Last week : Poisson bracket, Scattering

$AdS_5 \times S^5$ superstring

This week : SL(2) sector

large spin limit, transcendentality

BFKL limit

Asymptotic Bethe Ansatz

[Beisert Staudacher, 0307042 ; Beisert, 0511082]

§ SL(2) sector

$\mathcal{N}=4$ SYM has three rank-1 subsectors

$su(2)$: $\mathrm{tr}(Z^{J_1} W^{J_2})$ + perm , $s = 1/2$ XXX

$su(1|1)$: $\mathrm{tr}(Z^{J} \psi^{M})$ + perm , $S_{12} = -1$ (free)

$sl(2)$: $\mathrm{tr}(D_+^S Z^J)$ + perm , $s = -1/2$ XXX

has lot of data to test Bethe Ansatz & AdS/CFT

Cusp dimensions, (conformal) Regge limit, transcendentality ...

One-loop Bethe Ansatz is determined by rationality + global symmetry ($PSU(2,2|4)$))

Bethe Ansatz for spin s XXX model:

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k \neq j}^{M} \frac{u_j - u_k + 2is}{u_j - u_k - 2is}$$

Anomalous dimensions: $\gamma = \frac{\lambda}{8\pi^2} \sum_{j=1}^{M} \frac{1}{u_j^2 + 1/4}$

$s \leq 0$: non-compact spin chain

How to obtain Hamiltonian ?

$$\mathcal{O} = \mathrm{tr}\left(D_+^S Z^J\right) + \cdots$$

$$= \mathrm{tr}\left([n_+^\mu D_\mu]^S Z^J \right) + \cdots$$

$\llcorner$ $n_+^\mu = (n^+, n^-, n^2, n^3)$

$= (1, 0, 0, 0)$

: null vector

naively

$$\langle \partial_+^S \bar{Z}(x) \, \partial_+^S Z(y) \rangle \sim \left(\frac{\partial}{\partial x^+}\right)^S \left(\frac{\partial}{\partial y^+}\right)^S \frac{1}{|x-y|^2}$$

$$\sim \frac{(x_+ - y_+)^{2S}}{|x-y|^{2+2S}}$$

CFT 2-point fn with spin

$$\langle \mathcal{O}_{\mu_1 \cdots \mu_s}(x) \, \mathcal{O}_{\nu_1 \cdots \nu_s}(y) \rangle = \frac{I_{\vec{\mu}, \vec{\nu}}(x-y)}{|x-y|^{2\Delta}}$$

$$I_{\vec{\mu}\vec{\nu}}(x) = \text{Sym} \left\{ \prod_{k=1}^{s} \left(\eta_{\mu_k \nu_k} - \frac{2 x_{\mu_k} x_{\nu_k}}{x^2} \right) \right\}$$

symmetric traceless part

If $\mathcal{O}_+ = n_+^{\mu_1} \cdots n_+^{\mu_s} \mathcal{O}_{\mu_1 \cdots \mu_s}$, $n_+^{\mu} n_+^{\nu} I_{\mu\nu} = \dfrac{-2(x_+)^2}{x^2}$

$$\langle \mathcal{O}_+(x) \, \mathcal{O}_+(0) \rangle \sim \frac{(x_+)^{2s}}{|x|^{2\Delta + 2s}}$$

$\simeq$ naive results

Operators in $SL(2)$ sector

= Components of $SL(2)$ irrep with $s = -1/2$

$$\left\{ Z, \quad D_+ Z, \quad \frac{1}{2!} D_+^2 Z, \quad \frac{1}{3!} D_+^3 Z, \quad \cdots \right\}$$

$$\left\{ |0\rangle, \quad a_+|0\rangle, \quad a_+^2|0\rangle, \quad a_+^3|0\rangle, \quad \cdots \right\}$$

oscillator rep. $\quad [a, a^+] = 1$

$$\left\{ 1, \quad z, \quad z^2, \quad z^3, \quad \cdots \right\}$$

differential rep. $\quad [\partial_z, z] = 1$

differential realization of $SL(2)$ generators

$$\begin{cases} S^- = -\partial_z \\ S^0 = z\partial_z + s \\ S^+ = z^2\partial_z + 2sz \end{cases}$$

$s = -\frac{1}{2} \longrightarrow SL(2)$

$s = \frac{1}{2}\mathbb{N} \longrightarrow SU(2)$

$$\mathcal{O} = \mathrm{tr}(D_+^S Z^J) + \cdots$$

$\rightsquigarrow$ spin chain with J sites

$\rightsquigarrow$ wave-fn $\psi(z_1, z_2, \cdots, z_J)$

Twist-two case $(J=2)$

$$z_1^k z_2^{n-k} \sim (a_1^\dagger)^k (a_2^\dagger)^{n-k} |0\rangle \sim \frac{\mathrm{tr}(D_+^k Z \, D_+^{n-k} Z)}{k! \, (n-k)!}$$

Compute one-loop operator mixing :

$$H_{12}\left(z_1^k z_2^{n-k}\right) = \sum_{k'=0}^{n} \left\{ \delta_{kk'} (h_k + h_{n-k}) + \frac{(1-\delta_{kk'})}{|k-k'|} \right\} z_1^{k'} z_2^{n-k'}$$

$$h_n = \sum_{j=1}^{n} \frac{1}{j} = \psi(j+1) - \psi(1) \quad : \text{ harmonic number}$$

[Beisert, 0307015] [Belitsky, Derkachov, Korchemsky, Manashov, 0311104]

Eigenstate of $H_{12} \longleftrightarrow SL(2)$ spin-j representation

$$|j\rangle \sim (z_1 - z_2)^j = \sum_{l=0}^{j} (-1)^l \binom{j}{l} z_1^{j-l} z_2^{l}$$

In fact, one can show : cf. $(S_1^- + S_2^-)|j\rangle = 0$, HWS

$$H_{12} (z_1 - z_2)^j = 2h(j) (z_1 - z_2)^j$$

$$\Rightarrow \quad H_{12} = \sum_{j=0}^{\infty} 2h(j) \underbrace{P_{12,j}}_{\text{projector to spin-}j\text{ rep}}$$

Hamiltonian of $XXX_{-1/2}$ with length L

$$H = \sum_{k=1}^{L} H_{k,k+1} = 2 \sum_{k} \left[\psi(\hat{J}_{k,k+1}) - \psi(1) \right]$$

$\hat{J}_{12}$ is the "square root" of Casimir

$$S_{12}^{a} = S_{1}^{a} + S_{2}^{a}, \quad C_{12} = S_{12}^{0}(S_{12}^{0} - 1) + S_{12}^{+} S_{12}^{-}$$

such that $C_{12} |j\rangle = j(j-1) |j\rangle$

$\Rightarrow$ formally define $\hat{J}_{12}$ by $C_{12} \equiv \hat{J}_{12}(\hat{J}_{12} - 1)$

Solving $XXX_{-1/2}$ Bethe Ansatz

$$\left(\frac{v+i/2}{v-i/2}\right)^J \prod_{k=1}^{S} \frac{v-u_k+i}{v-u_k-i} = -1 \qquad (k=1,2,\cdots,S)$$

for $\mathcal{O} \sim \mathrm{tr}(D_+^S Z^J)$

twist-two $(J=2)$

there is only one HWS at each spin S

Hamiltonian is diagonal : $H = H_{12} = 2h(S)$

Position of all Bethe roots is known :

$$Q(u) = {}_3F_2\left(-S, S+1, \tfrac{1}{2}+iu \,;\, 1, 1 \,;\, 1\right)$$

No "finite" bound states in SL(2) sector

Outermost roots scale as $u_j \sim O(S)$

$\rightsquigarrow$ energy; $\quad \gamma = \dfrac{\lambda}{8\pi^2} \displaystyle\sum_{j=1}^{S} \dfrac{1}{u_j^2 + 1/4} = \dfrac{\lambda}{2\pi^2} h(S)$

TQ-relation :

$$t(v)\, Q(v) = Q(v+i)\left(v+\tfrac{i}{2}\right)^J + Q(v-i)\left(v-\tfrac{i}{2}\right)^J$$

$t(v) = 2v^2 - S(S+1) - 1/2$ from symmetry

§ Large Spin Limit

$$h(S) \to \log S + \gamma_E + \mathcal{O}\left(\frac{1}{S}\right)$$

$$\Delta - S - 2 = f(\lambda) \log S + \cdots$$

$$f(\lambda) = \frac{\lambda}{2\pi^2} + \mathcal{O}(\lambda^2) \quad : \quad \text{"scaling function"}$$

$\log S$ behaviour agrees with string theory
No wrapping corrections at the leading order
related to the vev of Wilson loop with cusp

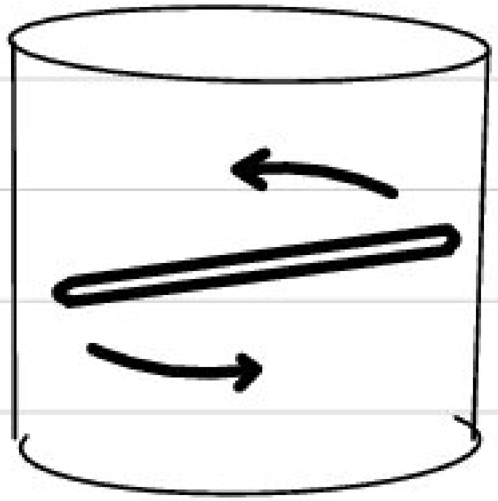
GKP folded string on AdS_3 :

$$ds^2 = -\cosh^2\rho \, dt^2 + d\rho^2 + \sinh^2\rho \, d\phi^2$$

(can solve Virasoro without S^5)

Ansatz : $t = \kappa\tau$, $\phi = \omega\tau$, $\rho = \rho(\sigma)$

Elliptic profile :



ρ_0 : turning point

$\rho_0 \to \infty$ if $\omega \to \kappa$

$$S \sim \sqrt{\lambda}\, e^{2\rho_0} , \quad E - S \sim \frac{\sqrt{\lambda}}{2\pi} 4\rho_0 \sim \frac{\sqrt{\lambda}}{\pi} \log S$$

$\because$) worldsheet has 4 segments $\rho \in [0, \rho_0]$

Therefore
$$E - S = f(\lambda)\,\log S + \cdots$$
$$f(\lambda) = \frac{\sqrt{\lambda}}{\pi} + \cdots \qquad (\lambda \gg 1)$$

The scaling function is related to the anomalous dimension of cusped Wilson loop

$$\begin{cases} W_{ren} = Z\, W_{bare} \sim \left(\dfrac{\Lambda_{UV}}{\Lambda_{IR}} \right)^{-\Gamma_{cusp}(\phi,\theta)} \\ \Gamma_{cusp} = -\dfrac{d \log Z}{d \log \mu} \end{cases}$$

Maldacena Wilson loop (in Euclidean space)

$$W = P \exp \oint ds \left(i A_\mu \frac{dx^\mu}{ds} + |\dot{x}^\mu| \phi^I n^I \right)$$

One-loop renormalization

$(\dot{x}_1^\mu, n_1^I)$ $(\dot{x}_2^\mu, n_2^I)$

$$\propto \left(\dot{x}_1 \cdot \dot{x}_2 - n_1 \cdot n_2 \right)$$

$$= \left(\cos\phi - \cos\theta \right)$$

Take the limit $\phi \equiv i\gamma \to i\infty$

$$x^\mu = (s\cosh\gamma , s\sinh\gamma , 0, 0) \longrightarrow (s e^\gamma , s e^\gamma , 0, 0)$$

lightlike limit

Weak coupling result :

[Drukker, Gross, Ooguri, 9904191][Drukker, Forini, 1105.5144]

$$\langle W_{cusp} \rangle \sim \exp\left(-T \sum_n \left(\frac{\lambda}{16\pi^2} \right)^n \Gamma_{cusp}^{(n)} \right)$$

(used conformal map to send cusp to spatial infinity)

$$\Gamma_{cusp}^{(1)} = 2 \, \frac{\cos\phi - \cos\theta}{\sin\phi} \, \phi \to 2\phi \qquad (\phi \to i\infty)$$

suggesting $\quad \Gamma_{cusp}(i\gamma, \theta) \to \gamma \, f(g) \qquad (\gamma \to \infty)$

Higher loop : $\gamma = \sum_{n=1}^{\infty} \left(\frac{\lambda}{16\pi^2} \right)^n \gamma_n$

$$\gamma_1 = 8 S_1$$

$$\gamma_2 = -16 \left\{ S_3 + S_{-3} - 2 S_{-2,1} + 2 S_1 \left(S_2 + S_{-2} \right) \right\}$$

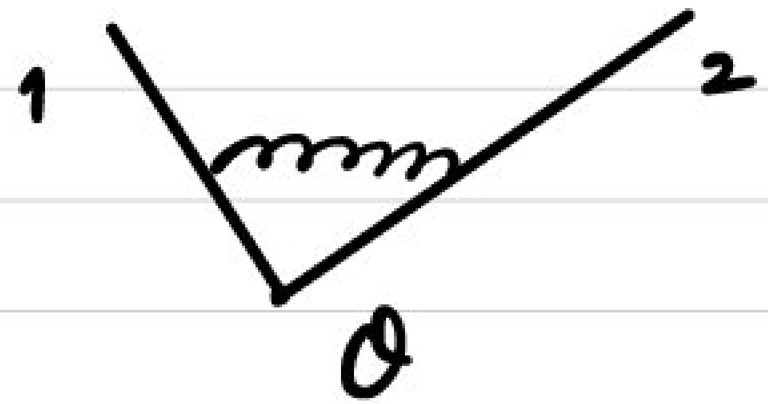
$$S_\phi = 1, \qquad S_{a_1 \cdots a_n} = \sum_{k=1}^{S} \frac{(\operatorname{sign} a_1)^k}{k^{|a_1|}} S_{a_2 \cdots a_n}$$

All terms of γ_n have the same transcendentality

$$wt(S_{a_1 a_2 \cdots}) = |a_1| + |a_2| + \cdots$$

related to max. transcendental terms in QCD

Why cusp dimension = scaling function ? (HARD)



cusp ~ insertion of defect op.

lightlike limit $\Rightarrow$ $\mathcal{O} \sim D_+^S Z$

One-loop in momentum space $\sim \int d^D k \dfrac{1}{(\dot{x}_1 \cdot k)(\dot{x}_2 \cdot k)}$

divergent if collinear ; $k^\mu \parallel \dot{x}_1^\mu$ or x_2^μ

collinear gluons are the main contribution for Γ_{cusp}, $f(g)$

$\Rightarrow$ should be true for any gauge theory, finite N

§ BFKL limit

$$\mathcal{U} = \mathrm{tr}\left(Z \overleftrightarrow{D}_+^{\,S} Z \right) \Rightarrow \gamma_1 = \frac{\lambda}{2\pi^2} h(S)$$

γ_1 diverges if $S \in \{-1, -2, \cdots\}$, Expand around $S = -1$

$$\gamma_1 = \frac{\lambda}{2\pi^2} h(-1+\omega) = \frac{\lambda}{2\pi^2} \left(-\frac{1}{\omega} + O(\omega^0) \right)$$

BFKL equation:

$$\omega = \frac{\lambda}{4\pi^2} \left\{ 2\psi(1) - \psi\left(\frac{1+\Delta}{2}\right) - \psi\left(\frac{1-\Delta}{2}\right) \right\}$$

[Kotikov, Lipatov
hep-ph/0208220]

$$= \frac{\lambda}{2\pi^2} \left(-\frac{1}{\Delta - 1} + \cdots \right)$$

→ agreement if $\Delta = 1 + \gamma_1$

ω_{BFKL} is the "singularity" of conformal 4pt function

[Costa, Gonçalves, Penedones, 1209.4355]

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle \sim g(u,v) \xrightarrow[\text{Mellin}]{} M(s,t)$$

$$M(s,t) = \sum_n \int d\nu \ b_{n,\nu} \ \underline{M_{n,\nu}(s,t)}$$

partial wave for $SL(2,\mathbb{C})$

replace $\sum_n \longrightarrow \int dJ$; $b(J,\nu) \sim \frac{b_0}{J - J_*}$

leading contribution in the Regge limit ($s \to \infty$, t fixed)

The pole position $J_* = J_*(\omega_{BFKL})$ is given by

"summing over all ladder diagrams"

$\rightsquigarrow$ BFKL equation

Why related to $\gamma(S)$ around $S = -1$?

- Singularity at the same quantum number & kinematics
- consistent with integrability predictions (all orders in λ)
- perhaps "trivial relation" in QED, no ladder sums

Historically, Regge limit of QCD → BFKL equations

$$\omega\, \Phi(k) \sim \int d^2k\; K_{BFKL}(k,k')\, \Phi(k')$$

⇝ H_{12} of integrable $SL(2,\mathbb{C})$ spin chain, $H = \sum_k H_{k,k+1}$

4D high energy scattering

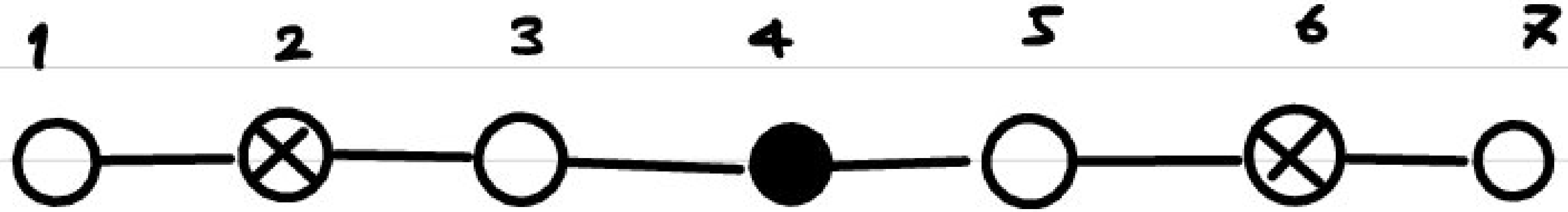
transverse 2D plane becomes "simple"

"BFKL Pomeron" for QCD ⇝ $N=4$ SYM ?

"Soft Pomeron" for hadrons ← real experiment

§ Asymptotic Bethe Ansatz

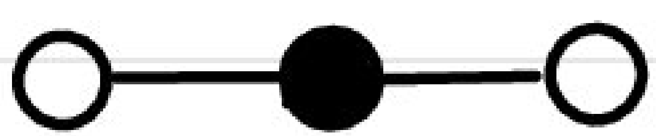
One-loop Bethe Ansatz for $psu(2,2|4)$



bosonic fermionic momentum-carrying

$V = [0,0,0,1,0,0,0]$: Dynkin label

vector rep. for $SO(6)$

example :  $V = [0,1,0]$

Cartan Matrix

$$M = \left(\begin{array}{cc|ccc|cc} -2 & 1 & & & & & \\ 1 & 0 & -1 & & & & \\ \hline & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & & \\ & & & -1 & 2 & -1 & \\ \hline & & & & -1 & 0 & 1 \\ & & & & & 1 & -2 \end{array}\right)\begin{array}{c} B \\ F \\ B \\ B \\ B \\ F \\ B \end{array}$$

Called "Beauty" diagram

Dynkin diagram of super Lie alg is not unique

Bethe Ansatz

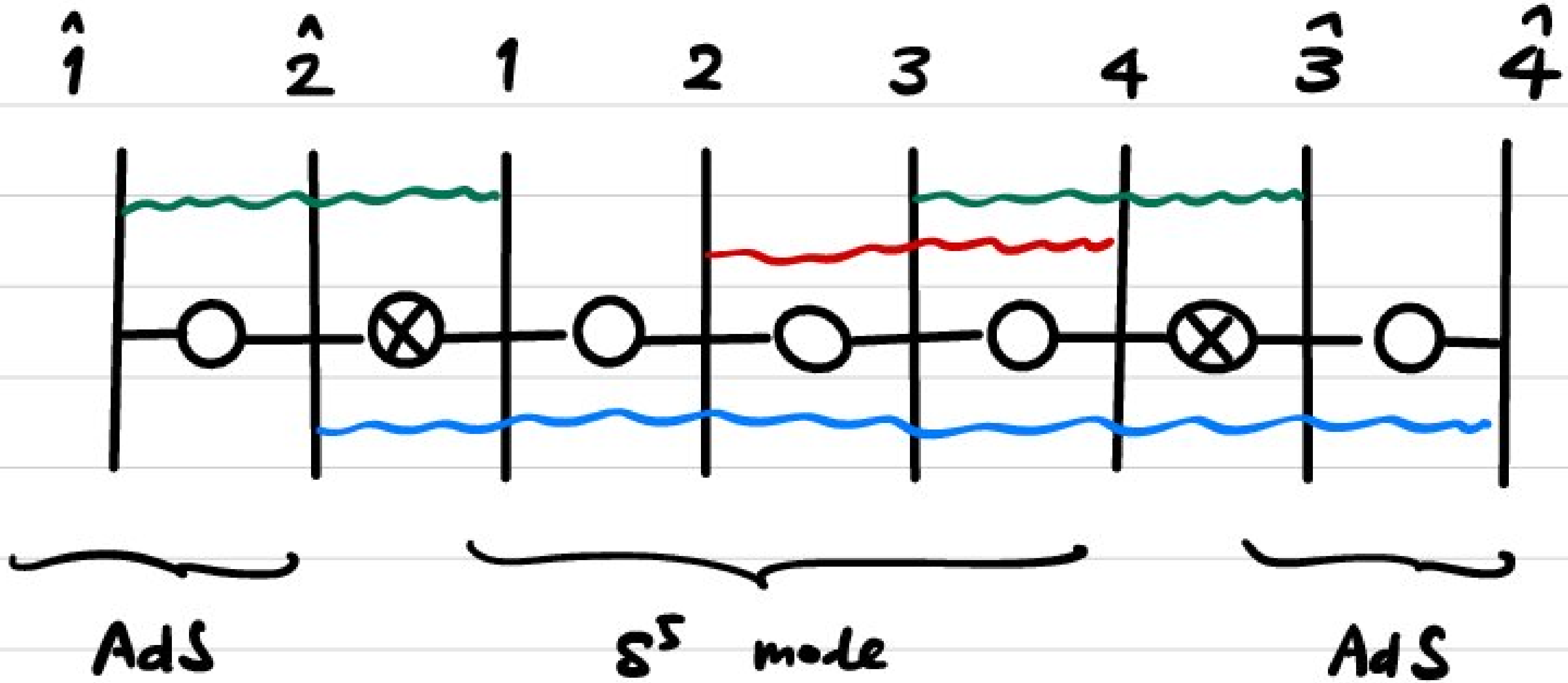
$$\left(\frac{u_j + \frac{i}{2} V_{k_j}}{u_j - \frac{i}{2} V_{k_j}} \right)^L = \prod_{l \neq j} \frac{u_j - u_l + \frac{i}{2} M_{k_j, k_l}}{u_j - u_l - \frac{i}{2} M_{k_j, k_l}}$$

Zero-momentum

$$\prod_j \frac{u_j + \frac{i}{2} V_{k_j}}{u_j - \frac{i}{2} V_{k_j}} = 1$$

Energy

$$E = \sum_j \left(\frac{i}{u_j + \frac{i}{2} V_{k_j}} - \frac{1}{u_j - \frac{i}{2} V_{k_j}} \right)$$



S^1 , J^1 J^2 J^3 , S^2 : Cartan

$$\left\{ \begin{matrix} \hat{1}\hat{2} & \hat{1}\hat{3} & \hat{1}\hat{4} \\ \hat{3}\hat{4} & \hat{2}\hat{4} & \hat{2}\hat{3} \end{matrix} \right\} \qquad \left\{ \begin{matrix} 12 & 13 & 14 \\ 34 & 24 & 23 \end{matrix} \right\} \qquad \left\{ \begin{matrix} \hat{1}a, \ \hat{2}a \\ a\hat{3}, \ a\hat{4} \end{matrix} \right\} \quad a = 1, 2, 3, 4$$

$\vec{y} \cdot \vec{y} = -1$ $\vec{x} \cdot \vec{x} = 1$ 16 fermions

AdS_5 S^5

Assuming that all roots are regular

(= HWS of $psu(2,2|4)$)

Number of Bethe roots & length

$(K^1, K^2, \cdots, K^7, L)$

$\longleftrightarrow$ Cartan charges of $psu(2,2|4)$

$(\Delta, S^1, S^2, J^1, J^2, J^3)$

Also "duality transformation" can change vacuum

$|0\rangle = \mathrm{tr}\, Z^L \longrightarrow |0\rangle = \mathrm{tr}\, F^L, \quad F = F_{1+i2,3+i4}$

3-loop in $SU(2)$ sector : [Beisert, Dippel, Staudacher, 0405001]
[Rej, Serban, Staudacher, 0512077]

higher loop $\to$ long-range interaction

Hubbard model : $n_{i\sigma} = c_{i\sigma}^{\dagger} c_{i,\sigma}$

$$H = -t \sum_{i} \sum_{\sigma=\uparrow,\downarrow} \underbrace{\left(c_{i,\sigma}^{\dagger} c_{i+1,\sigma} + c_{i+1,\sigma}^{\dagger} c_{i,\sigma} \right)}_{\text{hopping}} + U \sum_{i} \underbrace{n_{i\uparrow} n_{i\downarrow}}_{\text{Coulomb}}$$

Four states $\{ \phi, \underbrace{\uparrow, \downarrow}_{Z,\,W \text{ on anti-ferro vacuum}}, \uparrow\downarrow \}$

$U \to \infty$, expand hopping term

- Reproduces dilatation in SU(2) sector up to 3 loops

- Half-filling $\Rightarrow$ (dual) Lieb-Wu equations

- identify W as bound state of $(0, \updownarrow)$

$$\Rightarrow \quad e^{ip_nL} = \left(\frac{x(u_n + i/2)}{x(u_n - i/2)} \right)^L = \prod_{j \neq n}^{M} \frac{u_n - u_j + i}{u_n - u_j - i}$$

$$u(p) = x + \frac{1}{x} = \frac{1}{2} \cot \frac{p}{2} \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}}$$

$$\gamma = \frac{i\lambda}{8\pi^2} \sum_j \left\{ \frac{1}{x(u_j + i/2)} - \frac{1}{x(u_j - i/2)} \right\}$$

x is called Zhukovsky variable

$$x + \frac{1}{x} = u \iff x = \frac{1}{2}\left(u + \sqrt{u^2-4}\right)$$

$$x = \pm 1 \iff u = \pm 2 \quad : \text{ branch points}$$

"string" branch choice is $|x(u)| \geq 1$ for ${}^{\forall}u \in \mathbb{C}$

$$x^{\pm} \equiv x\left(u \pm \frac{i}{2}\right) = e^{\pm ip/2} \underbrace{\frac{1 + \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}}{\frac{\sqrt{\lambda}}{\pi}\sin\frac{p}{2}}}_{}$$

$\longrightarrow 1$ if $\lambda \gg 1$

4-loop (& strong coupling) $\Rightarrow$ Hubbard BAE incorrect

Two sources of corrections:

1) Dressing phase, $(\text{RHS}) = \dfrac{u_n - u_j + i}{u_n - u_j - i} e^{2i\theta(u_n, u_j)}$

2) Wrapping corrections, $(\text{Energy}) = E_{BAE} + \delta E$

"Asymptotic Bethe Ansatz" takes care of (1)

Beisert-Staudacher eq. with BES phase

[0504190, 0610251]

$$\left(\frac{x^+_{4,k}}{x^-_{4,k}}\right)^L = \prod_{j\neq k}^{K_4} \frac{x^+_{4,k} - x^-_{4,j}}{x^-_{4,k} - x^+_{4,j}} \; \frac{1 - \dfrac{1}{x^+_{4,k}\, x^-_{4,j}}}{1 - \dfrac{1}{x^-_{4,k}\, x^+_{4,k}}} \; \underbrace{\sigma^2(x_{4,k}, x_{4,j})}_{\text{dressing}}$$

$$\left(\sim \frac{u_k - u_j + i}{u_k - u_j - i} \right)$$

$$\times \prod_{j=1}^{K_3} \frac{x^-_{4,k} - y_{3,j}}{x^+_{4,k} - y_{3,j}} \prod_{j=1}^{K_5} \frac{x^-_{4,k} - y_{5,j}}{x^+_{4,k} - y_{5,j}}$$

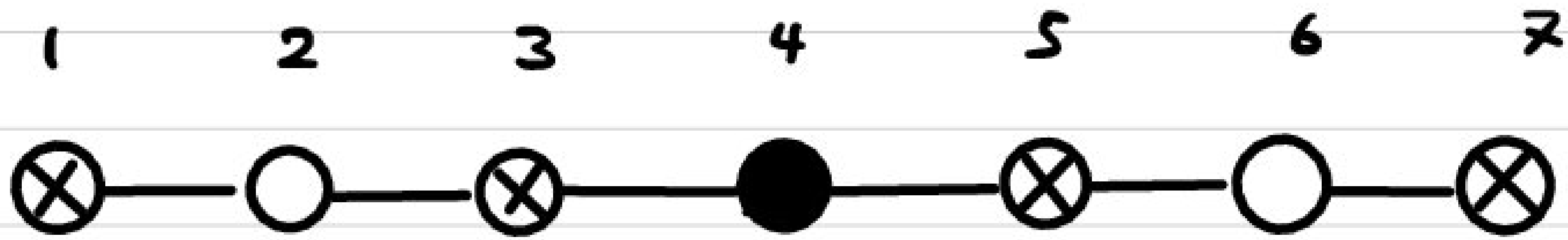
1, 3, 5, 7

fermionic roots

$y \leftrightarrow 1/y$

$$\times \prod_{j=1}^{K_1} \frac{1 - \dfrac{1}{x^-_{4,k}\, y_{1,j}}}{1 - \dfrac{1}{x^+_{4,k}\, y_{1,j}}} \prod_{j=1}^{K_7} \frac{1 - \dfrac{1}{x^-_{4,k}\, y_{7,j}}}{1 - \dfrac{1}{x^+_{4,k}\, y_{7,j}}}$$

$PSU(2,2|4)$ Dynkin :



$V = (0,0,0,1,0,0,0)$

6 more equations for auxiliary roots

$$y_1,\ x_2^{\pm},\ y_3,\quad y_5,\ x_6^{\pm},\ y_7$$

not "beauty" Dynkin :

fermionic dualities with Zhukovsky var. is complizated

At one-loop we know everything:

Dilatation operator, wave-fn, Factorized S-matrix

At higher loops:

$D \longrightarrow$ Possible to compute but tedious

Wrapping corrections from 4-loops

$\Psi \longrightarrow$ Length-changing effect: $\Psi\Psi \sim XYZ$

S-matrix $\longrightarrow$ Still simple $\longrightarrow$ BAE!

N.B. We don't know the operator D we're diagonalizing just its eigenvalues are conjectured.