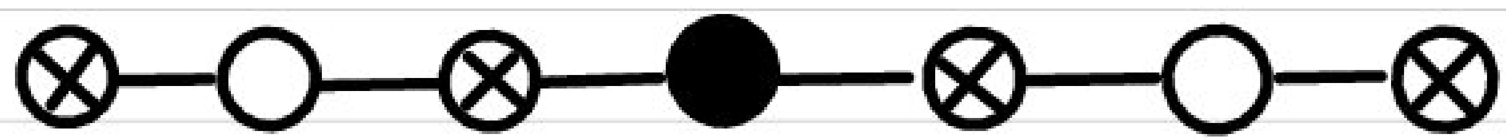
Last week : SL(2) sector

psu(2,2|4) Asymptotic Bethe Ansatz



+ Zhukovsky variable $x + \frac{1}{x} = u$

This week : (Nested) algebraic Bethe Ansatz

su(2|2) asymptotic spin chain

§ Review of algebraic Bethe Ansatz

Want to diagonalize the Hamiltonian of $XXX_{1/2}$ model :

$$H \,| s^{a_1} \cdots s^{a_L} \rangle = H^{a_1 \cdots a_L}{}_{b_1 \cdots b_L} \,| s^{b_1} \cdots s^{b_L} \rangle , \quad s^a = \uparrow, \downarrow$$

Introduce auxiliary space, define monodromy

$$M^{a_1 \cdots a_L , i}{}_{b_1 \cdots b_L , j}(\lambda) = \mathcal{L}^{a_L , i}{}_{b_L , i_L}(\lambda) \cdots \mathcal{L}^{a_1 , i_2}{}_{b_1 , j}(\lambda)$$

such that $\quad H^{\vec{a}}{}_{\vec{b}} = \dfrac{d}{d\lambda} \ln \operatorname{tr} M^{\vec{a}}{}_{\vec{b}} \Big|_{\lambda = \frac{i}{2}}$

Rewrite monodromy : $M^i{}_j = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$

Bethe vacuum :

$$\mathcal{L}_n(\lambda) \sim \begin{pmatrix} \lambda + i/2 & * \\ 0 & \lambda - i/2 \end{pmatrix}$$

$$(A+D)\,|\mathrm{vac}\rangle = \left(\lambda + \frac{i}{2}\right)^L + \left(\lambda - \frac{i}{2}\right)^L$$

Bethe state : $|u\rangle = B(u_1) \cdots B(u_n)\,|\mathrm{vac}\rangle$

Compute $(A+D)\,|u\rangle$ using RTT relations

$$A(\lambda)\,B(\mu) = \frac{\mu - \lambda + i}{\mu - \lambda}\,B(\mu)\,A(\lambda) + (\text{unwanted})$$

Nested Bethe Ansatz for $gl(N)$ spin chain

$$H|\vec{a}\rangle = H_{\vec{a},\vec{b}}\,|\vec{b}\rangle, \qquad a_k \in \{1,2,\cdots,N\}$$

Want to find $T(\lambda) = \mathrm{tr}\, M(\lambda)$ s.t.

$$T(\lambda) = \sum_n \left(\lambda - \frac{i}{2}\right)^n Q_n \rightsquigarrow H$$

Schematically,

$$M = \begin{pmatrix} A & B_1 & B_2 & \cdots & B_{N-1} \\ C_1 & & & & \\ \vdots & & D & & \\ C_{N-1} & & & & \end{pmatrix}$$

Bethe vacuum : $T(\lambda)\,|vac\rangle = \Lambda_0(\lambda)\,|vac\rangle$

Level - I Bethe states :

$$|u\rangle = B^{c_1}(u_1)\cdots B^{c_n}(u_n)\,F_{c_1\cdots c_n}\,|vac\rangle \quad \text{for } c_k = 1, 2, \cdots, N-1$$

RTT relations : (usually RTT $\longleftrightarrow$ Yangian)

$$A(\lambda)\,B^{c}(\mu) = r^{cc'}(\lambda,\mu)\,B^{c'}(\mu)\,A(\lambda) + (\text{unwanted})$$

Eigenvalue equation :

$$T_{c_1\cdots c_n}^{\;c_1'\cdots c_n'}(\lambda)\,F_{c_1'\cdots c_n'} = \Lambda(\lambda)\,F_{c_1\cdots c_n}$$

$\longrightarrow$ Diagonalize $gl(N-1)$ indices by nested Bethe Ansatz

But this doesn't work in $\mathcal{N}=4$ SYM beyond one-loop

$$S^a \longleftrightarrow \mathcal{V} = \{ \phi, \psi, F, D\phi, D\psi, DF, \cdots \}$$

singleton representation of $psu(2,2|4)$

non-compact spin chain $\longleftrightarrow$ $\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \cdots \otimes \mathcal{V}_L$

- Length-changing interaction : " $XYZ \longleftrightarrow \psi\psi 1$ "

[Beisert 0310252 0807.0799]

- Need to find R-matrix in x-variables
- Easy to guess rather than to prove

§ Asymptotic Spin Chain [Beisert, 0511082]

$su(2|3)$ sector $\{Z, \phi^1, \phi^2 \mid \psi^1, \psi^2\}$

Level-I vacuum , $|vac\rangle = |ZZ\cdots Z\rangle$

Level-I Bethe state (= level-II vacuum)

$$|x_1 \cdots x_K\rangle = \sum_{n_1 \ll n_2 \ll \cdots} e^{ip_1 n_1 + ip_2 n_2 + \cdots} |Z \cdots \underset{n_1}{\hat{\phi^1}} \cdots \underset{n_2}{\hat{\phi^1}} \cdots Z\rangle$$

Use coordinate Bethe Ansatz ; don't know $\vec{B}(u)$

Level-II, III $\longrightarrow$ same as nesting of Hubbard model

[Martins, Ramos, solv-int / 9712014]

To proceed, need more data:

- x-variable dependence
- S-matrix elements

Key idea: asymptotic chain $L \to \infty$ in Z^L

- add or remove Z without changing state

$\rightsquigarrow$ central extension

Important because supersymmetry predicts

$\Delta(\lambda)$ of non-BPS states to all orders of λ

$\mathrm{tr}\, Z^L \to$ BPS : $\mathrm{tr}\,(Z^\infty W^M) \to$ "almost" BPS

Algebra $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$

$$\left.\begin{aligned} [R^a{}_b, J^c] &= \delta_b^c J^a - \frac{1}{2}\delta_b^a \underline{J^c} \\ [L^\alpha{}_\beta, J^\gamma] &= \delta_\beta^\gamma J^\alpha - \frac{1}{2}\delta_\beta^\alpha \underline{J^\gamma} \end{aligned}\right\} \text{any generator}$$

$$\{Q^\alpha{}_a, S^b{}_\beta\} = \delta_a^b L^\alpha{}_\beta + \delta_\alpha^\beta R^b{}_a + \delta_a^b \delta_\alpha^\beta \underset{\text{center}}{\underline{C}}$$

Central extension:

$$\{Q^\alpha{}_a, Q^\beta{}_b\} = \epsilon^{\alpha\beta}\epsilon_{ab}\, P$$

$$\{S^a{}_\alpha, S^b{}_\beta\} = \epsilon^{ab}\epsilon_{\alpha\beta}\, K$$

$psu(2|2)$ on $\mathbb{R}^3$ is a limit of $\underline{d(2,1;\epsilon)}$

Kac classification of simple super Lie alg.

$$d(2,1;\epsilon) = \mathrm{Span}\{ \underbrace{R^a{}_b,\ L^\alpha{}_\beta,\ M^A{}_B}_{sl(2)^{\oplus 3}},\ \underbrace{Q^{a\beta C}}_{\text{fermions}} \}$$

$$[M^A{}_B, J^C] = \delta^C_B J^A - (\text{trace})$$

$$[R^a{}_b, Q^{c\delta E}] = \delta^c_b Q^{a\delta E} - (\text{trace})$$

$$\{Q^{a\beta C}, Q^{d\varepsilon F}\} = \alpha\, \epsilon^{ak}\epsilon^{\beta\varepsilon}\epsilon^{CF} R^d{}_k + \beta\, \epsilon^{ad}\epsilon^{\beta k}\epsilon^{CF} L^\varepsilon{}_k$$
$$+ \gamma\, \epsilon^{ad}\epsilon^{\beta\varepsilon}\epsilon^{CK} M^F{}_K$$

Jacobi identity $\Rightarrow$ $\alpha + \beta + \gamma = 0$

Rescaling $\Rightarrow$ $\epsilon = \gamma / \alpha$

$$\begin{pmatrix} Q^{a\beta 1} & Q^{a\beta 2} \end{pmatrix} = \begin{pmatrix} \epsilon^{ac} Q^{\beta}{}_{c} & \epsilon^{\beta\gamma} S^{a}{}_{\gamma} \end{pmatrix}$$

$$\begin{pmatrix} M^{1}{}_{1} & M^{1}{}_{2} \\ M^{2}{}_{1} & M^{2}{}_{2} \end{pmatrix} = \frac{1}{\epsilon} \begin{pmatrix} -C & P \\ -K & C \end{pmatrix}$$

Take the limit $\epsilon \longrightarrow 0$ $\Rightarrow$ $[M, M] = 0$.

$d(2,1:\epsilon) \rightsquigarrow$ Symmetry of $AdS_3 \times S^3 \times S^3 \times S^1$, $\epsilon \sim$ S^3 radius ratio

Representations of psu(2|2)

$$Q_{11}^{a_{11}} Q_{12}^{a_{12}} Q_{21}^{a_{21}} Q_{22}^{a_{22}} |\phi\rangle \qquad a_{i,j} \in \{0,1\}$$

$\Rightarrow$ dim. of minimum <u>long</u> rep is $2^4 = 16$

Minimum short rep. is $\{\phi^a | \psi^\alpha\}$ for $(2|2)$

$$R^a{}_b |\phi^c\rangle = \delta^c_b |\phi^a\rangle - \frac{1}{2} \delta^a{}_b |\phi^c\rangle$$

as in $J^c = \phi^c$

$$L^\alpha{}_\beta |\psi^\gamma\rangle = \delta^\gamma_\beta |\psi^\alpha\rangle - (\text{trace})$$

We want to identify

$$\begin{pmatrix} \Phi^{11} & \Phi^{12} \\ \Phi^{21} & \underbrace{\Phi^{22}}_{\text{dim } 1} \end{pmatrix} \sim \underbrace{\phi^a}_{\text{dim } \frac{1}{2}} \phi^b, \quad \begin{pmatrix} D_{11} Z & D_{12} Z \\ D_{21} Z & \underbrace{D_{22} Z}_{\text{dim } 2} \end{pmatrix} \sim \underbrace{\psi^\alpha}_{\text{dim } 1} \psi^\beta$$

$$\{\underbrace{Q}_{\text{dim } \frac{1}{2}}, Q^\dagger\} \sim \underbrace{P}_{\text{dim } 1} \qquad \{\underbrace{S}_{\text{dim } -\frac{1}{2}}, S^\dagger\} \sim \underbrace{K}_{\text{dim } -1}$$

$$\Rightarrow \begin{cases} Q^\alpha{}_a \, |\phi^b\rangle = a(x)\, \delta^b{}_a \, |\psi^\alpha\rangle & \text{dim} = 1 \\ Q^\alpha{}_a \, |\psi^\beta\rangle = b(x)\, \epsilon^{\alpha\beta} \epsilon_{ab} \, |\phi^b Z^\dagger\rangle & \text{dim} = \frac{3}{2} \end{cases}$$

$$\left\{\begin{array}{l}\text{Closure of the action of } Q, S \text{ on } \phi, \psi \\ \text{Matching } su(2) \otimes su(2) \text{ indices} \\ \text{Matching dimensions}\end{array}\right.$$

$\Rightarrow$ fixes the transformation rule up to coeff

$Z^{\pm}$ means adding / removing Z

$$\left\{\begin{array}{l} S^{a}{}_{\alpha} |\phi^{b}\rangle = c(x)\, \epsilon^{ab} \epsilon_{\alpha\beta} |\psi^{\beta} Z^{-}\rangle \\ S^{a}{}_{\alpha} |\psi^{\beta}\rangle = d(x)\, \delta^{\beta}_{\alpha} |\phi^{a}\rangle \end{array}\right.$$

Compute $\{Q, S\} = R + L + C$ on this basis :

$$Q^{\alpha}{}_{a} S^{b}{}_{\beta} \, |\phi^{c}\rangle = Q^{\alpha}_{a} \, c(x) \, \varepsilon^{bc} \varepsilon_{\beta\gamma} \, |\psi^{\gamma} Z^{-}\rangle$$

$$= b(x) c(x) \, \varepsilon^{bc} \varepsilon_{\beta\gamma} \varepsilon^{\alpha\gamma} \varepsilon_{ad} \, |\phi^{d} Z^{+} Z^{-}\rangle$$

$$= b(x) \, c(x) \, \delta^{\alpha}_{\beta} \left(\delta^{b}_{a} \, |\phi^{c}\rangle - \delta^{c}_{a} \, |\phi^{b}\rangle \right)$$

$$S^{b}_{\beta} \, Q^{\alpha}{}_{a} \, |\phi^{c}\rangle = S^{b}_{\beta} \, a(x) \, \delta^{c}_{a} \, |\psi^{\alpha}\rangle$$

$$= a(x) \, d(x) \, \delta^{c}_{a} \, \delta^{\alpha}_{\beta} \, |\phi^{b}\rangle$$

$$\delta^{\alpha}_{\beta} \, R^{b}{}_{a} \, |\phi^{c}\rangle = \delta^{\alpha}_{\beta} \left(\delta^{c}_{a} \, |\phi^{b}\rangle - \frac{1}{2} \delta^{b}_{a} \, |\phi^{c}\rangle \right)$$

Compare coefficients of $\delta^{\alpha}_{\beta}\delta^{b}_{a}|\phi^{c}\rangle$ and $\delta^{\alpha}_{\beta}\delta^{c}_{a}|\phi^{b}\rangle$

$|\phi^{c}\rangle$: $\quad b(x)c(x) = C - \frac{1}{2}$

$|\phi^{b}\rangle$: $\quad -b(x)c(x) + a(x)d(x) = 1$

Substitute back : $\quad C = \frac{ad-bc}{2} + bc = \frac{ad+bc}{2}$

Action of $\{Q, S\}|\psi\rangle$ is the same due to automorphism

$\phi \leftrightarrow \psi, \quad R \leftrightarrow L, \quad Q \leftrightarrow S$

The other relations :

$$\{Q, Q\} \sim P \quad \Rightarrow \quad P|\chi\rangle = a(x)\, b(x)\, |\chi\, Z^+\rangle$$

$$\{S, S\} \sim K \quad \Rightarrow \quad K|\chi\rangle = c(x)\, d(x)\, |\chi\, Z^-\rangle$$

Recall that $P = K = 0$ in the original $su(2|2)$

We impose $P = K = 0$ on the total state :

$$|\chi_1, \chi_2, \cdots\rangle = \sum_{n_1 \ll n_2 \ll} e^{ip_1 n_1 + ip_2 n_2 + \cdots} |\cdots Z \cdots \underset{\hat{n}_1}{\chi_1} \cdots \underset{\hat{n}_2}{\chi_2} \cdots Z\rangle$$

Then

$$P\,|x_1 \cdots x_M\rangle = \sum_{k=1}^{M} \underline{a_k b_k} \prod_{l=k+1}^{M} e^{-ip_l}\,|x_1 \cdots x_M z^+\rangle$$

rewrite $a_k b_k = \alpha\left(e^{-ip_k} - 1\right)$

Use:

$$\left(e^{-ip_k} \cancel{-1}\right) e^{-ip_{k+1}} + \left(\cancel{e^{-ip_{k+1}}} - 1\right) = e^{-i(p_k + p_{k+1})} - 1$$

$$P\,|x_1 \cdots x_M\rangle = \underline{\alpha\left(\prod_{k=1}^{M} e^{-ip_k} - 1\right)}\,|x_1 \cdots x_M z^+\rangle$$

periodicity conditions, $P_{tot} \equiv 0 \pmod{2\pi}$

Transporting the shift operator $Z^{\pm}$:

$$|Z^{\pm}x_1, x_2, \cdots\rangle = \sum_{n_1 \ll n_2} e^{ip_1 n_1 + ip_2 n_2 + \cdots} |\cdots \underset{n_1 \pm 1}{\hat{x_1}} \cdots \underset{n_2}{\hat{x_2}} \cdots\rangle$$

$$= \sum_{n_1 \ll n_2} e^{ip_1(n_1 \mp 1) + ip_2 n_2 + \cdots} |\cdots \underset{n_1}{\hat{x_1}} \cdots \underset{n_2}{\hat{x_2}} \cdots\rangle$$

$$= e^{\mp ip_1} |x_1 Z^{\pm}, x_2, \cdots\rangle$$

$$= e^{\mp ip_1} |x_1, Z^{\pm} x_2, \cdots\rangle$$

$$= e^{\mp ip_1 \mp ip_2} |x_1, x_2 Z^{\pm}, \cdots\rangle$$

Similarly

$$K|x_1 \cdots x_M\rangle = \beta\left(\prod_{k=1} e^{ip_k} - 1\right)|x_1 \cdots x_M z^-\rangle$$

$$\text{if} \quad c_k d_k = \beta\left(e^{ip_k} - 1\right)$$

In other words,
$$\begin{cases} P|x\rangle = \alpha\left(|z^+ x\rangle - |x z^+\rangle\right) \\ K|x\rangle = \beta\left(|z^- x\rangle - |x z^-\rangle\right) \end{cases}$$

Extra centers ~ Gauge transform for adding / removing z

Use $\begin{cases} a_k b_k = \alpha\left(e^{-ip_k} - 1\right) \\ c_k d_k = \beta\left(e^{ip_k} - 1\right) \end{cases}$ & $a_k d_k - b_k c_k = 1$

We have $C = \sum_k{}' C_k$

$$4C_k^2 = \left(a_k d_k + b_k c_k\right)^2 = 1 + 16\,\alpha\beta\,\sin^2\frac{p_k}{2}$$

Compare with $\quad \gamma = \sum_k \left(\sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p_k}{2}} - 1\right)$

gives $\quad 2C = 1 + \gamma, \qquad 16\alpha\beta = \lambda/\pi^2$

We also know that all-loop energy is written by Zhukovsky variables :

$$E \sim \frac{i}{g}\left(x^{+} - \frac{1}{x^{+}} - x^{-} + \frac{1}{x^{-}}\right)$$

$\Rightarrow$ a, b, c, d as a function of $x^{\pm}(p)$

$$(a,b,c,d) = \sqrt{\frac{g}{2}}\left(\gamma,\ \frac{-\alpha}{\gamma}\left(1-\frac{x^{-}}{x^{+}}\right),\ \frac{i\gamma}{\alpha x^{-}},\ \frac{x^{+}-x^{-}}{i\gamma}\right)$$

unitarity : $a = d^{*}$, $b = c^{*}$ $\Rightarrow$ $|\alpha| = 1$, $\gamma = \sqrt{i(x^{-}-x^{+})}$

Then

$$1 = ad - bc = \frac{g}{2i}\left(x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-}\right)$$

solved if $x + \frac{1}{x} = u$, $\quad f^\pm(u) = f(u \pm \frac{i}{g})$

Also

$$C = \frac{ad+bc}{2} = \frac{g}{2i}\left(x^+ - \frac{1}{x^+} - x^- + \frac{1}{x^-}\right)$$

In general, $\quad C = \sum_k \sqrt{1 + f(\lambda) \sin^2 \frac{p_k}{2}}$

fixed by localization / perturbative data

"g" depends on convention : $\quad g^2 = \frac{N g_{YM}^2}{n\pi^2}$ with $n = 4, 8, 16$

Crossing : relativistic case, $(E, p) = (m\cosh\theta, m\sinh\theta)$

$\theta \to \theta + \pi i$, $(E, p) \to (-E, -p) \Rightarrow$ anti-particle

$x^\pm$ parameterize rapidity torus. $E>0$ if $|x^\pm| \geq 1$

$$e^{ip} = \frac{x^+}{x^-}, \quad E = \frac{g}{i}\left(x^+ - \frac{1}{x^+} - x^- + \frac{1}{x^-}\right)$$

$$1 = \frac{g}{i}\left(x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-}\right)$$

$x^\pm \to 1/x^\pm$, $(E, p) \to (-E, -p) \Rightarrow$ anti-particle

S-matrix : defined as the coefficients of asymptotic wave-functions

$$|\psi\rangle \sim |\cdots \chi_1^a \cdots \chi_2^b \cdots\rangle + S^{ab}_{cd}(p_1,p_2)\,|\cdots \chi_2^c \cdots \chi_1^d \cdots\rangle$$

(+ when χ_1, χ_2 are nearby)

Then we constrain $\mathbb{S}$ by demanding

$$[\,\mathbb{J}_1 + \mathbb{J}_2,\ \mathbb{S}_{12}\,] = 0 \quad \text{for all } \mathbb{J} \text{ in } psu(2.2) \propto \mathbb{R}^3$$

$\Rightarrow$ automatically solves Yang-Baxter !

- Centrally-extended $psu(2|2) \ltimes \mathbb{R}^3$ is naturally extended to Yangian algebra

- However, S-matrix for bound states need more data
  - invariance under Yangian $[\hat{\mathbb{J}}_1 + \hat{\mathbb{J}}_2, \mathbb{S}_{12}] = 0$
  - unitarity, crossing, parity, CPT
  - consistency with fusion

[Arutyunov, Frolov, 0803.4323]

also fixes "gauge choice" $(a,b,c,d) \longrightarrow (\eta a, \frac{b}{\eta}, \eta c, \frac{d}{\eta})$

Example of computation :

$$\Delta(Q) = Q \otimes 1 + \underline{1 \otimes Q}$$

$\rightarrow e^{\mp ip} \otimes Q$ in string theory convention

$$[\Delta(Q), S_{12}] |\phi_1^a \phi_2^b\rangle = 0$$

$$S_{12} |\phi_1^a \phi_2^b\rangle = A'_{12} |\phi_2^a \phi_1^b\rangle + B'_{12} |\phi_2^b \phi_1^a\rangle$$

$$+ \frac{1}{2} C_{12} \varepsilon^{ab} \varepsilon_{\alpha\beta} |\psi_2^\alpha \psi_1^\beta Z^-\rangle$$

$$\Delta(Q_c^\alpha) |\phi_1^a \phi_2^b\rangle = a_1 \delta_c^a |\psi_1^\alpha \phi_2^b\rangle + a_2 \delta_c^b |\phi_1^a \psi_2^\alpha\rangle$$

- Compute $S_{12}\,\Delta(Q)\,|\phi\phi\rangle$

$$S_{12}|\phi_1\psi_2\rangle = G\,|\psi_2\phi_1\rangle + H\,|\phi_2\psi_1\rangle$$

$$S_{12}|\psi_1\phi_2\rangle = L\,|\phi_2\psi_1\rangle + K\,|\psi_2\phi_1\rangle$$

Thus $$S_{12}\,\Delta Q|\phi_1\phi_2\rangle = (a_1L + a_2H)\,|\phi_2\psi_1\rangle + (a_1K + a_2G)\,|\psi_2\phi_1\rangle$$

- Compute $\Delta(Q)\,S_{12}\,|\phi\phi\rangle$

$$\Delta(Q^\gamma_c)\,|\psi_2^\alpha\psi_1^\beta Z^-\rangle = b_2\,\varepsilon^{\gamma\alpha}\varepsilon_{cd}\,|\phi_2^d Z^+\psi_1^\beta Z^-\rangle + b_1\,\varepsilon^{\gamma\beta}\varepsilon_{cd}\,|\psi_2^\alpha\phi_1^d Z^+Z^-\rangle$$

Use $\quad z^+ z^- = 1. \quad z^+ \chi_k = e^{-ip_k} \chi_k z^+$

$$\Delta(Q)\,|\psi\psi z^-\rangle = b_2 e^{-ip_1}\,|\phi_2\psi_1\rangle + b_1\,|\psi_2\phi_1\rangle$$

Also

$$(Q\otimes 1)\left\{A'\,|\phi_2^a\phi_1^b\rangle + B'\,|\phi_2^b\phi_1^a\rangle + C\,|\psi\psi z^-\rangle\right\}$$

$$= a_2\left\{A'\,\delta_c^a\,|\psi_2^\beta\phi_1^b\rangle + B'\,\delta_c^b\,|\psi_2^\beta\phi_1^a\rangle + \cdots\right\}$$

$$= a_2 A\left\{\delta_c^a\,|\psi_2^\beta\phi_1^b\rangle + (a\leftrightarrow b)\right\}$$

$$+\, a_2 B\left\{\delta_c^a\,|\psi_2^\beta\phi_1^b\rangle - (a\leftrightarrow b)\right\} + \cdots$$

if we use

$$\begin{cases} A' = A+B \\ B' = A-B \end{cases}$$

Compare coefficient of four terms:

$$\delta^a_c |\psi_2^\beta \phi_1^b\rangle,\ \delta^b_c |\psi_2^\beta \phi_1^a\rangle,\ \delta^a_c |\phi_2^b \psi_1^\beta\rangle,\ \delta^b_c |\phi_2^a \psi_1^\beta\rangle$$

$\Rightarrow$ equations for $\{A, B, C, G, H, K, L\}$ & $\{a_k, b_k\}$

Result: $$A_{12} = S_{12}^0 \frac{x_1^+ - x_2^-}{x_1^- - x_2^+}$$

$$\frac{A_{12} - B_{12}}{2} = S_{12}^0 \frac{x_1^+ - x_2^-}{x_1^- - x_2^+} \frac{x_2^+ - x_1^+}{x_2^+ - x_1^-} \frac{1 - 1/x_2^- x_1^+}{1 - 1/x_2^- x_1^-}$$

$\vdots$

$$L_{12} = S_{12}^0 \frac{x_2^- - x_1^-}{x_2^- - x_1^+}$$

The S-matrix elements = Shastry's R-matrix of 1D Hubbard Model

Can diagonalize S-matrix using nested Bethe Ansatz

- coordinate BA [Beisert] [de Leeuw, 0705.2369]
- algebraic BA [Martins, Melo, 0703086]

$$|Z\cdots Z\rangle \longrightarrow |\phi^1\cdots\phi^1\rangle \longrightarrow |\psi^{\alpha_1}\cdots\psi^{\alpha_K}\rangle, \quad \phi^2 \sim \psi^1\psi^2$$

$(\alpha = 1,2)$ SU(2) symmetry

Composite excitation

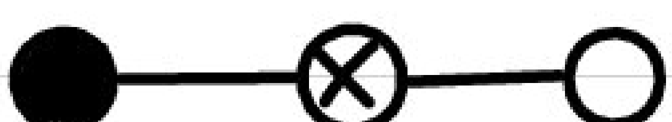
su(2|3) Bethe eq: $\quad (v_k = y_k + 1/y_k)$

$$\left(\frac{x_k^+}{x_k^-}\right)^{K_0} = \prod_{l\neq k}^{K_1} S^0(x_l,x_k)\,\frac{x_k^+ - x_l^-}{x_k^- - x_l^+}\prod_{l=1}^{K_2}\frac{x_k^- - y_l}{x_k^+ - y_l}$$

$$1 = \prod_{l=1}^{K_1}\frac{y_k - x_l^+}{y_k - x_l^+}\prod_{l=1}^{K_3}\frac{v_k - w_l + i/g}{v_k - w_l - i/g}$$

$$1 = \prod_{l}^{K_2}\frac{w_k - v_l + i/g}{w_k - v_l - i/g}\prod_{l\neq k}^{K_3}\frac{w_k - w_l - 2i/g}{w_k - w_l + 2i/g}$$

Same as XXX$_{1/2}$

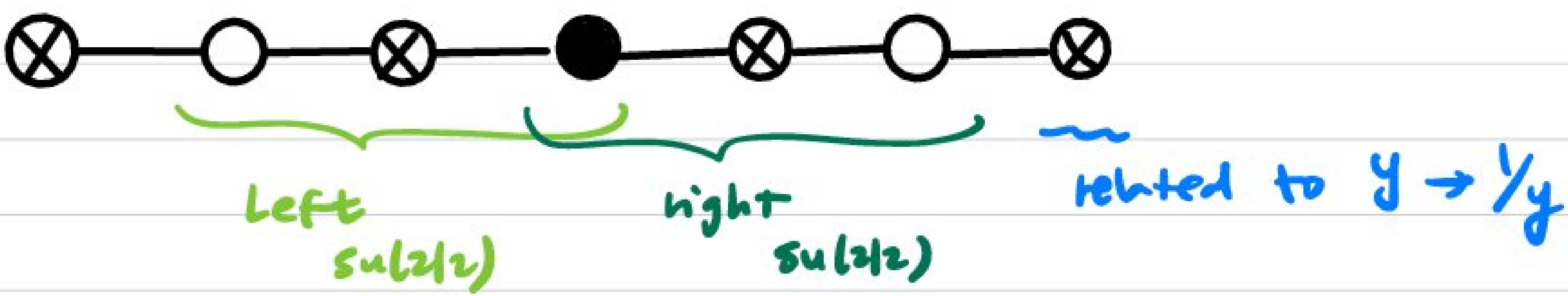


To get BAE for $\mathcal{N}=4$ SYM,

we need to glue two wings of $su(2|2)$

$$psu(2,2|4) \longrightarrow \left(psu(2|2) \ltimes \mathbb{R}^3 \right)^2$$

by identifying center C (or momentum-carrying node)



Can derive similar eqs from $su(1|1)^2$ for AdS_3

but one copy of $su(1|1)$ for AdS_2 is less strong

Symmetry doesn't fix the scalar $S_0(x_1, x_2)$

$\rightsquigarrow$ dressing factor

We need crossing relations

- if we assume "effective 2-dim QFT" describing asymptotic chain, we should have "crossing"
- or we impose that S-matrix scatters trivially against singlet = (particle, anti-particle composite)