Week 5 : RTT relations $\longrightarrow$ Bethe Ansatz, SoV

Week 3 : Particle motion on $AdS_5 \times S^5$

This week : Classical integrable systems

Sigma model on $\mathbb{R} \times S^3 \subset AdS_5 \times S^5$

($\longleftrightarrow$ SU(2) sector in $\mathcal{N}=4$ SYM)

Pohlmeyer reduction and giant magnons

Classical spectral curve

§ Classical Integrable System

EoM in classical mechanics $\longrightarrow$ nonlinear PDE

$$\dot{q}_i = f_i(p,q), \quad \dot{p}_i = g_i(p,q)$$

SoV (if possible) $\longrightarrow$ Action-angle variables

$$\dot{I}_i = 0, \quad \theta_i = \omega_i t$$

$\longrightarrow$ Liouville integrability, but how to find it?

A: No axiomatic "proof", many working examples

Idea: express eom as a linear system & find one-parameter generalization

1) Lax pair

$$\frac{d}{dt} L(\lambda) = [L(\lambda), M(\lambda)]$$

$\lambda \in \mathbb{C}$: spectral parameter

$L(\lambda)$, $M(\lambda)$: matrix or differential op.

$$\frac{d}{dt} \operatorname{tr} L^n = 0 \implies \operatorname{tr} L^n = \sum_n \lambda_n Q^n$$

2) Zero curvature equation

(Lax connection, Zakharov - Shabat eq.)

$$[\partial_t - L_t(\lambda), \underbrace{\partial_x - L_x(\lambda)}_{= M(\lambda) \text{ of Lax pair}}] = 0$$

compatibility condition of auxiliary linear problem

$$\left(\partial_\alpha - L_\alpha(\lambda)\right)\psi(t,x) = 0, \quad \alpha = t, x$$

Consider Wilson line (monodromy matrix)

$$W(b,a) = \underline{P\exp} \int_a^b dx\, L_x(t,x,\lambda)$$

path ordered exponential

$$\partial_a W = -W\, L_x(t,a)\,, \qquad \partial_b W = L_x(t,b)\, W$$

$\Rightarrow$ $\psi'(x) = W(x,y)\,\psi(y)$ satisfies

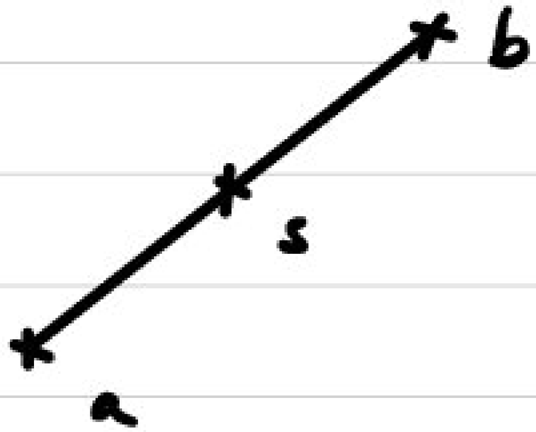
$$(\partial_x - L_x)\,\psi' = 0$$

$W(x,y)$ is a gauge transformation

Consider time derivative

$$\partial_t W = \int_a^b ds \; W(b,s) \, \underline{\partial_t L_x(t, x=s)} \; W(s,a)$$

$$= \partial_x L_t + [L_t, L_x]$$



This is total derivative:

$$\partial_s \Big(W(b,s) \, L_t(s) \, W(s,a) \Big)$$

$$= -W L_x L_t W + W \partial_x L_t W + W L_t L_x W$$

$$= W \left(\partial_x L_t + [L_t, L_x] \right) W$$

Thus

$$\partial_t W(b,a) = L_x(x=b)\, W(b,a) - W(b,a)\, L_x(x=a)$$

Wilson loop (transfer matrix)

$$T_c(\lambda) = \mathrm{tr}\left(P\exp\left[\oint_c dx\, L_x(t,x,\lambda) \right] \right)$$

generates the conserved charges,

$$\dot{T}_c = 0, \quad T_c(\lambda) = \sum_n \lambda_n Q^n$$

zero-curvature $\Rightarrow$ $T_c(\lambda) = T_{c+\delta c}(\lambda)$

as long as connection is smooth.

Examples

1) KdV equation, $\begin{cases} L = -\partial_x^2 + u \\ M = 4\partial_x^3 - 6u\partial_x - 3u_x \end{cases}$

$$\partial_t L = [L, M] \quad \Rightarrow \quad u_t + 6uu_x + u_{xxx} = 0$$

$L\psi = \lambda\psi$: "Schrödinger equation"

$M = M_2$ defines "energy" ; $\exists$ higher charges M_n

2) sine- Gordon

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{\beta^2}\left(\cos(\beta\phi) - 1\right)$$

eom: $$0 = \left(-\partial_t^2 + \partial_x^2\right)\phi - \frac{m^2}{\beta}\sin(\beta\phi)$$

rescale $(t, x, \phi) \longrightarrow m = \beta = 1$

Lax connection :

$$\begin{cases} L_t = \dfrac{i}{4}\begin{pmatrix} -\phi_x & \lambda e^{-i\phi/2} - \lambda^{-1} e^{i\phi/2} \\ \lambda e^{i\phi/2} - \lambda^{-1} e^{-i\phi/2} & \phi_x \end{pmatrix} \\ L_x = \dfrac{i}{4}\begin{pmatrix} -\phi_t & \lambda e^{-i\phi/2} + \lambda^{-1} e^{-i\phi/2} \\ \text{''} & \phi_t \end{pmatrix} \end{cases}$$

Finding one-soliton solution of sG:

$$T = \frac{t+x}{2}, \quad X = \frac{t-x}{2}$$

$$\partial_t^2 \phi - \partial_x^2 \phi + \sin\phi = 0 \Rightarrow \partial_T \partial_X \phi + \sin\phi = 0$$

Bäcklund transform: ϕ_0 is sol $\Leftrightarrow$ ϕ_1 is sol

$$\begin{cases} \partial_T (\phi_1 - \phi_0) = 2a \sin\left(\dfrac{\phi_1 + \phi_0}{2}\right) \\ \partial_X (\phi_1 - \phi_0) = -\dfrac{2}{a} \sin\left(\dfrac{\phi_1 - \phi_0}{2}\right) \end{cases} \quad (\forall a \in \mathbb{C})$$

$\Rightarrow$ linear in ∂_T, ∂_X

Substitute $\phi_0 = 0$

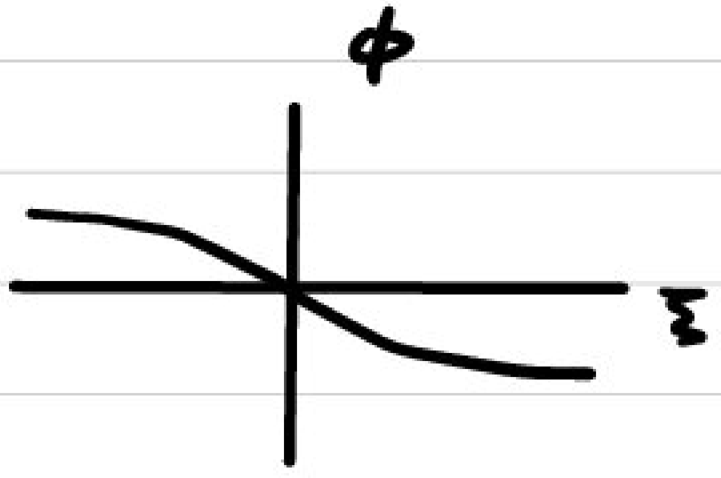
$$\frac{1}{2a}\frac{\partial}{\partial T}\phi_1 = \sin\frac{\phi_1}{2}, \qquad \frac{a}{2}\frac{\partial}{\partial X}\phi_1 = -\sin\frac{\phi_1}{2}$$

$$\xi \equiv \frac{X}{a} - aT, \qquad \frac{1}{2}\frac{\partial}{\partial \xi}\phi_1 = -\sin\frac{\phi_1}{2}$$

using

$$\int d\log\tan\frac{\phi}{4} = \int \frac{1}{2}\frac{d\phi}{\sin \phi/2} = -\xi$$

$$\phi_1 = 4\arctan e^{-\xi}$$



In the original (x, t) coordinate

$$-\xi = \frac{x - vt}{\sqrt{1-v^2}}, \qquad a = \frac{1-v}{\sqrt{1-v^2}}$$

velocity $\longleftrightarrow$ spectral param. $\lambda = a$

Energy :

$$E = \int \frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\partial_x \phi)^2 + (1-\cos\phi)$$

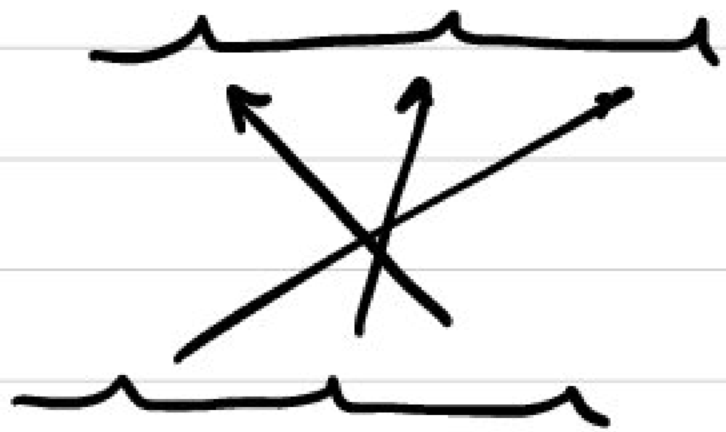
$$= \frac{4}{\sqrt{1-v^2}} \int d\xi \, \frac{1}{\cosh^2 \xi} = \frac{8}{\sqrt{1-v^2}}$$

energy concentrates at $\xi = 0$

Note

- Energy of N solitons

$$E = \sum_{i=1}^{N} \frac{8}{\sqrt{1-v_i^2}}$$



- Complicated to find the profile of N-solitons

(dressing method is easier than Bäcklund)

§ Classical sigma model on $\mathbb{R}_t \times S^3$ [Dorey, Vicedo 0601194]

(nonlinear) sigma model:

$$S = \frac{R^2}{4\pi\alpha'} \int_M d^2\sigma \sqrt{-\gamma}\, \gamma^{ab} G_{MN} \partial_a X^M \partial_b X^N$$

Sigma model on $M \simeq G/H$ is classically integrable

If $M \simeq \mathbb{R} \times S^3 \simeq \mathbb{R} \times SU(2)$

$$G \partial_a X \partial_b X = \mathrm{tr}\left[(g^{-1}\partial_a g)(g^{-1}\partial_b g) \right], \quad g \in SU(2)$$

"principal chiral model (PCM)"

Parameterization of $g \in SU(2)$

$$g = \begin{pmatrix} X_1 + iX_2 & X_3 + iX_4 \\ -X_3 + iX_4 & X_1 - iX_2 \end{pmatrix} = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$$

$$\text{with} \quad 1 = \sum_{i=0}^{4} |X_i|^2 = \sum_{\alpha=1,2} |z_\alpha|^2$$

OR

$$g = e^{i\phi\sigma_3} e^{i\theta\sigma_2} e^{i\psi\sigma_3} = \begin{pmatrix} \cos\theta\, e^{i(\phi+\psi)} & \sin\theta\, e^{i(\phi-\psi)} \\ -\sin\theta\, e^{-i(\phi-\psi)} & \cos\theta\, e^{-i(\phi+\psi)} \end{pmatrix}$$

Use

- $X_0 = \kappa\tau$ for time in $\mathbb{R}_t \times S^3$ (Euclidean)

- $\sqrt{\lambda} \equiv R^2/\alpha'$ is 't Hooft coupling

- $j_a = g^{-1}\partial_a g$, su(2) current (Maurer-Cartan)

  $*j_a = \epsilon^{ab} j_b$ with $a, b = 1, 2$

- choose conformal gauge in Euclidean w.s.

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int \left[dX_0 \wedge *dX_0 + \frac{1}{2} \mathrm{tr}(j \wedge *j) \right]$$

Flatness condition :

from $j = g^{-1} dg$, $dj = dg^{-1} \wedge dg = -j \wedge j$

EoM : from $g \to (1+\epsilon) g$, $g^{-1} \to g^{-1} (1-\epsilon)$

$$j \to j + d\epsilon$$

integration by parts $\rightsquigarrow$ $d * j = 0$

Virasoro condition : $\mathrm{tr}(j_+)^2 = \mathrm{tr}(j_-)^2 = -2k^2$

$$j_\pm = j_0 \pm j_1$$

Spacetime energy : $E = \frac{\sqrt{\lambda}}{2\pi} \oint d\sigma \; \partial_0 t = \sqrt{\lambda}\, K$

$SU(2)_L \times SU(2)_R$ symmetry ;

$$g \to U_L \, g \, U_R , \quad U = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} , \quad \begin{cases} z_1 \to e^{i\alpha_R + i\alpha_L} z_1 \\ z_2 \to e^{i\alpha_R - i\alpha_L} z_2 \end{cases}$$

Nöther charge :

$$J_1 + J_2 = \frac{\sqrt{\lambda}}{4\pi i} \oint \mathrm{tr}\left(g^{-1} \partial_0 g \cdot \sigma_3 \right)$$

$$J_1 - J_2 = \frac{\sqrt{\lambda}}{4\pi i} \oint \mathrm{tr}\left(\partial_0 g \, g^{-1} \cdot \sigma_3 \right)$$

Lax connection :

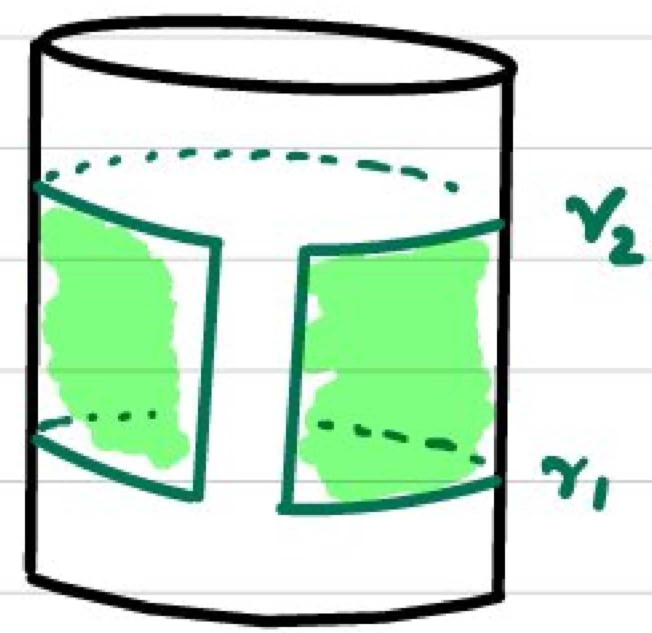
$$J(x) \equiv \frac{j - x * j}{1 - x^2} \Rightarrow dJ(x) - J(x) \wedge J(x) = 0$$

flatness :

$$J_+(\tau, \sigma, x) = \frac{1}{2}\left(\frac{j_+}{1-x} - \frac{j_-}{1+x} \right)$$

Monodromy :

$$\Omega(x) = P\exp\left(\oint_\gamma d\sigma \; J_+(\tau, \sigma, x) \right)$$



$\Omega(x)$ is independent of reference point

(τ, σ) because of flatness condition

$$\int d\sigma d\tau \, (dJ - J \wedge J) = 0$$

Classification of <u>all</u> solutions of PCM

$\longleftrightarrow$ all possible forms of quasi-momentum $p(x)$

We skip construction of string profile $X^\mu(\tau,\sigma)$

(but not easy to compute $p(x)$ from a given $X^\mu(\tau,\sigma)$)

Profile $\longrightarrow$ Baker - Akhiezer function

(Fay identity on Riemann Θ function + uniqueness)

$\Omega = P\exp\left(\oint J_+\right)$ is 2×2 matrix

Matrix elements are meromorphic in x (except $x = \pm 1$)

Diagonalize $\Omega \longrightarrow p(x)$: quasi-momentum

$$U \Omega U^{-1} = \mathrm{diag}\left(e^{ip(x)}, e^{-ip(x)}\right)$$

$p(x)$ has $\begin{cases} \text{simple poles at } x = \pm 1 \\ \text{square-root branch cuts, } \quad p \longleftrightarrow -p \end{cases}$

§ Pohlmeyer reduction and giant magnons

[Hofman, Maldacena 0604135]

Operators in $\mathcal{N}=4$ SYM

$$\mathcal{O} \sim \sum_{n_1 < n_2} e^{ip_1 n_1 + ip_2 n_2} \, \mathrm{tr}\,(ZZZ \cdots \underset{\hat{n}_1}{W} \cdots \underset{\hat{n}_2}{W} \cdots ZZZ)$$

W is "excitation" or "magnon"

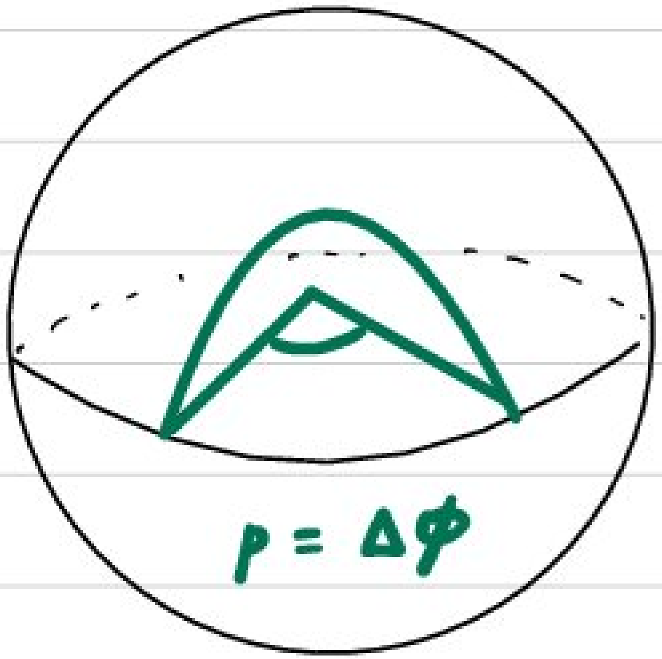
In classical string theory, we expect

magnon $\longleftrightarrow$ soliton solution in $AdS_5 \times S^5$

"giant magnon"

"Soliton" on $\mathbb{R} \times S^2$ $\xrightarrow[\text{Pohlmeyer reduction}]{}$ soliton of Sine-Gordon eq.

$$\partial_+ \partial_- \vec{X} + \cos\phi \, \vec{X} = 0 \qquad \partial_+ \partial_- \phi + \sin\phi = 0$$



$\xleftarrow[\text{inverse}]{}$ $\phi = 4 \arctan e^{\frac{x - vt}{\sqrt{1-v^2}}}$

$v = \cos \frac{p}{2}$

v : spectral parameter

p : magnon momentum , $E \sim \frac{\sqrt{\lambda}}{2} \sin \frac{p}{2}$

Derivation of Pohlmeyer reduction

Write the sigma model action as

$$L \sim \partial_+ \vec{X} \cdot \partial_- \vec{X} + \underset{\sim}{\Lambda} (\vec{X} \cdot \vec{X} - 1)$$

Lagrange multiplier

Constraint : $\vec{X} \cdot \vec{X} = 1 \quad \Rightarrow \quad \partial_+ \vec{X} \cdot \vec{X} = \partial_- \vec{X} \cdot \vec{X} = 0$

Virasoro conditions : $(\partial_+ \vec{X})^2 = (\partial_- \vec{X})^2 = \kappa^2$

from $t = \kappa \tau$, choose $\kappa = 1$

eom: $\partial_+ \partial_- \vec{X} + \Lambda \vec{X} = 0$

multiply $\vec{X}$: $\vec{X} \cdot \partial_+ \partial_- \vec{X} + \Lambda = 0$

differentiate $\vec{X} \cdot \partial_- \vec{X} = 0 \rightsquigarrow \Lambda = \partial_+ \vec{X} \cdot \partial_- \vec{X} = \cos\phi$

$\Rightarrow \partial_+ \partial_- \vec{X} + \cos\phi \, \vec{X} = 0$

Main idea:

use $(\vec{X}, \partial_+ \vec{X}, \partial_- \vec{X})$ as a basis of $S^2 \subset \mathbb{R}^3$

derive eom for $\partial_+ \vec{X} \cdot \partial_- \vec{X} = \cos\phi$

Ansatz : $\partial_+^2 \vec{X} = a \vec{X} + b\, \partial_+ \vec{X} + c\, \partial_- \vec{X}$

Apply $\vec{X}$; $\vec{X} \cdot \partial_+^2 \vec{X} = \partial_+ (\vec{X} \cdot \partial_+ \vec{X}) - (\partial_+ \vec{X})^2$

$\rightsquigarrow a = -1$

" $\partial_+ \vec{X}$; $\partial_+^2 \vec{X} \cdot \partial_+ \vec{X} = b + c \cos\phi = 0$

" $\partial_- \vec{X}$: $\partial_+^2 \vec{X} \cdot \partial_- \vec{X} = \partial_+ (\partial_+ \vec{X} \cdot \partial_- \vec{X}) + \partial_+ \vec{X} \cdot \vec{X} \cos\phi$

$\partial_+ (\cos\phi) = b \cos\phi + c$ $\rightsquigarrow$ solve b, c

Same equations by interchanging $+ \leftrightarrow -$

Result :
$$\partial_+^2 \vec{X} = -\vec{X} + \frac{\cos\phi \, \partial_+\phi}{\sin\phi} \partial_+ \vec{X} - \frac{\partial_+\phi}{\sin\phi} \partial_- \vec{X}$$

$$\partial_-^2 \vec{X} = -\vec{X} + \cdots$$

Then :

$$\partial_- \partial_+^2 \vec{X} = \left\{ -1 - \partial_- \left(\frac{\partial_+\phi}{\sin\phi} \right) \right\} \partial_- \vec{X}$$

$$+ \ \partial_- \partial_+ (\log \sin\phi) \, \partial_+ \vec{X} + \partial_+ (\log \sin\phi) \cdot \cos\phi \, \vec{X}$$

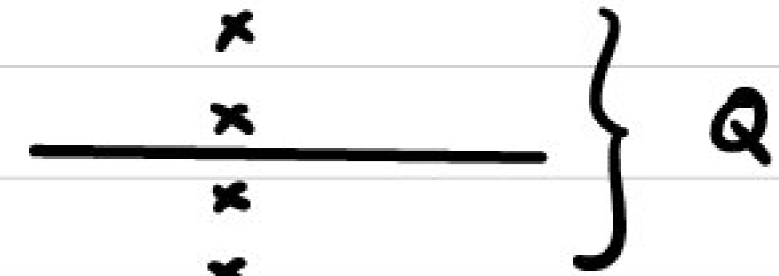
$$- \frac{\partial_+\phi}{\sin\phi} \underline{\partial_-^2 \vec{X}}$$

substitute the above eq.

compare coeffs of $(\vec{X}, \partial_+\vec{X}, \partial_-\vec{X})$

(skip details) $\rightsquigarrow$ $\partial_+\partial_-\phi + \sin\phi = 0$

Dyonic giant magnon $\longleftrightarrow$ magnon bound states

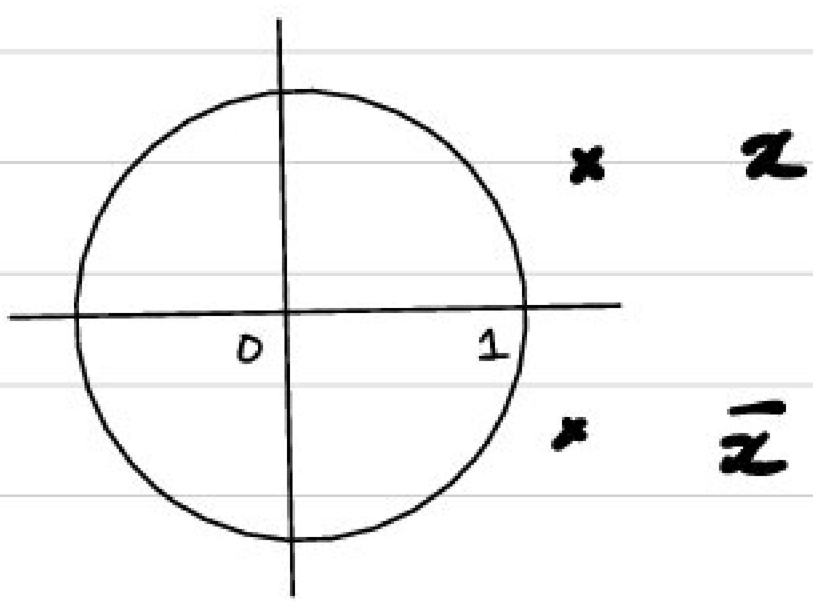


angular momentum

$J_1 \to \infty$, $J_2 = Q$

$\longleftrightarrow$ sigma model on $\mathbb{R} \times S^3$ $\longleftrightarrow$ complex sine Gordon

Classical spectral curve = poles in x-plane



$$\begin{cases} E - J_1 = \dfrac{\sqrt{\lambda}}{4\pi}\left(x - \dfrac{1}{x} - \bar{x} + \dfrac{1}{\bar{x}}\right) \\[2ex] J_2 = \dfrac{\sqrt{\lambda}}{4\pi}\left(x + \dfrac{1}{x} - \bar{x} - \dfrac{1}{\bar{x}}\right) \end{cases}$$

Use $\frac{x}{\bar{x}} = e^{ip} \Rightarrow E - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}}$

Recall in XXX Bethe Ansatz :

$$\frac{u + i/2}{u - i/2} = e^{ip} \Rightarrow u = \frac{1}{2} \cot \frac{p}{2}$$

$$\Delta = \frac{\lambda}{2\pi^2} \sum_j \frac{1}{4u_j^2 + 1} = \frac{\lambda}{2\pi^2} \sin^2 \frac{p}{2}$$

→ agrees with $E - J_1$ at small λ

← "BPS" relation

§ Finite gap solution

Find various relation for $p(x)$

$$\Omega(x) = P\exp \oint \frac{1}{2}\left(\frac{j_+}{1-x} - \frac{j_-}{1+x}\right)$$

$$\mathrm{tr}\, j_\pm = 0, \quad \mathrm{tr}\, j_\pm^2 = -2k^2$$

$\Rightarrow$ diagonalize $j_\pm \sim \begin{pmatrix} ik & 0 \\ 0 & -ik \end{pmatrix}$ near $x = \pm 1$

Thus $p(x) = -\dfrac{\pi k}{x \mp 1} + \cdots \quad (x \to \pm 1)$

Near $x \to \infty$, we find $SU(2)_R$ global charge

$$\Omega(x) = P\exp \oint \left(-\frac{j_0}{x} + \cdots \right) \qquad (x \to \infty)$$

Using $\mathrm{tr}(j_0) = 0$, $\quad P\exp \oint \mathrm{tr}(j_0 \sigma_3) = J_1 + J_2$

$$p(x) = -\frac{2\pi}{x\sqrt{\lambda}} (J_1 + J_2) \qquad (x \to \infty)$$

$$= -\frac{2\pi}{x\sqrt{\lambda}} (L - 2J) \qquad \begin{cases} L \equiv J_1 - J_2 \\ J = -J_2 \end{cases}$$

Near $x \to 0$, we find $SU(2)_L$ global charge

$$\partial_a - J_a \sim g^{-1}(\partial_a - x\, l_a + \cdots)\, g$$

$l_a = (\partial_a g) g^{-1}$ gives $J_1 - J_2 = L$

$$\rightsquigarrow \quad p(x) = \underline{2\pi m} + \frac{2\pi x}{\sqrt{\lambda}} L + \cdots \qquad (x \to 0)$$

$m \in \mathbb{Z}$, mode number (or winding number)

$p(x)$ is defined modulo $2\pi\mathbb{Z}$, so mode number can be removed from $p(x \to \infty)$

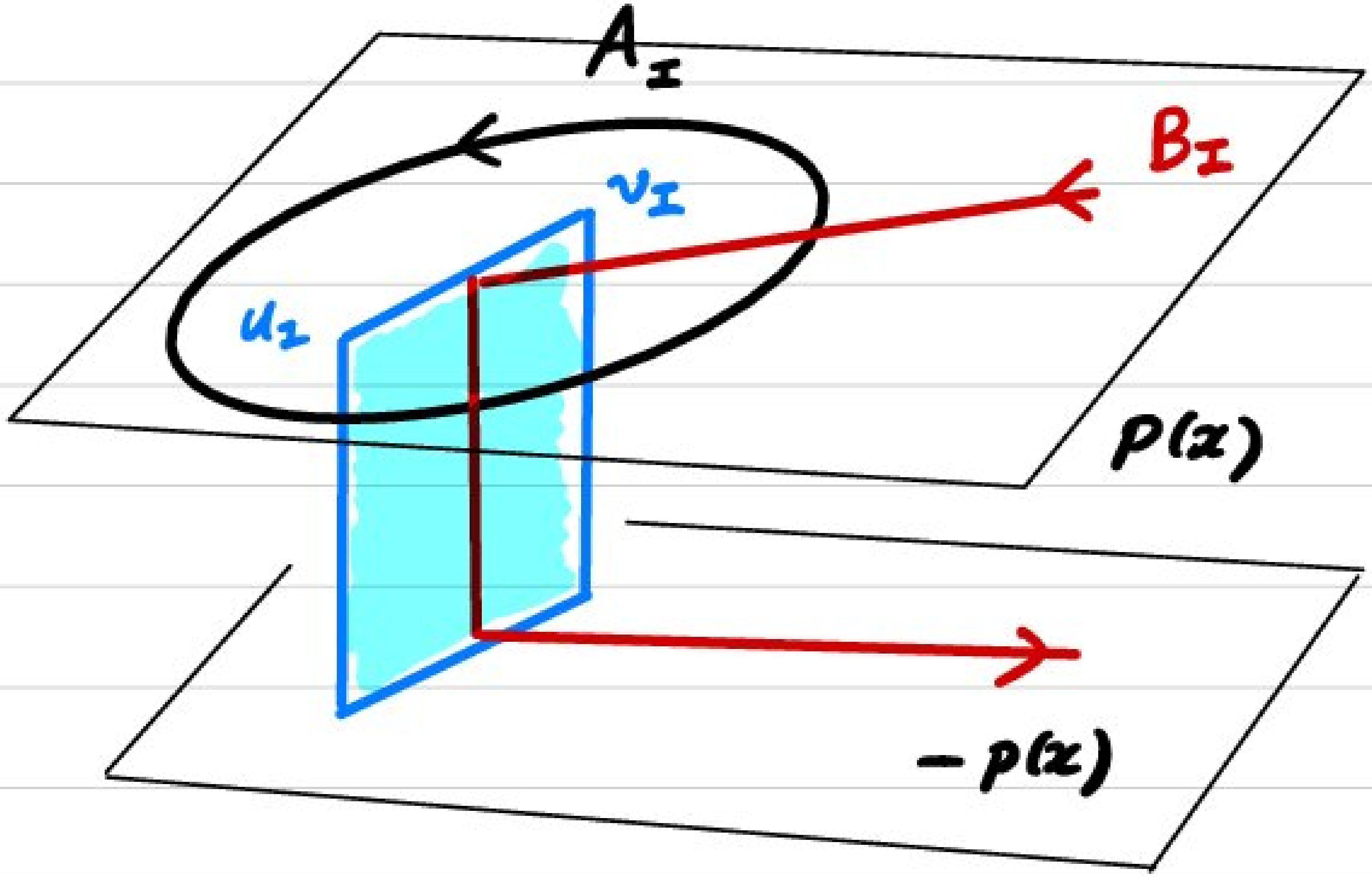
Classical spectral curve :

$$\Sigma \ : \quad \det \begin{pmatrix} y - p'(x) & 0 \\ 0 & y - p'(x) \end{pmatrix} = 0$$

for example, genus-g hyperelliptic curve is

$$y^2 = \prod_{I=1}^{g+1} (x - u_I)(x - v_I)$$

N.B. unlike SW curve, $dp = \frac{dx}{y}$, we fix $p(x)$ from the consistency conditions & analyticity



hyperelliptic curve

has $2g$ cycles

$\{A_I, B_I\}$

$C_I = (u_I, v_I)$

branch cut

period integrals are quantized

$$\oint_{A_I} d\varphi = 2\pi m_I \,, \quad \oint_{B_I} d\varphi = 2\pi n_I \quad (m_I, n_I \in \mathbb{Z})$$

Examples of finite-gap solutions [KMMZ 0402207]

Two formulations look "similar" in both sides:

Classical spectral curve $\longleftrightarrow$ XXX BAE

String side:
$$G(x) = p(x) + \frac{\pi\Delta}{\sqrt{\lambda}}\left(\frac{1}{x-1} + \frac{1}{x+1}\right)$$

$$p(x) = 2\pi m + \frac{2\pi L}{\sqrt{\lambda}}\, x + O(x^2)$$

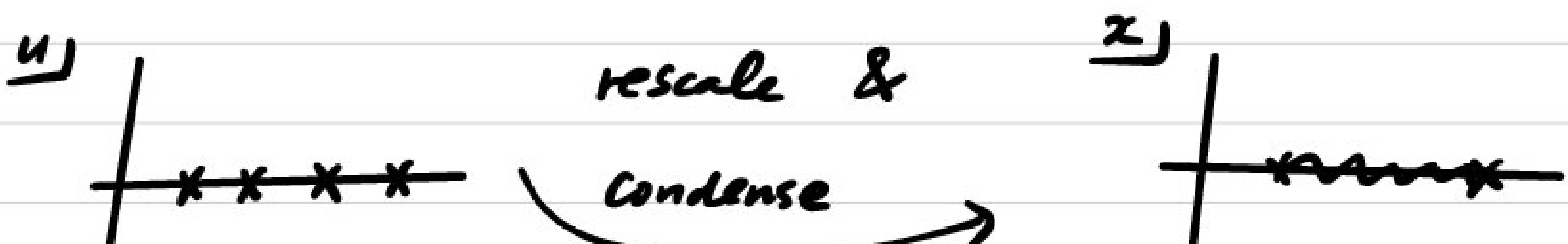
$$\Rightarrow \quad \Delta - L = -\frac{\sqrt{\lambda}}{2\pi}\oint \frac{dx}{2\pi i}\,\frac{G(x)}{x^2} \qquad (\lambda \gg 1)$$

Gauge side : $\gamma = \frac{\lambda}{2\pi^2} \sum_j \frac{1}{4u_j^2+1}$

rescale $u_j = L x_j$, $L \to \infty$, x_j fixed

define resolvent $G(x) = \frac{1}{L} \sum_i \frac{1}{x - x_j} = \int_C \frac{d\xi \, \rho(\xi)}{x - \xi}$

$\rho(\xi)$: density of roots



$$\gamma = \frac{\lambda}{8\pi^2 L} \int_C \frac{dx\, \rho(x)}{x^2} = \frac{-\lambda}{8\pi^2 L} \oint_C \frac{dx}{2\pi i} \frac{G(x)}{x^2} \quad (\lambda \ll 1)$$

$\Rightarrow$ $\Delta - L$ $(\lambda \gg 1)$ and γ $(\lambda \ll 1)$ are given by

$$\oint \frac{dx}{2\pi i} \frac{G(x)}{x^2}, \qquad G(x): \text{ resolvent}$$

we can compare other quantities

$$\int dx\, \rho(x) = \frac{J}{L} : \text{ filling fraction (number of roots)}$$

Comment

- Spectral parameter in XXX BAE is u

$T(u)$ $\quad u = i/2 \rightsquigarrow$ local charge

$\quad u = \infty \rightsquigarrow$ global charge

- Spectral parameter in classical string is x

$e^{\pm i p(x)}$ $\quad x = \pm 1 \rightsquigarrow$ energy

$\quad x = 0, \infty \rightsquigarrow$ global charge

Related by $u = x + \frac{1}{x}$

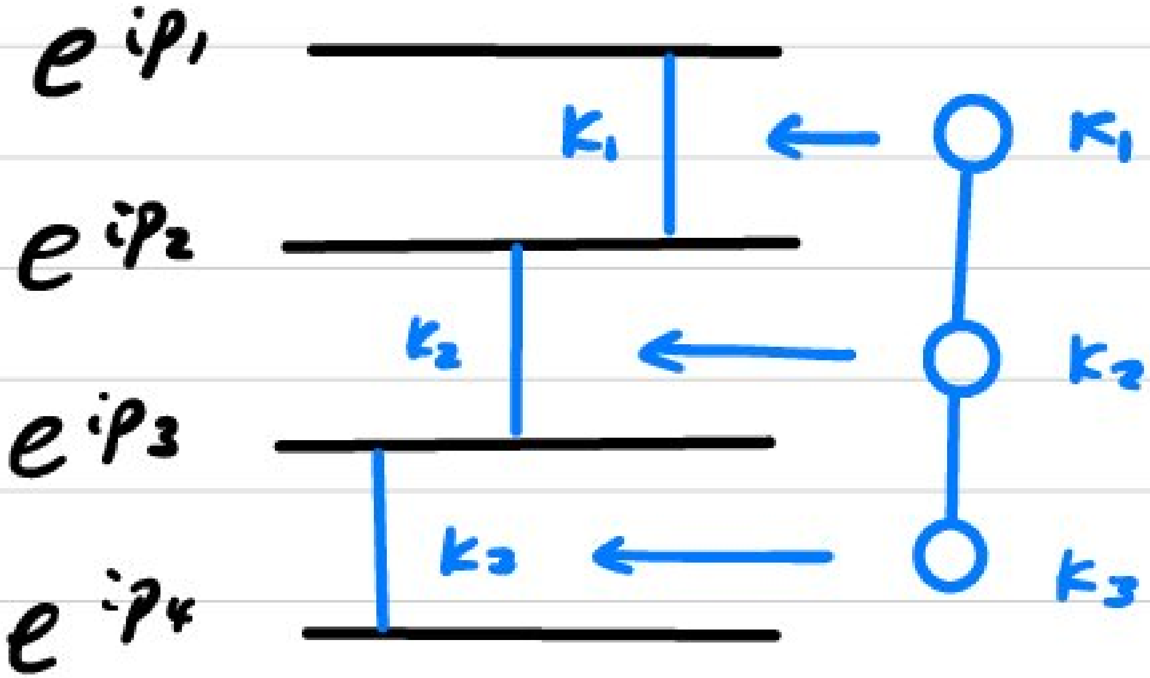
Q: In Bethe Ansatz, only the density of roots is used

Why introduce 2-sheet complex plane in PCM?

Quasi-momentum is the fundamental rep. of SU(N)

Bethe root lives in the Dynkin node of SU(N)

e.g. $N=4$



"going to another sheet" doesn't exist in XXX