Last week : Classical Integrability

PCM & sine-Gordon

Spectral curve

This week : Semiclassical Quantization

Scattering

Superstring on $AdS_5 \times S^5$

§ Semiclassical Quantization

Quantize classical integrable systems

- historically how QISM was found

(Covariant) quantization of $AdS_5 \times S^5$ is hard

- no 'proof' of quantum integrability

PCM $\rightsquigarrow$ $\{L, L\} = \delta'$: non-ultralocal

$\longrightarrow$ also hard !

PCM Lagrangian :

$$\mathcal{L} = \frac{1}{2} G_{MN} \partial_\alpha X^M \partial^\alpha X^N = \frac{1}{2} \mathrm{tr} \left(g^{-1} \partial_\alpha g \; g^{-1} \partial^\alpha g \right)$$

polar coordinates in S^3 : $(q^1, q^2, q^3) = (\theta, \phi, \psi)$

$$ds^2 = d\theta^2 + d\phi^2 + d\psi^2 + 2 \cos\theta \, d\phi \, d\psi$$

$$\rightsquigarrow \quad p_i = \frac{\delta \mathcal{L}}{\delta \partial_0 q^i} \, , \quad \{ q^i(\sigma), p_j(\sigma') \} = \delta_i^j \, \delta(\sigma - \sigma')$$

Want to compute the Poisson bracket

$$\{ j_\alpha(\sigma), j_\beta(\sigma') \} = \cdots$$

We can derive Poisson brackets in many ways,

but they are invariant under canonical transform

$$\{q^i, p_j\} \rightarrow \{Q^i, P_j\}$$

If we use polar coordinates,

$$j_\alpha = g^{-1}\partial_\alpha g = \underbrace{T^a}_{\text{SU(2) generators}} \underbrace{E^a_i}_{\text{vierbein of } S^3} \partial_\alpha q^i$$

$j_0^a \sim E_i^a \partial_0 q^i$: momentum , $E_i^a q^i$: "coordinate"

Eventually we obtain

$$\{ j_1^a(\sigma), j_1^b(\sigma') \} = 0$$

(no momentum)

(Poisson-Lie)

$$\{ j_0^a(\sigma), j_0(\sigma') \} = -f^{abc} j_0^c \delta(\sigma-\sigma')$$

$$\{ j_0^a(\sigma), j_1(\sigma') \} = -f^{abc} j_1^c \delta(\sigma-\sigma') - \delta^{ab} \delta'(\sigma-\sigma')$$

non-ultralocal term

from $j_1 \sim g^{-1}\partial_1 g$

Lax connection of PCM:

$$J_\sigma(\sigma,\tau,x) = \frac{j_0 - x j_1}{1-x^2}$$

Poisson bracket is [Maillet, Dorey-Vicedo 0606287]

$$\{ J_\sigma(\sigma,\tau,x), J_\sigma(\sigma',\tau,x') \}$$

$$= [r(x,x'), J_\sigma \otimes 1 + 1 \otimes J_\sigma] \delta(\sigma-\sigma')$$

$$- [s(x,x'), J_\sigma \otimes 1 + 1 \otimes J_\sigma] \delta(\sigma-\sigma')$$

$$- 2 s(x,x') \delta'(\sigma-\sigma')$$ still non-ultralocal

r: antisymmetric, $\hbar \to 0$ limit of R-matrix

$$R = I + \frac{\hbar}{u} P \equiv I + \hbar r$$

s: symmetric $\rightsquigarrow$ ambiguity in $P\exp$, Jacobi id, etc.

Maillet introduced "symmetric brackets" to show

$$\{ \mathrm{tr}\,\Omega(x), \quad \mathrm{tr}\,\Omega(x') \} = 0$$

But how to quantize PCM remains unclear

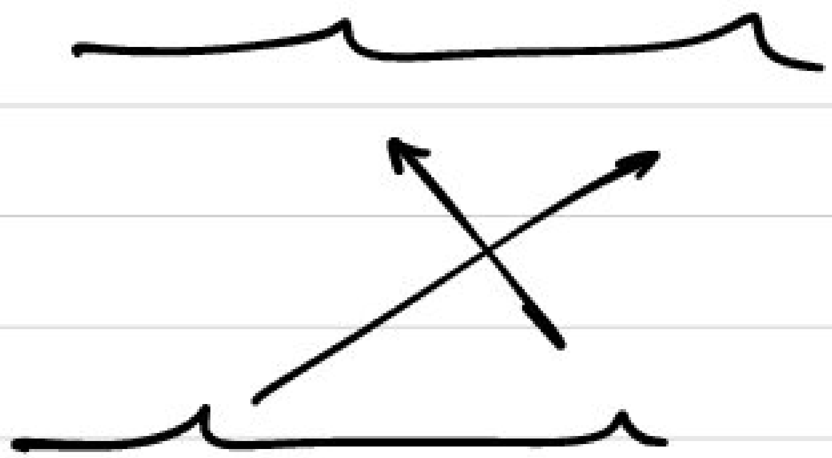
$\Rightarrow$ needs different method to find R-matrix

In sine-Gordon, there is a gauge transformation

which removes $\partial_\alpha \phi$ from Lax connection

- Poisson brackets are ultralocal
- Use QISM to find R-matrix
- (inverse) Pohlmeyer doesn't solve PCM problem

§ Scattering of giant magnons [Hofman, Maldacena]



Compute classical S-matrix from soliton scattering

Compare with "Bethe Ansatz" like prediction (AFS phase)

This is worldsheet scattering

$\sim$ wave-function for dilatation eigenstates

Not spacetime scattering $\sim$ 4pt fn of AdS/CFT

N giant magnons in $\mathbb{R}_t \times S^3$

$\updownarrow$ Pohlmeyer

N soliton solutions in (complex) sine-Gordon

$\uparrow$

Dressing method ; for $N=2$

$$\begin{cases} \phi = 4 \arctan\left(\dfrac{a_1+a_2}{a_1-a_2} \dfrac{e^{\xi_1} - e^{\xi_2}}{1+e^{\xi_1+\xi_2}} \right) \\ \partial_X \partial_T \phi = \sin\phi \, , \quad \xi_i = a_i X + \dfrac{T}{a_i} \end{cases}$$

Lax connections in lightcone coordinate:

$$L_T = i\lambda\sigma_3 + \frac{i\partial_T\phi}{2}\sigma_1 \quad , \quad L_X = \frac{-i}{4\lambda} e^{\frac{i\phi}{2}\sigma_1}\sigma_3 e^{-\frac{i\phi}{2}\sigma_1}$$

$$(\partial_\alpha - L_\alpha)\psi = 0 \qquad \text{for} \quad \alpha = X, T$$

The vacuum solution: $\phi = 0$

$$(\partial_X - i\lambda\sigma_3)\psi_0 = (\partial_T + \frac{i}{4\lambda}\sigma_3)\psi_0 = 0$$

$$\Rightarrow \psi_0 = \exp\left[\left(i\lambda X - \frac{iT}{4\lambda}\right)\sigma_3\right]$$

Dressing transform :

$$\psi = g(\lambda)\,\psi_0 \,, \qquad g(\lambda) = 1 + \sum_{i=1}^{N} \frac{R_i}{\lambda - \lambda_i}$$

$$0 = (\partial_\alpha - L_\alpha)\,\psi_0 = g(\partial_\alpha - L_\alpha)\,g^{-1}\,\psi$$

$$= (\partial_\alpha + g^{-1}\partial_\alpha g + g L_\alpha g^{-1})\,\psi$$

$$= (\partial_\alpha - L_\alpha^{(g)})\,\psi$$

ψ solves eom $\Longleftrightarrow$ $L_\alpha^{(g)}$ takes the standard form

Suppose R_i is a projector, $R_i^2 = R_i$, $R_i^\dagger = R_i$

$$g^{-1}(\lambda) = 1 - \sum_{i=1} \frac{R_i}{\lambda - \bar{\lambda}_i} = g(\bar{\lambda})^\dagger$$

Then

$$g(\lambda)\, g^{-1}(\lambda) = 1$$

$L_\alpha(\lambda)$ and $L_\alpha^{(g)}(\lambda)$ should be analytic except $\lambda = 0, \infty$

But $L_\alpha^{(g)}(\lambda)$ has a simple pole at $\lambda = \lambda_i$

$$\operatorname*{Res}_{\lambda = a_i} L_\alpha^{(g)} = 0 \quad \text{for all } i \Rightarrow \text{fixes } R_i$$

Take $\lambda \to 0$ in the dressed connection:

$$L_x^{(g)} = \frac{-i}{4\lambda} e^{\frac{i\phi^{(g)}}{2}\sigma_1} \sigma_3 \, e^{-\frac{i\phi^{(g)}}{2}\sigma_1}$$

$$= \frac{-i}{4\lambda} g \sigma_3 g^{-1} \underbrace{- g^{-1}\partial_x g}_{\text{regular}}$$

$$\Rightarrow \quad g(\lambda) = \begin{pmatrix} \cos\phi/2 & i\sin\phi/2 \\ i\sin\phi/2 & \cos\phi/2 \end{pmatrix} + \mathcal{O}(\lambda) = 1 + \sum_i \frac{R_i}{\lambda - \lambda_i}$$

We can recover $\tan\left(\frac{\phi}{4}\right)$ from R_i's.

Kink-kink solution $\left(\lambda_k = i a_k = i\sqrt{\frac{1-v_k}{1+v_k}} \right)$

$$\tan\frac{\phi}{4} = \frac{a_1+a_2}{a_1-a_2}\,\frac{e^{\xi_1}-e^{\xi_2}}{1+e^{\xi_1+\xi_2}}$$

kink-antikink solution : $\xi_2 \to \xi_2 + i\pi$

kink-kink

kink-antikink

center-of-mass frame : $a_1 = 1/a_2 = \sqrt{\frac{1-v_1}{1+v_1}}$.

$$X = \frac{x+t}{2}, \quad T = \frac{x-t}{2}, \quad \xi_1 = \gamma(x-vt) = \frac{x-vt}{\sqrt{1-v^2}}$$

$$\tan\frac{\phi}{4} = \frac{1}{v}\frac{\sinh(\gamma v t)}{\cosh(\gamma x)}$$; take $x, t \to \pm\infty$

$$\to \frac{1}{v}\exp\left(\mp\frac{x-vt}{\sqrt{1-v^2}}\right) \equiv \exp\left(\mp\frac{x-v(t+\Delta T/2)}{\sqrt{1-v^2}}\right)$$

$\Delta T = \frac{2}{\gamma v}\log v$: time delay in CoM

In the original giant magnon solution in $\mathbb{R}_t \times S^3$

$$E - J = \frac{\sqrt{\lambda}}{\pi} \sin\frac{p}{2}\,, \qquad v = \frac{\partial E}{\partial p} = \cos\frac{p}{2}$$

Thus

$$\xi = \frac{x - (\cos\frac{p}{2})\, t}{\sin\frac{p}{2}} \equiv \cosh\theta\, x - \sinh\theta\, t\,; \quad \text{rapidity}$$

$$\tanh\theta = \cos\frac{p}{2}\,, \qquad \frac{1}{\sinh\theta} = \gamma v$$

Time delay of particle 1 passing through particle 2 is

$$\Delta T_{12} = \sinh\theta_1 \log\tanh^2\left(\frac{\theta_1 - \theta_2}{2}\right)$$

Consider "wave-function" for particle 1

$$\psi \sim e^{-iEt + ip(E)x - i\delta(E)}$$

$$= e^{-iE(t+\Delta T) + ip(E)x} \equiv e^{i\Phi(E)}$$

$$\frac{\partial \delta}{\partial E} = t + \Delta T - \frac{\partial p}{\partial E} x = \underbrace{\left(t - \frac{x}{v}\right)}_{\text{free particle phase}} + \Delta T$$

Thus phase shift is $\delta_{12} = \int dE_1 \, \Delta T_{12}$

AFS phase :

$$\theta = \frac{\sqrt{\lambda}}{2\pi} \sum_{r=2}^{\infty} \left\{ q_r(p_1)\, q_{r+1}(p_2) - q_{r+1}(p_1)\, q_r(p_2) \right\}$$

$$q_{r+1}(p) = \frac{i}{r} \left\{ \frac{1}{(x^+)^r} - \frac{1}{(x^-)^r} \right\}$$

$$\begin{cases} \text{in XXX model} \quad x^\pm \sim u \pm \frac{i}{2} & (\lambda \ll 1) \\ \text{in string theory} \quad x^\pm = \exp(\pm i p/2) & (\lambda \gg 1) \end{cases}$$

sum $\longrightarrow$ $\theta(p_1, p_2) = \frac{\sqrt{\lambda}}{\pi} \left(\cos\frac{p_1}{2} - \cos\frac{p_2}{2} \right) \log(\cdots)$

Substitute p-variable:

$$\Delta T_{12} = \tan\frac{p_1}{2} \cdot \log\left(\frac{1-\cos\frac{p_1-p_2}{2}}{1+\cos\frac{p_1-p_2}{2}} \right)$$

time delay is related to the phase shift by

$$\delta_{12} = \int dE_1 \, \Delta T_{12} = \frac{\sqrt{\lambda}}{2\pi}\int dp_1 \sin\frac{p_1}{2} \log(\cdot\cdot)$$

$$= -\frac{\sqrt{\lambda}}{\pi}\left\{ \underbrace{\left(\cos\frac{p_1}{2} - \cos\frac{p_2}{2}\right)\log(\cdot\cdot)}_{\text{AFS phase}} + \underbrace{p_1 \sin\frac{p_2}{2}}_{\sim p_1 E_2 \text{, gauge choice}} \right\}$$

We computed scattering phase from 2-magnon sol

& agrees with AFS phase

We don't get "matrix structure" of S-matrix

because we started from $\mathbb{R}_t \times S^2 \subset AdS_5 \times S^5$

Global charges $(\cancel{E}, \underbrace{S_1, S_2}, \cancel{J_1}, \underbrace{J_2, J_3})$

residual $psu(2|2)^2$ symmetry

fixes the matrix structure

Giant magnon at $p \to 0$

$\rightsquigarrow$ BMN or point-particle limit , $t = \phi = \tau$

Expand $AdS_5 \times S^5$ action around this 'vacuum'

Light cone gauge : $X^0 = t, \ X^9 = \phi$

$X^i = y^i / \lambda^{1/4} \ (i = 1 \sim 8)$: physical

$$\mathcal{L} = \sqrt{\lambda}\, G_{MN}(X)\, \partial X^M \bar{\partial} X^N$$

$$= \partial y^i \bar{\partial} y^i + \frac{1}{\sqrt{\lambda}} \underline{V_4(y^i, y^j)} + \cdots$$

Worldsheet 4pt also gives "S-matrix" in BMN limit

§ Superstring on $AdS_5 \times S^5$

[GSW textbook, Berkovits lectures, Arutyunov Frolov 0901.4937]

Bosonic : $\mathcal{L} = \left(\sqrt{-\gamma}\, \gamma^{ab} G_{MN} + \epsilon^{ab} B_{MN} \right) \partial_a X^M \partial_b X^N$

$a, b = 0, 1$

Superstring $\begin{cases} \text{worldsheet SUSY : RNS} \quad X^M, \psi^M \\ \qquad M = 0, 1, \cdots, 9 \\ \text{spacetime SUSY : GS, PS} \\ \qquad X^M, \theta^A \\ \qquad A = 1, 2, \cdots, 16 \end{cases}$

In RNS, Bosonization $\rightsquigarrow$ Spacetime fermions

$e^{-\varphi}$

need to introduce 'picture' and PCO

D-brane background (as in AdS / CFT)

RR flux $\rightsquigarrow$ Non-zero torsion

not Levi-Civita connection, common in SUGRA

Supersymmetrize : $\Pi_a{}^{\mu} = \partial_a X^{\mu} - \frac{1}{2} \partial_a \theta^A (\Gamma^{\mu})_{AB} \theta^B$

but θ^A has too many d.o.f.

Green-Schwarz flat space action

$$S_{GS} = S_{kin} + S_{WZ}$$

$$S_{kin} = -\frac{1}{2\pi} \int d^2\sigma \sqrt{-\gamma}\, \gamma^{ab}\, \Pi_a{}^M \cdot \Pi_b{}^M$$

$$S_{WZ} = -\frac{i}{2\pi} \int d^2 z\ \epsilon^{ab}\, \partial_a X^M \cdot \theta^A \not{\partial}_{AB}\, \theta^B$$

(this is heterotic, needs $(\theta_1{}^A, \theta_2{}^A)$ for type II)

$S_{WZ} \sim$ "fermionic B-field"

S_{GS} is invariant under global & local SUSY

local SUSY ~ κ-symmetry ~ $\sqrt{\text{Viracoro}}$

$$\begin{cases} \delta\theta^A = (\Gamma^M)^A{}_B \, \Pi^M_a \, K^{Aa} \\ \delta X^M = i\,\theta^A (\Gamma^M)^A{}_B \, \delta\theta^B \end{cases}$$

Then $\delta\Pi^M_a = -\partial_a\theta^A (\Pi_b)_{AB} \, K^{Bb}$

$$\delta S_{kin} = -\frac{1}{\pi}\int \sqrt{-\gamma}\,\gamma^{ab}\,\Pi^M_a\,\delta\Pi^M_a$$

The second term was added such that

$$\delta S_{WZ} = -\frac{1}{\pi}\int \epsilon^{ab}\, \partial_a \theta^A (\not{\Pi}_b)_{AB}\, K^{Bb}$$

(using Fierz identity if needed)

Define the projector :

$$P_\pm \equiv \frac{1}{2}\left(\gamma^{ab} \pm \frac{\epsilon^{ab}}{\sqrt{-\gamma}}\right), \quad P_\pm^2 = P_\pm, \quad P_\pm P_\mp = 0$$

which can be written as $P_\pm = \frac{1}{2}(1 \pm \Gamma)$

We gauge-fix $SO(1,9)$ rotation s.t.

$$P_+ \kappa^1 = \kappa^1, \qquad P_- \kappa^2 = \kappa^2$$

Now $\delta S_{GS} = -\frac{2}{\pi} \int \sqrt{-\gamma}\, \partial_a \theta\, \not{\Pi}\, \underline{P_+ \kappa^a}$

$= 0$ for κ^2

Thus half of θ^A go away by κ-symmetry

Also $\delta^2 X \propto (\not{\Pi})^2 = 0$ by Virasoro constraints

$\Rightarrow$ $(\kappa\text{-symmetry})^2 =$ Virasoro

GS action on supercoset is simple

flat 10d : $\dfrac{\text{superpoincaré }(P, Q, M)}{SO(1,9)\ \ (M)}$

$AdS_5 \times S^5$: $\dfrac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$

$\mathbb{Z}_4$ grading : $g \in PSU(2,2|4)$, $\quad j = g^{-1} dg$

split j by the eigenspace of Ω

$\Omega^4 = 1 \Rightarrow$ eigenvalues $\{+1, i, -1, -i\}$

An example:

$$\mathfrak{su}(4) = \mathrm{Span}\{ T \in \mathrm{Mat}(4, \mathbb{C}) \mid T^\dagger = T,\ \mathrm{tr}(T) = 0 \}$$

$$\dim_{\mathbb{R}} \mathfrak{su}(4) = 15$$

$$\mathfrak{so}(5) = \mathrm{Span}\{ T \in \sqrt{-1}\ \mathrm{Mat}(4, \mathbb{R}) \mid T^T = T \}$$

$$\dim_{\mathbb{R}} \mathfrak{so}(5) = 10$$

$$\mathfrak{su}(4) = \underbrace{\text{"symmetric"}}_{S^5} \oplus \underbrace{\text{"anti symmetric"}}_{\mathfrak{so}(5)}$$

$psu(2,2|4) \subset sl(4|4)$;

$$M = \begin{pmatrix} B_1 & F_2 \\ F_3 & B_4 \end{pmatrix} \begin{matrix} \}4 \\ \}4 \end{matrix}$$
$$\qquad\qquad\quad\;\; \underbrace{}_{4} \;\; \underbrace{}_{4}$$

$$K = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}, \quad \mathcal{K} = \mathrm{diag}(K, K)$$

$$\Omega(M) = -\mathcal{K} M^{st} \mathcal{K}, \qquad M^{st} = \begin{pmatrix} B_1^t & -F_3^t \\ F_2^t & B_4^t \end{pmatrix}$$

$$\Omega(J_k) = i^k J_k \; : \; k = 0,1,2,3, \quad \dim J_0 = 20 \quad (\text{gauge away})$$
$$\dim J_2 = 10 \quad (\text{bosons})$$
$$\dim J_1 = \dim J_3 = 16 \quad (\text{fermions})$$

Lagrangian for $AdS_5 \times S^5$ Green–Schwarz

$$\mathcal{L} = -\frac{\sqrt{\lambda}}{4\pi}\, \mathrm{str}\left(\sqrt{-\gamma}\, \gamma^{ab}\, \underline{J_a^{(2)} J_b^{(2)}} + \kappa\, \epsilon^{ab}\, \underline{J_a^{(1)} J_b^{(3)}} \right)$$

Bosonic kinetic

Wess-Zumino term

but bosonic $B_{MN} = 0$

- $\mathrm{Str}(J + c\mathbb{1}) = \mathrm{str}(J)$

choose c s.t. $\mathrm{tr}(J^{(2)}) = 0 \longrightarrow$ "p" in $psu(2,2|4)$

- $\kappa = \pm 1$ to have κ-symmetry

kill half of fermionic d.o.f.

$$\begin{cases} P_- J^{(1)} = 0 \\ P_+ J^{(3)} = 0 \end{cases}$$

Lax connections

BPR form : $$J = \frac{(1-z^2)^2}{2z^2} J^{(2)} - \frac{1}{2}\left(z^2 - \frac{1}{z^2}\right) * J^{(2)}$$

[0305116]

$$+ (z-1) J^{(1)} + \left(\frac{1}{z} - 1\right) J^{(3)}$$

- gauge transformation : $J' = h L h^{-1} + dh \cdot h^{-1}$

  introduces z-independent terms including $J^{(0)}$

- Keep K-symmetry & flatness without eom : [0910.0221]

$$J_{strong} = J_{BPR} + \frac{1-z^4}{2}\left(C^{(0)} + \frac{C^{(1)}}{z^3} + \frac{C^{(3)}}{z} \right)$$

Coset coordinate

$$G = \exp(\chi)\ \exp(X)$$

$$X = \begin{pmatrix} 0 & Y & & \\ Y^{\dagger} & 0 & & \\ & & 0 & iZ \\ & & iZ^{\dagger} & 0 \end{pmatrix}, \quad \chi = \begin{pmatrix} & & \theta_1 & \theta_2 \\ & & \theta_3^{\dagger} & \theta_4 \\ -\theta_1^{\dagger} & \theta_3 & & \\ -\theta_2^{\dagger} & \theta_4^{\dagger} & & \end{pmatrix}$$

using $\quad su(2,2) \oplus su(4) \supset su(2)^{\oplus 4}$

$$Z = \begin{pmatrix} z^{1\dot{2}} & -z^{1\dot{1}} \\ z^{2\dot{2}} & -z^{2\dot{1}} \end{pmatrix}, \quad Y = \begin{pmatrix} y^{3\dot{4}} & -y^{3\dot{3}} \\ y^{4\dot{4}} & -y^{4\dot{3}} \end{pmatrix}$$

If we use $\quad z^{ai} \to (z^1, z^2, z^3, z^4)$

relatoon to the embedding coordinates of $S^5 \subset \mathbb{R}^6$

$$X^{2k-1} + \sqrt{-1}\, X^{2k} = \frac{z^{2k-1} + \sqrt{-1}\, z^{2k}}{1 + z^2/4}, \quad X^{5+i6} = \frac{1 - z^2/4}{1 + z^2/4}\, e^{i\phi}$$

similarly for $AdS_5 \subset \mathbb{R}^{4,2}$

Metric :

$$ds^2 = -\left(\frac{1+y^2/4}{1-y^2/4}\right) dt^2 + \left(\frac{1-z^2/4}{1+z^2/4}\right) d\phi^2 + \frac{(dz^i)^2}{(1+z^2/4)^2} + \frac{(dy^i)^2}{(1-y^2/4)^2}$$

$\to$ easy to expand around $y^i = z^i = 0$.