$A(\lambda)\, B(\mu) = \underbrace{\frac{\lambda-\mu-i}{\lambda-\mu}}_{= 1 - \frac{i}{\lambda} + \cdots} B(\mu)\, A(\lambda) + \underbrace{\frac{i}{\lambda-\mu} B(\lambda)\, A(\mu)}_{\mathcal{O}(\lambda^{N-2})}$$

Compare $\mathcal{O}(\lambda^{N-1})$:

$$i S^3 B(\mu) = i B(\mu) S^3 - i B(\mu) \Rightarrow [S^3, B(\mu)] = -B(\mu)$$

Similarly, (2,3) component of RTT relation:

$$(u-v)\, B(u)\, C(v) + i\, D(u) A(v) = i\, D(v) A(u) + (u-v)\, C(v) B(u)$$

large u limit & collect $O(u^L)$ using $C \to i u^{L-1} S^+$

$$[S^+, B(v)] = A(v) - D(v)$$

For the Bethe vacuum, $|0\rangle = |\uparrow\uparrow \cdots \uparrow\rangle$

$$S^3 |0\rangle = \sum_{n=1}^{L} S_n^3 |0\rangle = \frac{L}{2} |0\rangle$$

$$S^3 |\vec{u}\rangle = \left(\frac{L}{2} - M\right) |\vec{u}\rangle$$

Thus (regular) Bethe states are the SU(2) HWS

$$S^+ |\vec{u}\rangle = 0 .$$

Add $u_{M+1} = \infty$ to a solution of BAE $\{u_k\}$

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{j\neq k}^{M} \frac{u_k - u_j + i}{u_k - u_j - i} \underbrace{\frac{u_k - \infty + i}{u_k - \infty - i}}_{=1}$$

$$|u'\rangle \equiv \underbrace{B(\infty)}_{= i v^{L-1} S^- \ (v\to\infty)} B(u_1) \cdots B(u_M) |0\rangle$$ is still a solution

$= i v^{L-1} S^- \ (v \to \infty)$; SU(2) descendants

$$S^+ |\vec{u}\rangle = \sum_{k=1}^{M} B(u_1)\cdots B(u_{k-1}) \underline{\left(A(u_k) - D(u_k) \right)} \cdots B(u_M) |0\rangle$$

bring right

$$\equiv \sum_{l=1}^{M} C_l \; B(u_1)\cdots \left(\text{no } B(u_l)\right)\cdots B(u_M)\,|0\rangle$$

Using RTT relations,

$$\left(A(u_1) - D(u_1) \right) B(u_2) = B(u_2) \left(\frac{u_1 - u_2 + i}{u_1 - u_2} A(u_1) - \frac{u_1 - u_2 - i}{u_1 - u_2} D(u_1) \right) + \cdots$$

$$C_1 = \prod_{j=2}^{M} \frac{u_1 - u_j + i}{u_1 - u_j} \left(u_1 + \frac{i}{2}\right)^L - \prod_{j=2} \frac{u_1 - u_j - i}{u_1 - u_j} \left(u_1 - \frac{i}{2}\right)^L = 0$$

permutation symmetry $\Rightarrow$ $C_l = 0$ from BAE for u_l

[Marboe, Volin, 1608.06504]

§ Singular solutions :

Bethe states $|\vec{u}\rangle$ $\Rightarrow$ BAE for $\{u_k\}$

(~ PBC for particles on length-L circle)

$\Leftarrow$ is not true !

e.g.

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{j \neq k} \frac{u_k - u_j + i}{u_k - u_j - i}$$

if $(u_1, u_2) \simeq (i/2, -i/2)$, we get $\infty = \infty$

Can regularize by ϵ, but needs $O(\epsilon^L)$ term

There are other meaningless solutions like $u_j = u_k$

'meaningless' because Gaudin norm vanishes

$$\langle u|u\rangle = \det_{j,k}\left(\frac{\partial}{\partial u_j}\log BAE_k\right)$$

But there can be "almost degenerate" solutions

for twisted boundary conditions

$$e^{i\phi}\left(\frac{u_k+i/2}{u_k-i/2}\right)^L = e^{i\phi+iLp_k} = \prod_{j\neq k}\frac{u_k-u_j+i}{u_k-u_j-i}$$

Numerically, better to solve Baxter TQ relations

$$T(v)\,Q(v) = \phi(v-\tfrac{i}{2})\,Q(v+i) + \phi(v+\tfrac{i}{2})\,Q(v-i)$$

$$\phi(v) = v^L, \quad Q(v) = \prod_{j=1}^{M} (v-u_j)$$

polynomiality of $T(v) \Rightarrow$ BAE

But Baxter TQ relations with $(u_1, u_2) = (\tfrac{i}{2}, -\tfrac{i}{2})$ may contain "unphysical" solutions (overcounting)

$\rightsquigarrow$ need to solve QQ-system for a complete sol.

Remark : in coordinate Bethe Ansatz,

$$\psi(p_1, p_2) = \sum_{n_1 < n_2} \left\{ e^{ip_1 n_1 + ip_2 n_2} + S(p_1, p_2)\, e^{ip_1 n_2 + ip_2 n_1} \right\}$$

$$\mathrm{tr}\left(Z \cdots \underset{\hat{n}_1}{W} \cdots \underset{\hat{n}_2}{W} \cdots Z \right)$$

From trace cyclicity, ψ is invariant under the shift

$$(n_1, n_2, \cdots) \mapsto (n_1 + 1, n_2 + 1, \cdots) \Rightarrow e^{iP_{tot}} = 1$$

"momentum quantization" $e^{iLP_{tot}} = 1$

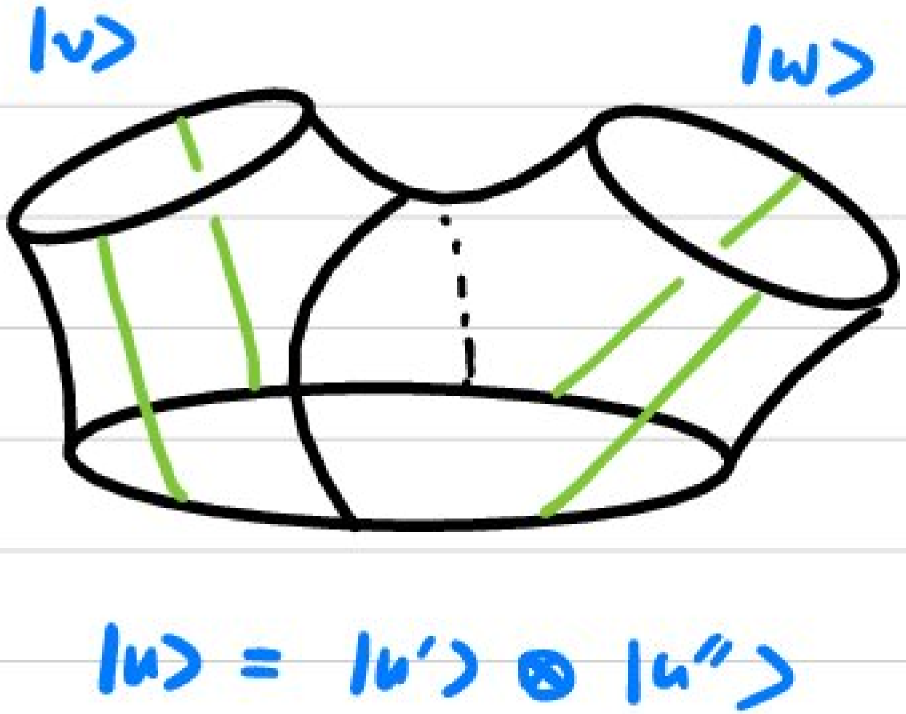
$\longrightarrow$ take a product of all BAE, $\prod_k e^{iLp_k} = 1$

§ Separation of Variables

3pt functions of $\mathcal{N}=4$ SYM :

- Split Bethe state $|u\rangle = |u'\rangle \otimes |u''\rangle$
- take inner product $\langle \mathcal{O}_u \mathcal{O}_v \mathcal{O}_w \rangle \sim \sum_{u = u' \cup u''} \langle v | u' \rangle \langle w | u'' \rangle$

$L = L_1 + L_2$, $M = M_1 + M_2$



$$\{u_j\}_{j=1}^{M} = \underline{\{u_j'\}_{j=1}^{M_1}} \cup \{u_j''\}_{j=M_1+1}^{M_1+M_2}$$

Solves BAE for L , not L_1

$\Rightarrow$ off-shell

For (on-shell) Bethe states

$$\langle u|v\rangle \propto \delta(\{u\},\{v\}) \times (\text{Gaudin det})$$

For off-shell Bethe states

$$\langle u|v\rangle \sim (\text{Slavnov determinant})$$

$\leftarrow$ Separation of Variables (SoV) basis

$$\langle x|u\rangle = \prod_{j=1}^{L}\prod_{k=1}^{M}(x_j - u_k)$$

What is this X ?

cf. [Ryan Volin 1810.10996]

[Gromov Levkovich-Maslyuk, Sizov, 1610.08032]

SoV basis :

complete basis which diagonalizes (twisted) B-operator

$$\langle x | B(u) = \prod_{j=1}^{L} (u - x_j)$$

B(u) is nilpotent (not diagonalizable) in periodic chain

Twist : $T_a^K(u) = \mathrm{tr}\left(K \, \mathcal{L}_{L,a}(u) \cdots \mathcal{L}_{1,a}(u) \right)$

$$= \begin{pmatrix} A^K(u) & B^K(u) \\ C^K(u) & D^K(u) \end{pmatrix}$$

$K \sim \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$

Expand $\quad B^k(u) = \sum_{n=0}^{L} u^n B^k_n$

$$B^k_n \sim \mathrm{diag}\{ b_n^{(1)}, b_n^{(2)}, \cdots, b_n^{(2^L)} \}$$

For each eigenvalue $\quad b^{(\alpha)}(u) = \sum_{n=0}^{L} u^n b_n^{(\alpha)} \quad (\alpha = 1, \cdots, 2^L)$

define the polynomial roots $\quad b^{(\alpha)}(u) = \prod_{j=1}^{L} (u - x_j^{(\alpha)})$

Left eigenvector $\quad \langle x^{(\alpha)} | B^k(u) = \langle x^{(\alpha)} | \prod_j (u - x_j^{(\alpha)})$

Off-shell Bethe state : $B(u_1)\cdots B(u_M)|0\rangle \equiv |u\rangle$

Wave-function for one magnon : $\psi(x|u) \equiv \prod_{j=1}^{L}(u-x_j)$

Some people introduces "quantum position operator"

$$\langle x_1, x_2, \cdots, x_L|\hat{x}_j = \langle x_1, x_2, \cdots, x_L| x_j$$

but $\{\hat{x}_j\}$ is complicated ($\hat{x}_j \sim (E_{11})_j$ for simple case)

Spectrum of $\hat{x}_j$ is $\underline{x_j = \theta_j \pm \frac{i}{2}}$ (θ_j : inhomogeneity)

SoV points

Evaluate transfer matrix :

$$\langle x^{(\alpha)} | T(v) | u \rangle = \langle x | \big(A(v) + D(v) \big) B(u_1) \cdots B(u_M) | 0 \rangle$$

RTT relations :

$$\big(A(v) + D(v) \big) B(u_1) = \frac{B(u_1)}{v - u_1} \left\{ (v - u_1 - i) A(v) + (v - u_1 + i) D(v) \right\}$$

$$+ \frac{B(v)}{v - u_1} \left\{ i A(u_1) - i D(u_1) \right\}$$

unwanted term vanishes if $v = x_j^{(\alpha)}$

$$\langle x^{(\alpha)} | B(v) = \langle x^{(\alpha)} | \prod_{j=1}^{L} (v - x_j^{(\alpha)})$$

$$\langle x^{(\alpha)} | T(x_j^{(\alpha)}) | u \rangle = \psi(x^{(\alpha)}|u_1) \cdots \psi(x^{(\alpha)}|u_M)$$

$$\left\{ \frac{Q(v-i)}{Q(v)} \left(v+\frac{i}{2}\right)^L + \frac{Q(v+i)}{Q(v)} \left(v-\frac{i}{2}\right)^L \right\}_{v=x_j^{(\alpha)}}$$

Using

$$\Psi(x) \equiv \prod_{k=1}^{M} \psi(x|u_k) = \prod_{k}^{M} \prod_{j=1}^{L} (u_k - x_j) = (-1)^{LM} \prod_{j=1}^{L} Q(x_j)$$

$$\Psi(x) \frac{Q(x_j^{(\alpha)} \mp i)}{Q(x_j^{(\alpha)})} = (-1)^{LM} \prod_{\ell \neq j} Q(x_\ell^{(\alpha)}) \cdot Q(x_j \mp i)$$

$$= \Psi(x_1, \cdots, x_j \mp i, \cdots x_L)$$

Therefore

$$\langle x^{(a)} | T(x_j^{(a)}) | u \rangle = (x_j^{(a)} + \tfrac{i}{2})^L \, \Psi(x_1 \cdots, x_j - i, \cdots x_L)$$

$$+ (x_j^{(a)} - \tfrac{i}{2})^L \, \Psi(x_1 \cdots, x_j + i, \cdots x_L) \qquad (\forall j = 1, \cdots, L)$$

For an on-shell state :

$$\langle x^{(a)} | T | u \rangle = \left(A(x_j^{(a)}) + D(x_j^{(a)}) \right) \Psi(x^{(a)})$$

$$\equiv t(x_j^{(a)}) \, \Psi(x^{(a)})$$

$$\Psi(x_1 \cdots x_L) = (-1)^{LM} \prod_j Q(x_j) \longrightarrow$$ equation factorizes for each j

Finding S.V points (Spectrum of B-operator)

If you introduce inhomogeneity

$$T(v) = \mathrm{tr}\left(\mathcal{L}_{L,a}(v-\theta_L)\cdots\mathcal{L}_{1,a}(v-\theta_1)\right)$$

transfer matrix eigenvalue with θ's :

$$\begin{cases} t(v) = \phi(v+\frac{i}{2})\dfrac{Q(v-i)}{Q(v)} + \phi(v-\frac{i}{2})\dfrac{Q(v+i)}{Q(v)} \\ \phi(v) = \prod_{j=1}^{L}(v-\theta_j) \end{cases}$$

Difference equation for the total wave function :

$$t(x_j^{(\alpha)})\, \Psi(x_1^{(\alpha)}, \cdots, x_L^{(\alpha)})$$

$$= \phi(x_j^{(\alpha)} + \tfrac{i}{2})\, \Psi(x_1^{(\alpha)}, \cdots, x_j^{(\alpha)} - i, \cdots, x_L^{(\alpha)})$$

$$+ \phi(x_j^{(\alpha)} - \tfrac{i}{2})\, \Psi(x_1^{(\alpha)}, \cdots, x_j^{(\alpha)} + i, \cdots, x_L^{(\alpha)})$$

or its factorized form for each j, α :

$$t(v)\, Q(v) = \phi(v + \tfrac{i}{2})\, Q(v-i) + \phi(v - \tfrac{i}{2})\, Q(v+i)$$

at $v = x_j^{(\alpha)}$, any j, α

polynomiality $\Rightarrow$ valid for any $v \in \mathbb{C}$

We should be able to define the wave-function only at 2^L SoV points.

if $x_j^{(\alpha)} = \theta_j \pm \frac{i}{2}$, then $\phi(x_j^{(\alpha)} \mp \frac{i}{2}) = 0$.

$$\begin{cases} t(\theta_j + \frac{i}{2})\, \Psi(\cdots x_j = \theta_j + \frac{i}{2} \cdots) = \phi(\theta_j + i)\, \Psi(\cdots x_j = \theta_j - \frac{i}{2} \cdots) \\ t(\theta_j - \frac{i}{2})\, \Psi(\qquad \theta_j - \frac{i}{2} \qquad) = \phi(\theta_j - i)\, \Psi(\cdots x_j = \theta_j + \frac{i}{2} \cdots) \end{cases}$$

$\Rightarrow$ determines Ψ at different X's consistently

These SoV points are consistent w. construction of twisted B-op.