Higher-point Correlation Functions in AdS/CFT and Background Independence

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1. **Motivation**
2. **LLM geometry and background independence**
3. **Free n-pt function from permutations**
4. **Free 3pt function at finite Nc**
5. **Summary**

1. Motivation

- **AdS/CFT** の多点関数を知りたい
- 多点関数を知ることは $\mathbf{1}/\boldsymbol{Nc}$ 効果を知ること
- 可積分系の方法も使えるけど、全部の情報は入ってない
- 有限群の方法を使うと、自由多点関数を任意の $\boldsymbol{Nc}$ で計算できる
- **LLM** 幾何に対応する非自明な **large** $\boldsymbol{Nc}$ 極限が取れる

Motivation

- AdS/CFT has been intensively studied for decades
- The primary example of AdS/CFT is the correspondence between $\mathcal{N}=4$ super Yang-Mills ($\mathcal{N}=4$ SYM) in 4D, and superstring in $AdS_5 \times S^5$
- Both theories are believed to be integrable in the planar large Nc limit
- Using integrability, some quantities can be computed as a function of the 't Hooft coupling $\lambda = N_c\, g_{\rm YM}^2$, interpolating the weak and strong coupling regions

  - Energy spectrum (2-point function)
  - OPE coefficients (3-point function)
  - Wilson loop vev ~ Scattering amplitudes

n-point functions in AdS/CFT

- n-point functions are main local observables of QFT
- related to the fundamental problems of string theory in $AdS_5 \times S^5$
  - String interaction (g_s)
  - Needs an integration over the moduli space of worldsheet ($n \geq 4$) in covariant formalism (RNS or PS)
- computed by using conformal symmetry and integrability
  - Conformal bootstrap [Polyakov (1973)]]Ferrara, Gatto, Grillo (1973)]
  - Hexagonization [Fleury, Komatsu (2016)] [Eden, Sfondrini (2016)]

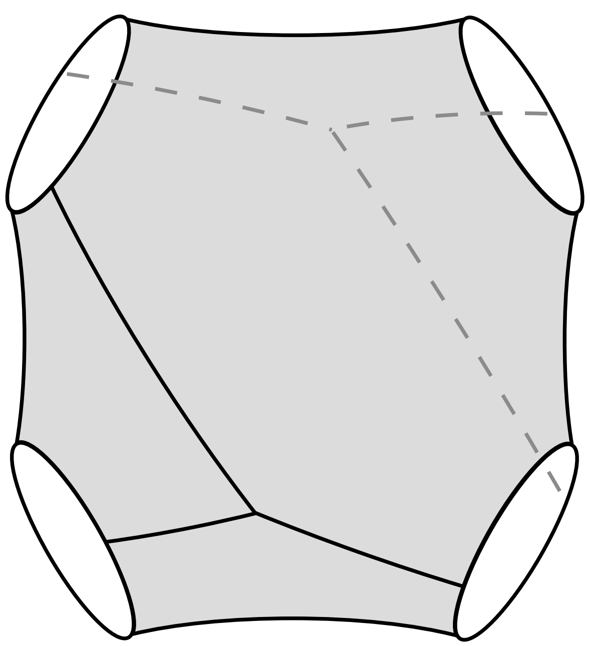
Hexagonization of n-point

- **Construct directly planar n-point functions ($n>2$)**
- **Generalized to non-planar corrections**

Idea: triangulation (or hexagonization) of Riemann surfaces

$\sim$ open string field theory

Example: planar 4pt = 4 hexagons glued together



Gluing

= insertion of the resolution of unity

$$1 = \sum_{\psi} \int dp \, |\psi(p)\rangle \langle \psi(p)|$$

[Fleury, Komatsu (2016)] [Eden, Sfondrini (2016)]

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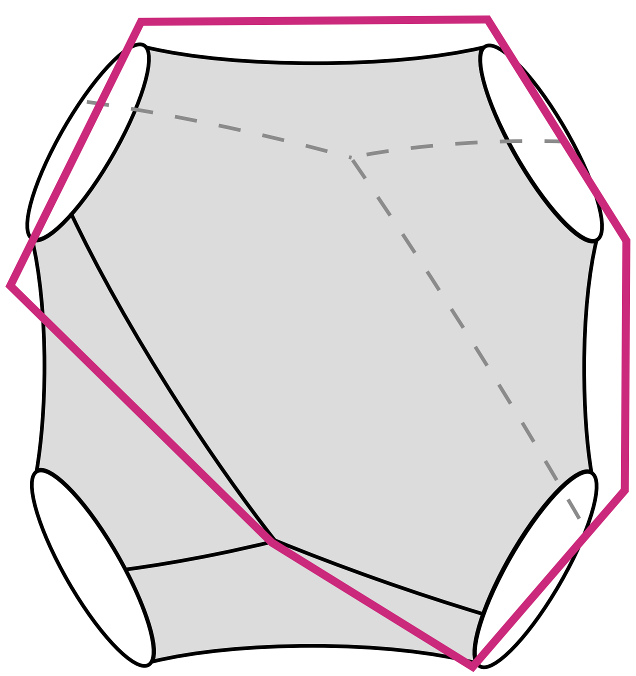

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[Fleury, Komatsu (2016)] [Eden, Sfondrini (2016)]

Generally: $\Sigma_{g,n} \rightarrow (4g + 2n - 4)$ hexagons

[Bargheer, Caetano, Fleury, Komatsu Vieira (2017, 2018)]

Hexagonization of n-point

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- Hexagonization & gluing procedures are quite complicated
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Complex scalars in $\mathcal{N}$=4 SYM $= (X, Y, Z, \overline{X}, \overline{Y}, \overline{Z})$

Half-BPS state at finite Nc

$$\mathcal{O}_\circ = \mathrm{tr}\,(Z^L) + N_c^{-1} \sum_m c_{1,m} \mathrm{tr}\,(Z^m)\,\mathrm{tr}\,(Z^{L-m}) + \dots$$

From $1/Nc$ to Finite Nc

New methods which complements integrability?

- **Back to perturbation of $\mathcal{N}$=4 SYM**
  - **Finite group method**

    [Brown, Heslop, Ramgoolam (2007)]
    [de Mello Koch, Smolic, Smolic (2007)]

    $\sim$ **character expansion in (multi-)matrix model**

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$\sim$ character expansion in (multi-)matrix model

$$\mathrm{tr}\, Z^L = \sum_{R \vdash L} c_R\, \mathcal{O}^R(Z)$$

Operator basis labeled by Young diagram

$$\langle \mathrm{tr}\,(Z^L)\mathrm{tr}\,(\overline{Z}^L)\rangle = \sum_{R,S \vdash L} c_R c_S \langle \mathcal{O}^R(Z)\mathcal{O}^S[\overline{Z}]\rangle$$

$c_R = 0, \pm 1$ for single-trace

$$= \sum_R c_R^2 \prod_{(i,j)\in R} (N_c - j + i)$$

Results are valid for any Nc

[Kostov, Staudacher (1996)] [Kostov, Wynter, Staudacher (1997)]

Main results

Computed n-pt functions using finite-group methods

- **Generalized to n-pt functions of free $\mathcal{N}$=4 SYM at any orders of $1/Nc$**

  [RS (2018)]

- **Free 3-pt of free $\mathcal{N}$=4 SYM at any Nc**

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- **After a finite Nc analysis, we can take non-trivial large Nc limit**
- **Huge operators (like determinants) are interesting examples**

$$\Delta \sim N_c \ \rightarrow \text{Giant gravitons (D-branes)}$$
$$\Delta \sim N_c^2 \ \rightarrow \text{Deformation of } AdS_5 \times S^5$$

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LLM geometry

2. LLM geometry and background independence

- **Lin-Lunin-Maldacena background と LLM 平面**
- **Schur 演算子**
- **Distant corners approximation と同心円の対応**
- **LLM の背景独立性**

LLM geometry

- General half-BPS non-singular solutions of IIB supergravity
- dual to half-BPS operators $\prod_i \mathrm{tr}\, Z^{n_i}$
- with the residual symmetry $\mathfrak{psu}(2|2)^2 \supset \mathfrak{so}(4)^2$

[Lin, Lunin, Maldacena (2004)]

$$ds^2 = -2y\cosh G\left(dt^2 + V_i\, dx^i\right)^2 + \frac{dy^2 + (dx^i)^2}{2y\cosh G} + ye^G d\Omega_{S^3} + ye^{-G} d\Omega_{\tilde{S}^3}$$

$$z = \frac{\tanh G}{2}\,, \quad \partial_i z = -\epsilon_{ij}\, y\partial_y V_i\,, \quad \partial_y z = y\epsilon_{ij}\,\partial_i V_j\,, \quad (i=1,2)$$

$$\Rightarrow \quad z(x^1, x^2, y=0) = \pm\frac{1}{2}, \quad \text{either } S^3 \text{ or } \tilde{S}^3 \text{ collapses}$$

The sign choice defines a droplet pattern in the LLM plane

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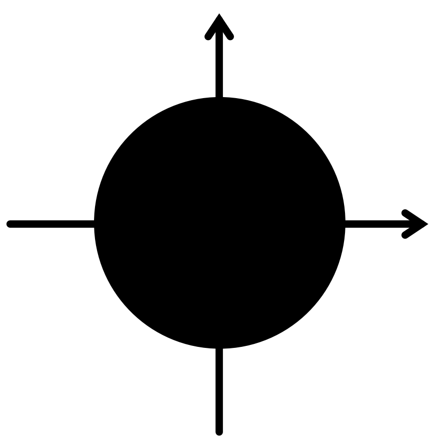
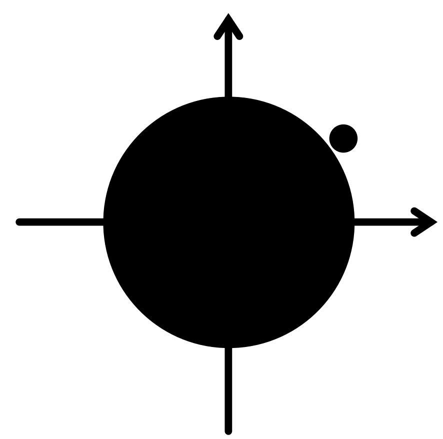
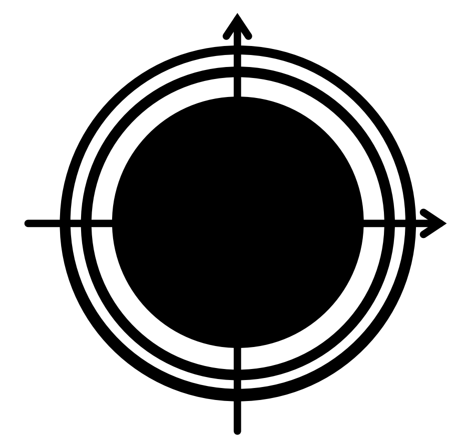


$AdS_5 \times S^5$



(non-maximal) giant graviton



Concentric circles (extra symmetry)

Schur operators

- **LLM geometries are dual to the half-BPS operators of $\mathcal{N}$=4 SYM with huge dimensions**
- **The "LLM operators" can be constructed by representation theory; Young diagram with huge boxes**

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- The "LLM operators" can be constructed by representation theory; Young diagram with huge boxes

$$\mathcal{O}^R(Z) = \frac{1}{L!} \sum_{\alpha \in S_L} \chi^R(\alpha)\, \mathrm{tr}_L \left(\alpha Z^{\otimes L}\right)$$

$\chi^R(\alpha) = S_L$ character of irrep R

$Z = \mathrm{diag}(z_1, z_2, \ldots, z_{N_c}) \;\Rightarrow\; \mathcal{O}^R(Z) =$ Schur polynomial of $\{z_i\}$

[Corley, Jevicki, Ramgoolam (2001)]

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Multi-trace structure of $\underline{\mathrm{tr}_L(\alpha Z^{\otimes L})} \leftrightarrow$ Cycle type of $\alpha \in S_L$

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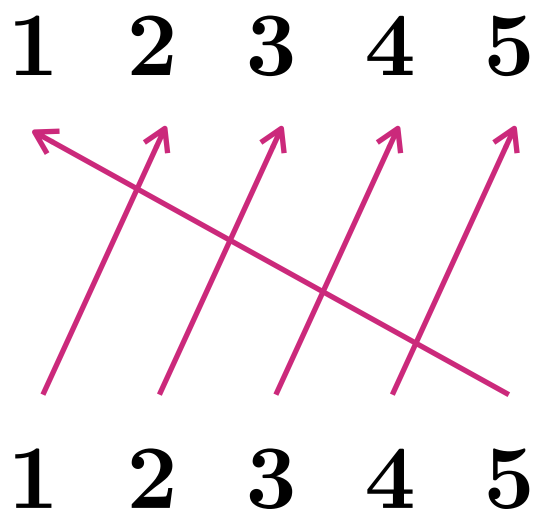
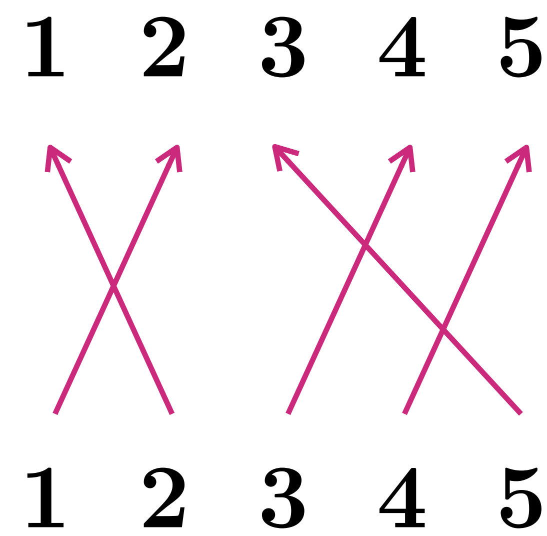


$\alpha_1 = (12345)$

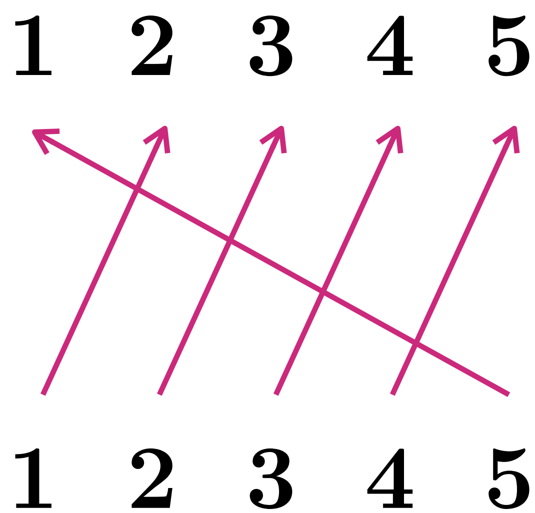
$\mathrm{tr}_5(\alpha_1 Z^{\otimes 5}) = \mathrm{tr}\,(Z^5)$

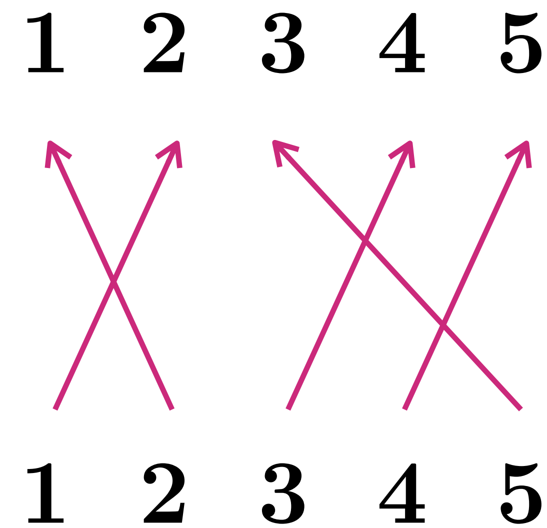


$\alpha_2 = (12)(345)$

$\mathrm{tr}_5(\alpha_2 Z^{\otimes 5}) = \mathrm{tr}\,(Z^2)\,\mathrm{tr}\,(Z^3)$

Multi-trace structure of $\mathrm{tr}_L(\alpha Z^{\otimes L})$ $\leftrightarrow$ Cycle type of $\alpha \in S_L$





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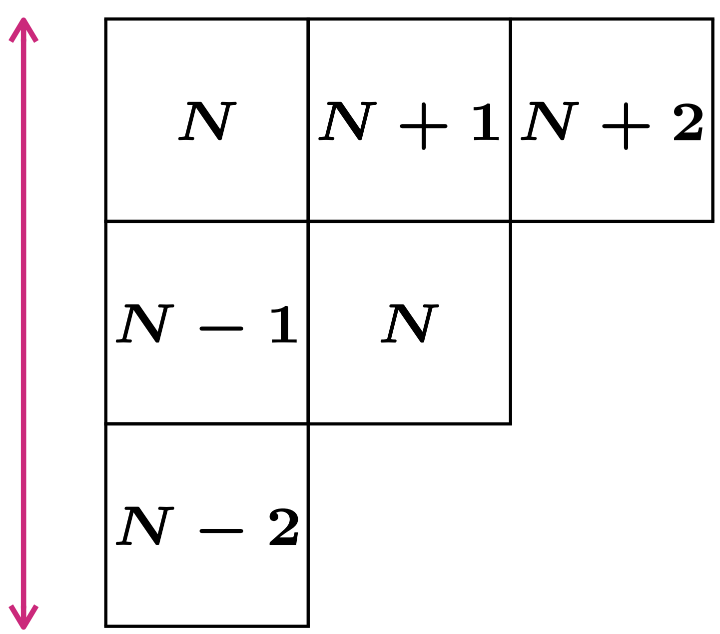
Finite Nc constraints on the height of Young diagram R



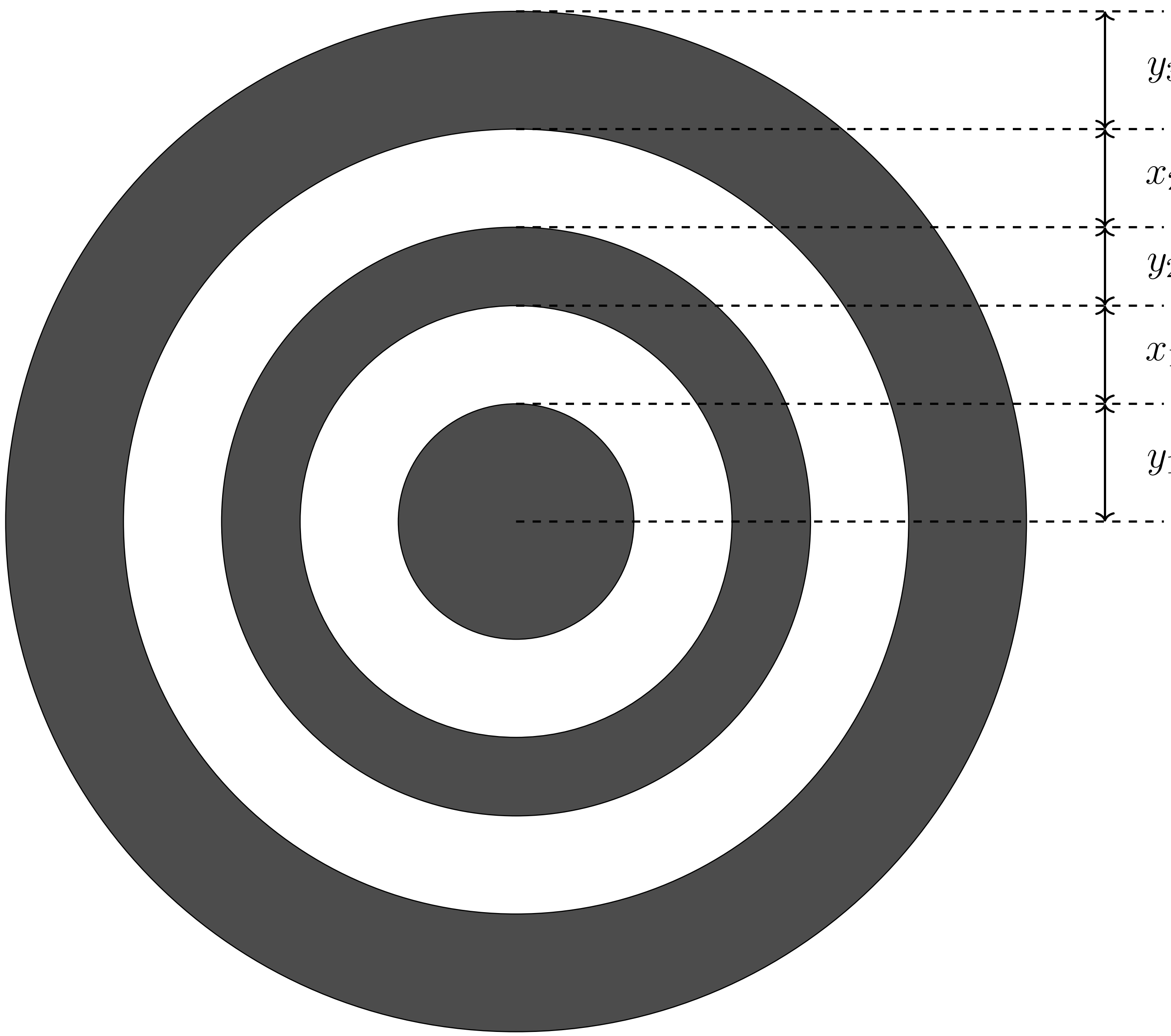
$$\langle \mathcal{O}^R(Z)\, \mathcal{O}^S(\overline{Z})\rangle = \delta^{RS}\, \mathrm{Wt}_{N_c}(R)$$

$$\Rightarrow \quad \mathrm{Wt}_N\left(\,\yng(3,2,1)\,\right) = N^2\left(N^2-1\right)\left(N^2-4\right)$$

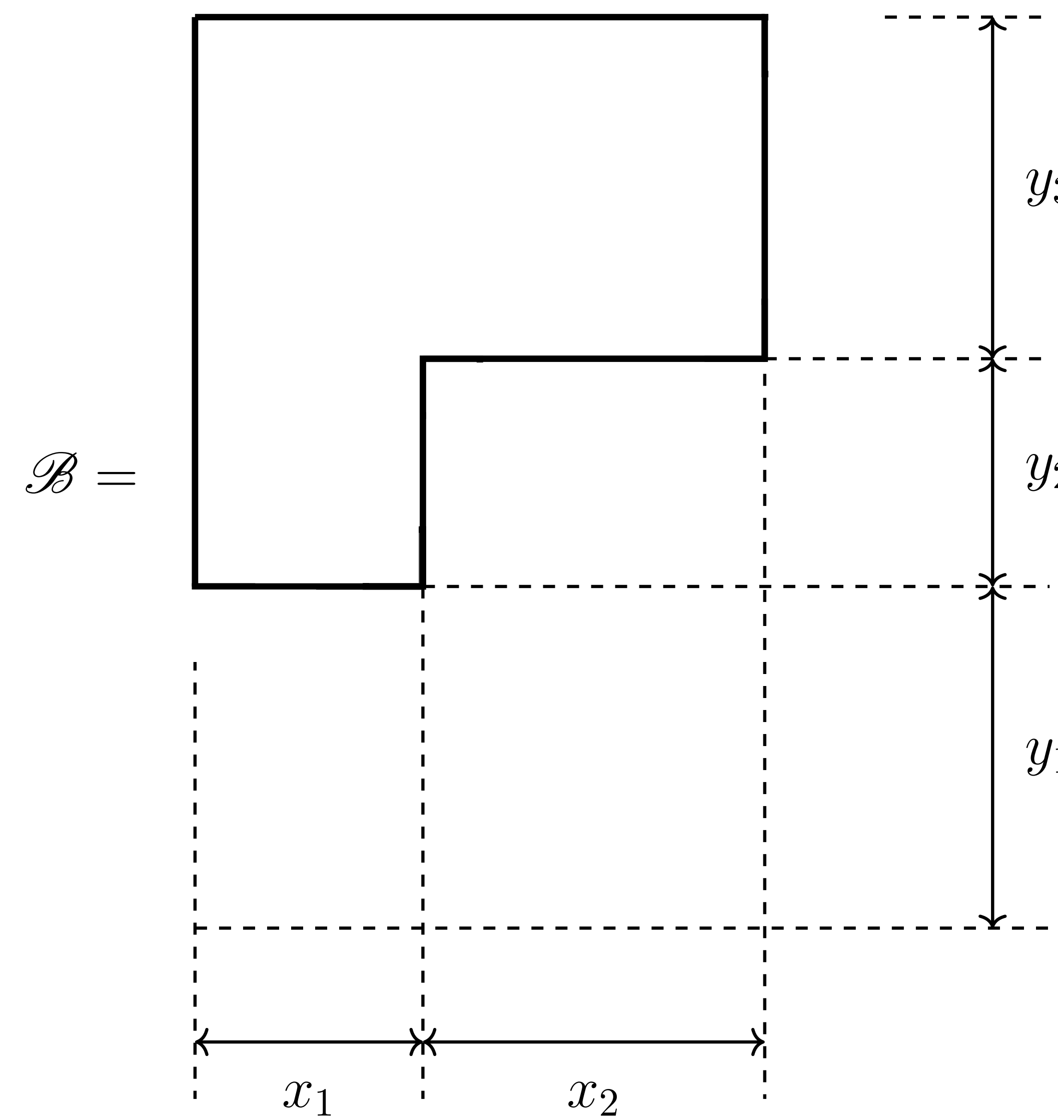
Weight vanishes if height $= Nc$

AdS/CFT correspondence

LLM geometry with concentric circles



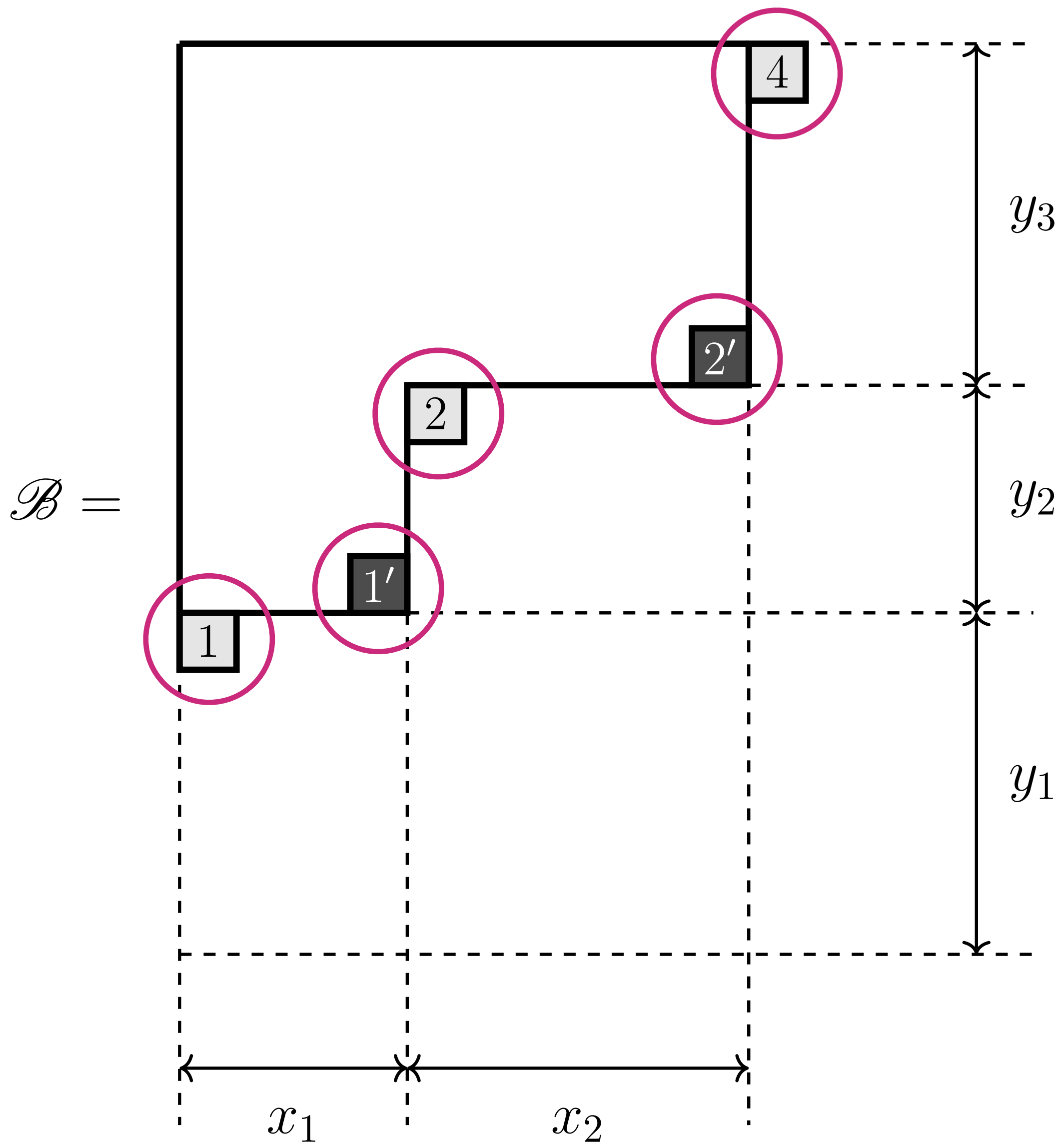
Schur operator with staircase Young diagram



[Lin, Lunin, Maldacena (2004)]

Adding excitations

- **We can add/remove boxes ($\mathcal{N}$=4 SYM scalars) to/from the corners of the staircase Young diagram**
- **This corresponds to stringy excitations over LLM geometry**



Adding excitations

- We can add/remove boxes ($\mathcal{N}$=4 SYM scalars) to/from the **corners** of the staircase Young diagram
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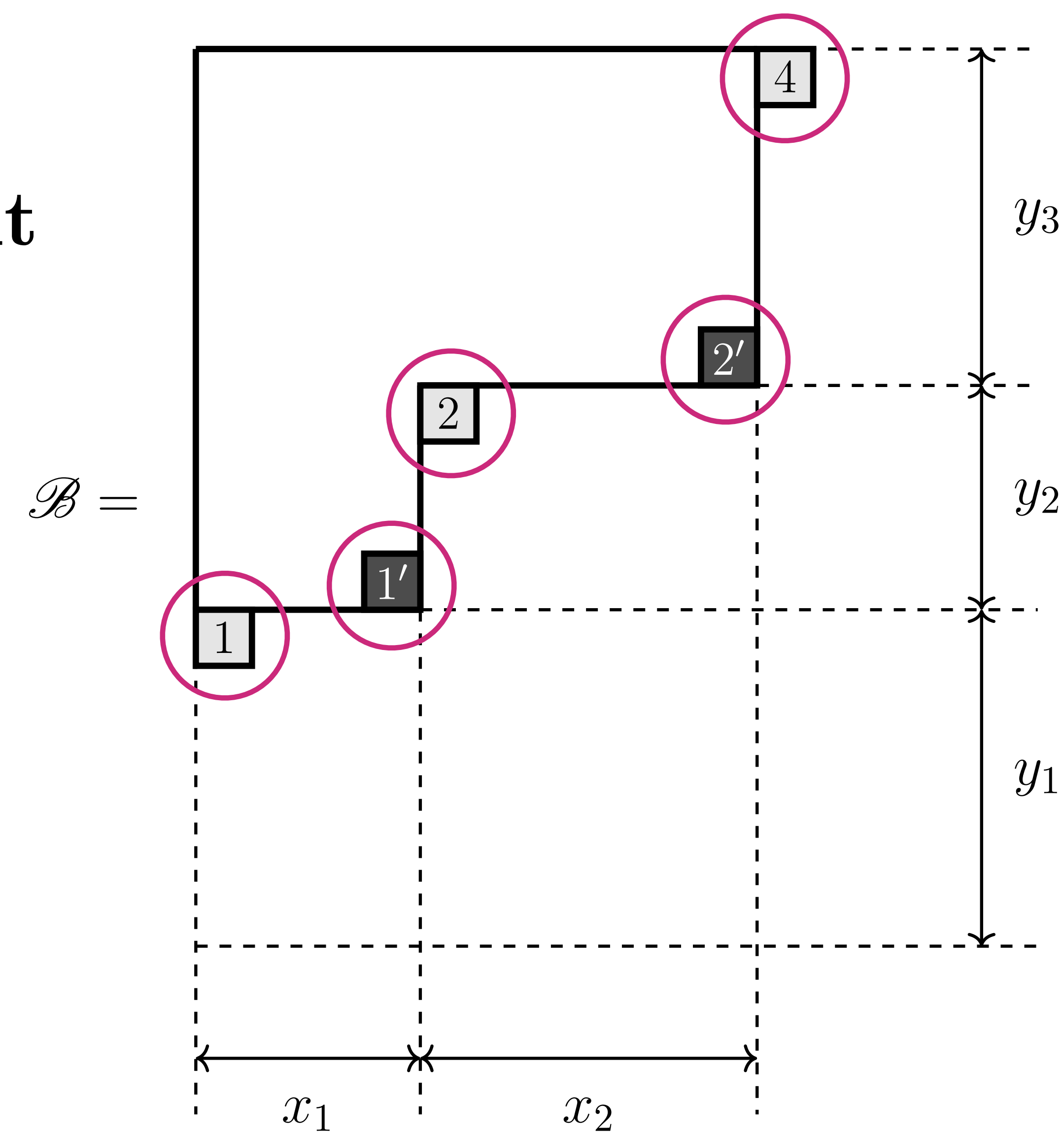
Distant corners approximation:
Assume that each corners are distant

$$x_i \sim y_j \sim \mathcal{O}(N_c) \gg 1$$

Excitations at different corners decouple in this **new large *Nc* limit**



[De Comarmond, de Mello Koch, Jefferies (2010)]
[Carlson, de Mello Koch, Lin (2011)]

LLM background independence

- **Excitations in the $su(2)$ sector (Z, W) attached to a corner of $\mathcal{B}$ can be expressed by restricted Schur operators**

$$\mathcal{O}^{R,(r_1,r_2),\mu_+,\mu_-} \leftarrow \mathrm{tr}_L(\alpha Z^{\otimes m_1} W^{\otimes m_2})$$

$$R \vdash L = (m_1 + m_2), \; r_1 \vdash m_1, \; r_2 \vdash m_2$$

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- The action of one-loop dilatation in the $su(2)$ sector on the restricted Schur operators has been computed

$$\mathcal{D}\,\mathcal{O}^{R,(r_1,r_2),\mu_+,\mu_-} = \sum_{T,t_{1,2},\nu_\pm} M^{R,(r_1,r_2),\mu_+,\mu_-}_{T,(t_1,t_2),\nu_+,\nu_-}\,\mathcal{O}^{T,(t_1,t_2),\nu_+,\nu_-}$$

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LLM background independence

- **In the distant corners approximation, the one-loop mixing matrix $M^{\mathcal{B}}{}_{ij}$ with background $\mathcal{B}$ coincides with M_{ij} without background $\mathcal{B}$, up to a simple rescaling depending on $\mathcal{B}$**

$$M^{+R,(+r_1,r_2),\mu_+,\mu_-}_{+T,(+t_1,t_2),\nu_+,\nu_-} = M^{R,(r_1,r_2),\mu_+,\mu_-}_{T,(t_1,t_2),\nu_+,\nu_-}(N_c \to N_{eff})$$

[de Mello Koch, Huang, Tribelhorn (2018)]

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- Therefore, there is a one-to-one map between small operators and LLM operators of $\mathcal{N}=4$ SYM ($\mathcal{B}$ independence)

$$\mathcal{O}_\bullet = \sum_{R,r_{1,2},\mu_\pm} \boxed{c_{R,(r_1,r_2),\mu_+,\mu_-}} \mathcal{O}^{R,(r_1,r_2),\mu_+,\mu_-} \quad \leftrightarrow$$

$$\mathcal{O}^{\mathcal{B}}_\bullet \simeq \sum_{R,r_{1,2},\mu_\pm} \boxed{c_{R,(r_1,r_2),\mu_+,\mu_-}} \mathcal{O}^{+R,(+r_1,r_2),\mu_+,\mu_-}$$

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3. Free n-pt function from permutations

- 置換群を使った自由２点関数の公式

- $\boldsymbol{n}$ 点に拡張するときの問題と解決法

- $\mathbf{BPS}$ ３点関数の例

- 置換群を使った自由 $\boldsymbol{n}$ 点関数の公式

Free 2pt in permutation basis

Write six scalar fields in $\mathcal{N}$=4 SYM: $\Phi^{A_p} \in (X, \overline{X}, Y, \overline{Y}, Z, \overline{Z})$

Permutation basis of multi-trace operators

$$\mathcal{O}_\alpha^{A_1 A_2 \dots A_L} = \mathrm{tr}_L(\alpha\, \Phi^{A_1} \otimes \Phi^{A_2} \otimes \cdots \otimes \Phi^{A_L})$$

$$\equiv \sum_{a_1, a_2 \dots, a_L = 1}^{N_c} (\Phi^{A_1})^{a_1}_{a_{\alpha(1)}} (\Phi^{A_2})^{a_2}_{a_{\alpha(2)}} \dots (\Phi^{A_L})^{a_L}_{a_{\alpha(L)}}$$

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Explicitly: $\mathcal{O}_\alpha^{(l,m,n)} = \mathrm{tr}_L \left(\alpha\, X^{\otimes l}\, Y^{\otimes m}\, Z^{\otimes n}\right)$

Flavor symmetry : $\mathcal{O}_\alpha^{(l,m,n)} = \mathcal{O}_{\gamma^{-1}\alpha\gamma}^{(l,m,n)} \quad (\forall \gamma \in S_l \otimes S_m \otimes S_n)$

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Tree-level 2pt functions

$$\langle \mathcal{O}_{\alpha_1}^{(l,m,n)}(x)\, \overline{\mathcal{O}}_{\alpha_2}^{(l,m,n)}(0) \rangle = |x|^{-2(l+m+n)} \sum_{\gamma \in S_l \otimes S_m \otimes S_n} N_c^{C(\alpha_1 \gamma \alpha_2 \gamma^{-1})}$$

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$$C(\lambda) = \#\text{ of cycles in } \lambda \in S_L,\ C(\mathrm{id}) = C\Big((1)(2)(3)\dots(L)\Big) = L$$

Free n-pt in permutation basis?

Problems

1. Expects $\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle \sim N_c^{C(\alpha_1 \ldots \alpha_n)}$, but $\alpha_i \in S_{L_i} \neq S_{L_j}$

2. Should sum over the bridge lengths $\ell_{ij} \in \mathbb{Z}_{\geq 0}$ if $n \geq 4$

$$\ell_{ij} = \#\text{ of contractions between } (\mathcal{O}_i, \mathcal{O}_j)$$

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Problems

1. Expects $\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle \sim N_c^{C(\alpha_1 \ldots \alpha_n)}$, but $\alpha_i \in S_{L_i} \neq S_{L_j}$

2. Should sum over the bridge lengths $\ell_{ij} \in \mathbb{Z}_{\geq 0}$ if $n \geq 4$

$$\ell_{ij} = \# \text{ of contractions between } (\mathcal{O}_i, \mathcal{O}_j)$$

Solutions

1. Extend the operator, $\hat{\mathcal{O}}_{\hat{\alpha}_i} \equiv \mathcal{O}_{\alpha_i} \times \mathrm{tr}\,(1)^{L-L_i} \quad (\hat{\alpha}_i \in S_L)$

$$L = \frac{1}{2} \sum_i L_i = \text{Total } \# \text{ of contractions}$$

2. Include the sum over $\{\ell_{ij}\}$ in the sum over permutations

Example (BPS 3pt)

Neglecting the trivial spacetime dependence, 3pt=OPE

$$C_{\circ\circ\circ} = \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \left\langle \mathrm{tr}_{L_1}\left(\alpha_1 Z^{\otimes L_1}\right) \mathrm{tr}_{L_2}\left(\alpha_2 \tilde{Z}^{\otimes L_2}\right) \mathrm{tr}_{L_3}\left(\alpha_3 \overline{Z}^{\otimes L_3}\right) \right\rangle$$

$$\contraction{}{Z}{{}^b_a\,}{\overline{Z}} Z^b_a\, \overline{Z}^d_c = \contraction{}{Z}{{}^b_a\,}{\tilde{Z}} Z^b_a\, \tilde{Z}^d_c = \contraction{}{\tilde{Z}}{{}^b_a\,}{\overline{Z}} \tilde{Z}^b_a\, \overline{Z}^d_c = \delta^d_a\, \delta^b_c\,, \quad \tilde{Z} = (Z + \overline{Z} + Y - \overline{Y})$$

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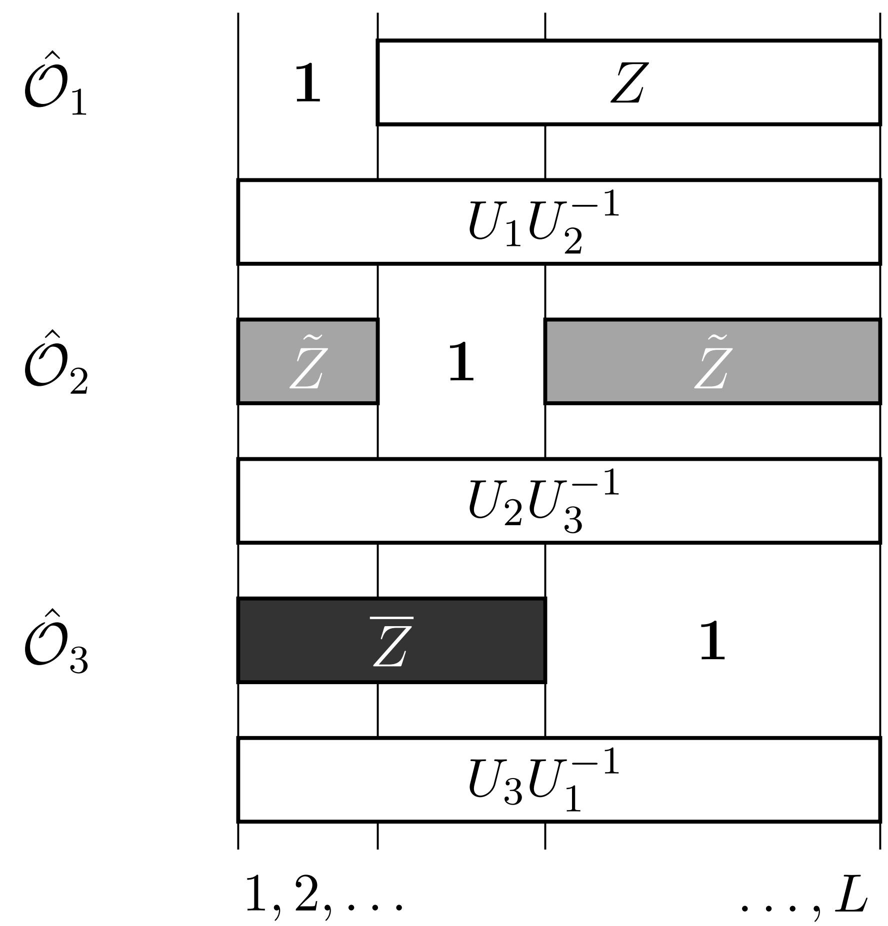
$$C_{\circ\circ\circ} = \sum_{(U_1, U_2, U_3) \in S_L}$$



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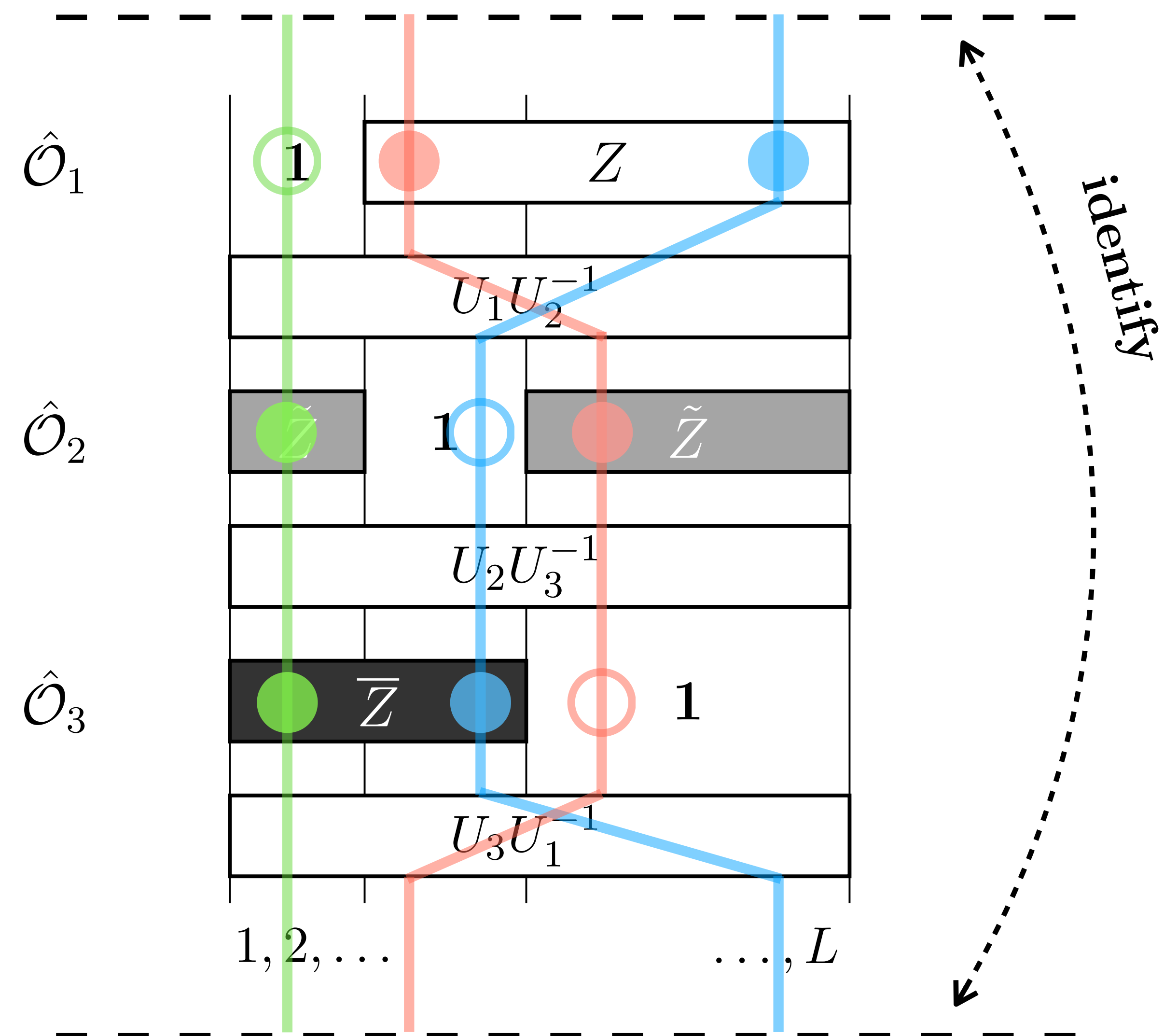

- Each line represent Wick contractions
- There are L lines in total
- Wick contraction hits two Z's and one 1
- Line positions are permuted by U's

Free n-pt in permutation basis

1) Define the extended operator by adding identity fields

$$\hat{\mathcal{O}}_i \equiv \mathcal{O}_{\alpha_i} \times \mathrm{tr}\,(1)^{\overline{L}_i} \equiv \prod_{p=1}^{L} (\Phi^{\hat{A}_p^{(i)}})^{a_p}_{a_{\hat{\alpha}_i(p)}}$$

$$\hat{\alpha}_i = \alpha_i \circ 1_{\overline{L}_i} \in S_L \,, \quad \overline{L}_i = L - L_i \,, \quad \Phi^{\hat{A}_p^{(i)}} \in (X, \overline{X}, Y, \overline{Y}, Z, \overline{Z}, 1)$$

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2) Introduce n-tuple Wick contractions

$$\overbrace{(\Phi^{\hat{A}_1})^{b_1}_{a_1} (\Phi^{\hat{A}_2})^{b_2}_{a_2} (\Phi^{\hat{A}_3})^{b_3}_{a_3} \ \ldots \ (\Phi^{\hat{A}_n})^{b_n}_{a_n}} = \underline{h^{\hat{A}_1 \hat{A}_2 \ldots \hat{A}_n}} \ \delta^{b_2}_{a_1} \delta^{b_3}_{a_2} \ \ldots \ \delta^{b_n}_{a_1}$$

equal to $g^{A_i A_j}$ if $\hat{A}_k = 1 \ (\forall k \neq i, j)$, otherwise vanish

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3) Take all n-tuple Wick contractions, specified by

$$(U_1\,, U_2\,, \ldots\,, U_n) \in S_L^{\otimes n}$$

Free n-pt in permutation basis

Formula

$$\left\langle \prod_{i=1}^{n} \mathcal{O}_i(x_i) \right\rangle = \frac{1}{L!\prod_{i=1}^{n}(L-L_i)!} \times$$

$$\sum_{\{U_i\}\in S_L^{\otimes n}} \left(\prod_{p=1}^{L} h^{\check{A}_p^{(1)}\,\check{A}_p^{(2)}\ldots\check{A}_p^{(n)}} \right) N_c^{C(\check{\alpha}_1\,\check{\alpha}_2\,\ldots\,\check{\alpha}_n)}$$

Notation: $\check{\alpha}_k = U_k^{-1}\,\hat{\alpha}_k\,U_k\,,\quad \check{A}_p^{(k)} = \hat{A}_{U_k(p)}^{(k)}$

Redundancy: $U_k \;\rightarrow\; U_k V \quad (\forall k,\; V \in S_L)$

[RS (2018)]

Free n-pt in permutation basis

Formula

$$\left\langle \prod_{i=1}^{n} \mathcal{O}_i(x_i) \right\rangle = \frac{1}{\underbrace{L!}_{\text{Redundancy of } V} \underbrace{\prod_{i=1}^{n}(L-L_i)!}_{\text{Redundancy of permuting identity fields}}} \times$$

Redundancy of V — Redundancy of permuting identity fields

$$\underbrace{\sum_{\{U_i\}\in S_L^{\otimes n}}}_{\text{All possible Wick}} \underbrace{\left(\prod_{p=1}^{L} h^{\check{A}_p^{(1)} \check{A}_p^{(2)} \ldots \check{A}_p^{(n)}} \right)}_{\text{Flavor factor to remove unwanted Wick}} N_c^{C(\check{\alpha}_1 \check{\alpha}_2 \ldots \check{\alpha}_n)}$$

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4. Free 3pt function at finite N_c

- 置換群から表現へ
- 多重トレース演算子の **Restricted Schur** 基底
- 簡計算法
- ３点関数の **Fourier** 変換
- **BPS** ３点関数の例
- 背景独立性

From Permutation to Representation

- Needs representation basis to impose finite Nc
- Fourier transform on finite groups:

  Permutations (algebra) ↔ Representations (matrix)

$$\alpha \in S_L \;\leftrightarrow\; D^R_{ij}(\alpha),\; R \vdash L,\; (i, j = 1, 2, \ldots, \underline{d_R})$$

dimension of S_L irrep R

- Various useful identities
  - Grand orthogonality
  - Character expansion of class functions
  - Schur/Weyl duality

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$$\sum_{\sigma \in S_L} D^R_{ij}(\sigma) D^S_{kl}(\sigma^{-1}) = \frac{L!}{d_R} \delta_{il} \delta_{jk}$$

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dimension of S_L irrep R

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$$\delta(\beta) = \frac{1}{L!} \sum_{R \vdash L} d_R \, \chi^R(\beta), \quad \chi^R(\beta) = \sum_{I=1}^{d_R} D^R_{II}(\beta)$$

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dimension of S_L irrep R

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$$N^{C(\sigma)} = \sum_{R \vdash L} \underline{\mathrm{Dim}_N(R)}\, \chi^R(\sigma)$$

dimension of SU(Nc) irrep R

Restricted Schur Basis

Define the standard basis (Young-Yamanouchi patterns)

$$\left| \begin{matrix} R \\ I \end{matrix} \right\rangle, \quad (I = 1, 2, \ldots, d_R), \quad \text{e.g.} \left\{ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array} \right\}$$

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Irreducible decomposition of R by the restriction to subgroup

$$S_L \downarrow (S_l \otimes S_m \otimes S_n), \quad R = \bigoplus_{\substack{r_1 \vdash l \\ r_2 \vdash m \\ r_3 \vdash n}} \bigoplus_{\mu=1}^{g(r_1, r_2, r_3; R)} (r_1 \otimes r_2 \otimes r_3)_\mu$$

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Multiplicity label

Littlewood-Richardson (LR) coeff.

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Define branching coefficients as the split-standard transform

$$B_{I \to (i_1, i_2, i_3)}^{R \to (r_1, r_2, r_3), \mu} = \left\langle \begin{matrix} R \\ I \end{matrix} \middle| \begin{matrix} r_1 & r_2 & r_3 \\ i_1 & i_2 & i_3 \end{matrix} \; \mu \right\rangle, \quad (B^T)_{I \to (i_1, i_2, i_3)}^{R \to (r_1, r_2, r_3), \mu} = \left\langle \begin{matrix} r_1 & r_2 & r_3 \\ i_1 & i_2 & i_3 \end{matrix} \; \mu \middle| \begin{matrix} R \\ I \end{matrix} \right\rangle$$

Restricted Schur Basis

Define the restricted Schur character by

$$\chi^{R,(r_1,r_2,r_3),\nu_+,\nu_-}(\sigma) \equiv \sum_{I,J} \sum_{i,j,k} B^{R\to(r_1,r_2,r_3)\nu_+}_{I\to(i,j,k)} \, (B^T)^{R\to(r_1,r_2,r_3),\nu_-}_{J\to(i,j,k)} \, D^R_{IJ}(\sigma)$$

Define operators in the restricted Schur basis by

$$\mathcal{O}^{R,(r_1,r_2,r_3),\nu_+,\nu_-}[X,Y,Z]$$
$$= \frac{1}{m_1!\, m_2!\, m_3!} \sum_{\alpha\in S_M} \chi^{R,(r_1,r_2,r_3),\nu_+,\nu_-}(\alpha)\, \mathrm{tr}_M \left(\alpha\, X^{\otimes m_1} Y^{\otimes m_2} Z^{\otimes m_3}\right)$$

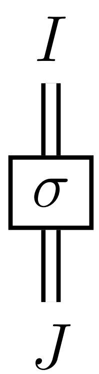
The res. Schur respects the symmetry (or redundancy)

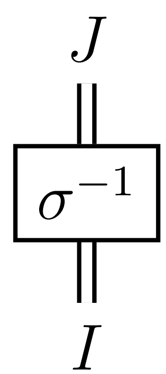
$$\mathcal{O}^{(l,m,n)}_{\alpha} = \mathrm{tr}_L \left(\alpha X^{\otimes l} Y^{\otimes m} Z^{\otimes n}\right) = \mathcal{O}^{(l,m,n)}_{\gamma^{-1}\alpha\gamma} \quad (\forall \gamma \in S_l \otimes S_m \otimes S_n)$$

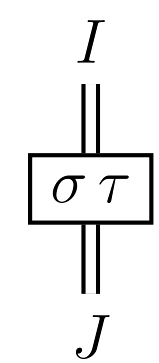
Quiver calculus

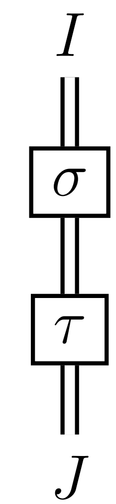
Diagrams for simplifying representation-theoretical quantities

- **Multiplication of representation matrices**

$D^R_{IJ}(\sigma) =$ 

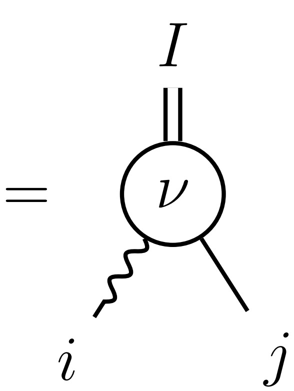
$=$ 

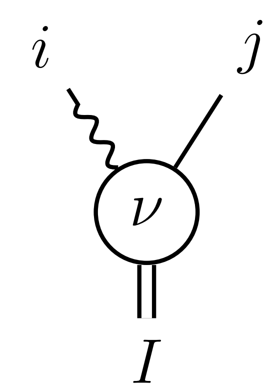
$D^R_{IJ}(\sigma\tau) = \sum_{K=1}^{d_R} D^R_{IK}(\sigma) D^R_{KJ}(\tau) =$ 

$=$ 

- **Branching coefficients with a multiplicity label**

$\boldsymbol{R = \bigoplus_{\nu} (r_1 \otimes r_2)_{\nu}}$

$B^{R\to(r_1,r_2)\nu}_{I\to(i,j)} =$ 

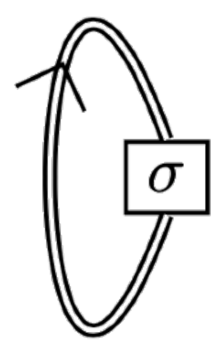
$=$ 

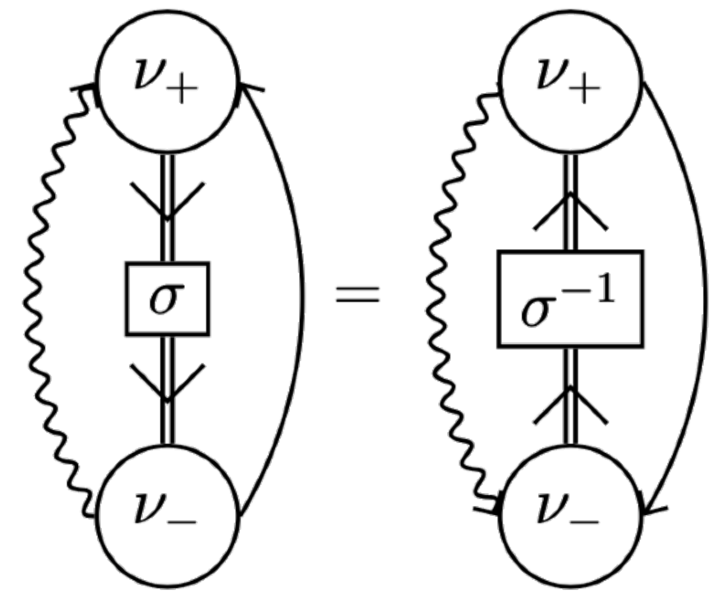
[de Mello Koch, Smolic, Smolic (2007)] [Paskounis, Ramgoolam (2013)] [Mattioli, Ramgoolam (2016)]

Quiver calculus

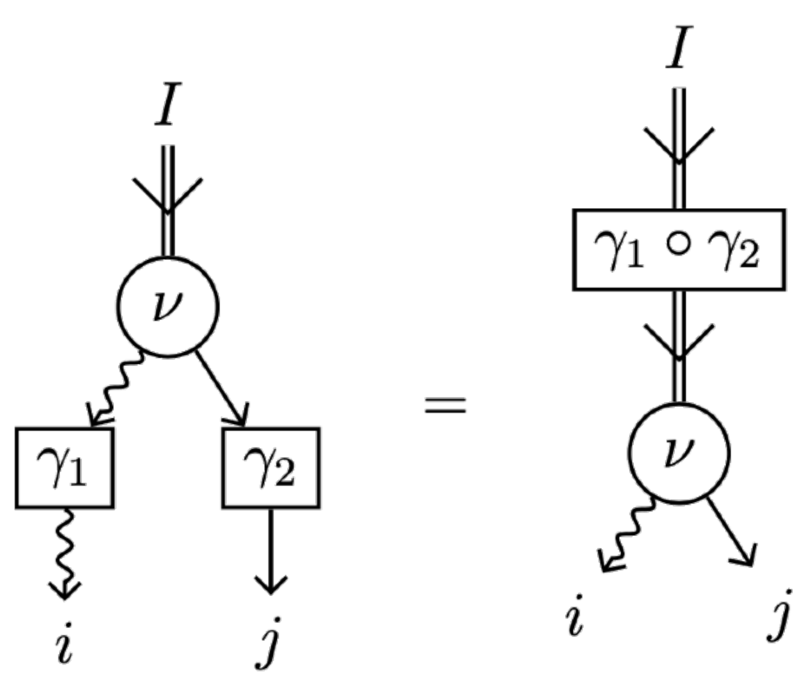
Diagrams for simplifying representation-theoretical quantities

- **Characters and restricted characters**

$\chi^R(\sigma) = \chi^R(\sigma^{-1}) =$ 

$\chi^{R(r_1,r_2)(\nu_+,\nu_-)}(\sigma) =$ 

- **Identity with branching coefficients**



If $\gamma \in S_L$ is written as

$\gamma = \gamma_1 \circ \gamma_2 \in S_l \times S_m$,

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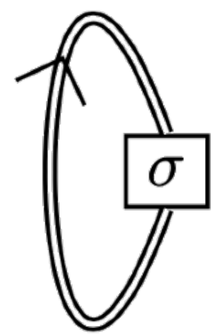
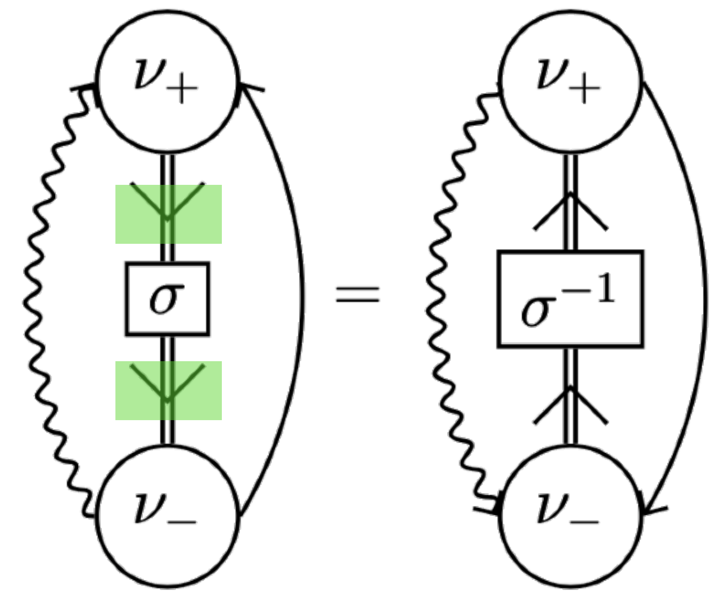
$\chi^R(\sigma) = \chi^R(\sigma^{-1}) =$ 

$\chi^{R(r_1,r_2)(\nu_+,\nu_-)}(\sigma) =$ 

$\boldsymbol{\gamma\sigma\gamma^{-1}}$

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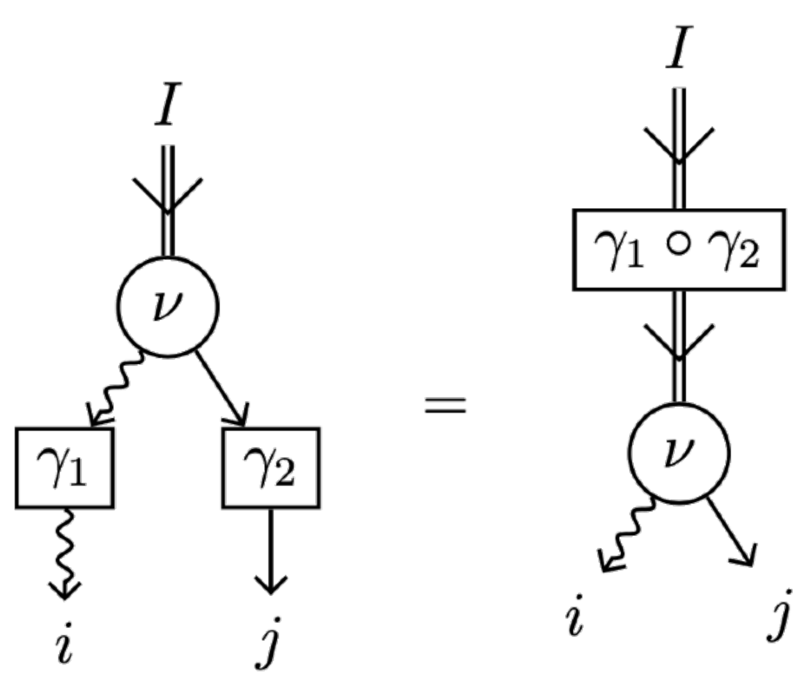

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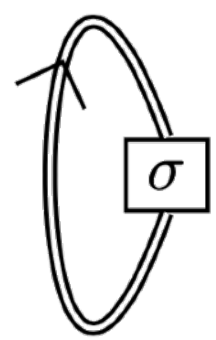
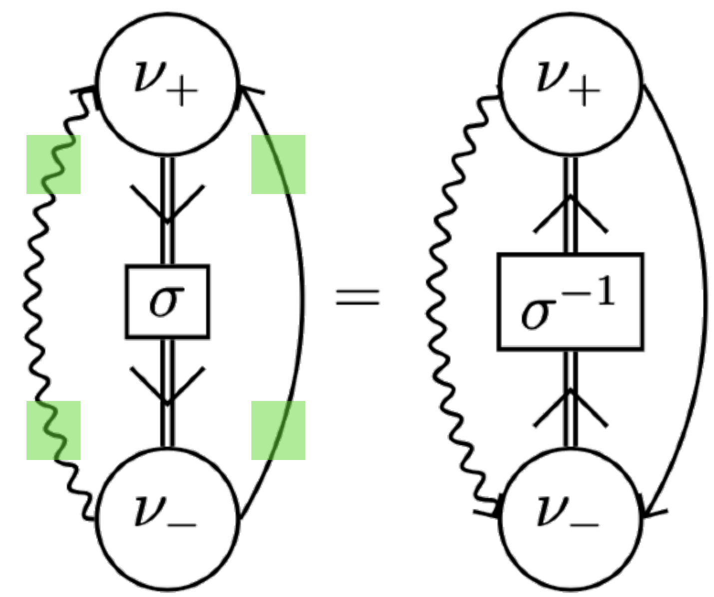
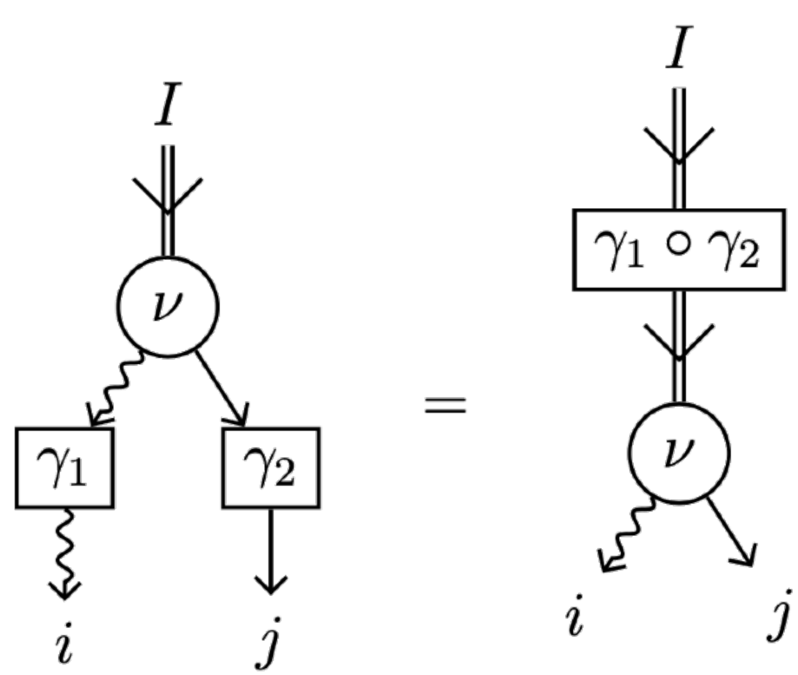
$\chi^R(\sigma) = \chi^R(\sigma^{-1}) =$ 

$\chi^{R(r_1,r_2)(\nu_+,\nu_-)}(\sigma) =$ 

- **Identity with branching coefficients**

$(\gamma_1 \circ \gamma_2)\sigma(\gamma_1 \circ \gamma_2)^{-1}$



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3pt function in the res. Schur basis

BPS 3pt in permutation basis

$$C_{\circ\circ\circ} = \left\langle \mathrm{tr}_{L_1}(\alpha_1 Z^{\otimes L_1})\ \mathrm{tr}_{L_2}(\alpha_2 \tilde{Z}^{\otimes L_2})\ \mathrm{tr}_{L_3}(\alpha_3 \overline{Z}^{\otimes L_3}) \right\rangle$$

We derived 3pt formula by using extended operators

$$\mathcal{O}_{\hat{\alpha}_i}^{(L_i,\overline{L}_i)}[Z,1] = \mathrm{tr}_{L_i}\left(\alpha_i Z^{\otimes L_i}\right) \times \mathrm{tr}\,(1)^{\overline{L}_i}, \quad \overline{L}_i + L_i = L$$

Partial Fourier transform (only for non-identity fields)

$$\hat{\mathcal{O}}^{R_i\,(\overline{L}_i)}[Z,1] = \frac{1}{L_i!} \sum_{\alpha_i \in S_{L_i}} \chi^{R_i}(\alpha_i)\, \mathcal{O}_{\hat{\alpha}_i}^{(L_i,\overline{L}_i)}[Z,1]$$

3pt function in the res. Schur basis

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$$C_{\circ\circ\circ} = \left\langle \mathrm{tr}_{L_1}\big(\alpha_1 Z^{\otimes L_1}\big)\ \mathrm{tr}_{L_2}\big(\alpha_2 \tilde{Z}^{\otimes L_2}\big)\ \mathrm{tr}_{L_3}\big(\alpha_3 \overline{Z}^{\otimes L_3}\big) \right\rangle$$

We derived 3pt formula by using extended operators

$$\mathcal{O}_{\hat{\alpha}_i}^{(L_i,\overline{L}_i)}[Z,1] = \mathrm{tr}_{L_i}\Big(\alpha_i Z^{\otimes L_i}\Big) \times \mathrm{tr}\,(1)^{\overline{L}_i}, \quad \overline{L}_i + L_i = L$$

Complete Fourier transform

$$\hat{\mathcal{O}}^{R_i\,(\overline{L}_i)}[Z,1] = \frac{1}{L_i!\,\overline{L}_i!} \sum_{t_i\,\vdash \overline{L}_i} \ \sum_{\hat{\alpha}_i \in S_{L_i} \times 1_{\overline{L}_i}} d_{t_i}\,\chi^{R_i \otimes t_i}(\hat{\alpha}_i)\,\mathcal{O}_{\hat{\alpha}_i}^{(L_i,\overline{L}_i)}[Z,1]$$

- **Fourier Transform of the δ-function = uniform distribution**
- **FT of identity fields = sum over all representations**

3pt function in the res. Schur basis

BPS 3pt in permutation basis

$$C_{\circ\circ\circ} = \left\langle \mathrm{tr}_{L_1}\!\left(\alpha_1 Z^{\otimes L_1}\right)\, \mathrm{tr}_{L_2}\!\left(\alpha_2 \tilde{Z}^{\otimes L_2}\right)\, \mathrm{tr}_{L_3}\!\left(\alpha_3 \overline{Z}^{\otimes L_3}\right) \right\rangle$$

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Complete Fourier transform

$$\hat{\mathcal{O}}^{R_i\,(\overline{L}_i)}[Z, 1] = \frac{1}{L_i!\,\overline{L}_i!} \sum_{t_i \vdash \overline{L}_i} \sum_{\hat{\alpha}_i \in S_{L_i} \times 1_{\overline{L}_i}} d_{t_i}\, \chi^{R_i \otimes t_i}(\hat{\alpha}_i)\, \mathcal{O}^{(L_i, \overline{L}_i)}_{\hat{\alpha}_i}[Z, 1]$$

We compute BPS 3pt in the res. Schur basis, defined by

$$\tilde{C}_{\circ\circ\circ} = \left\langle \hat{\mathcal{O}}^{R_1 (\overline{L}_1)}_1[Z, 1]\, \hat{\mathcal{O}}^{R_2 (\overline{L}_2)}_2[\tilde{Z}, 1]\, \hat{\mathcal{O}}^{R_3 (\overline{L}_3)}_3[\overline{Z}, 1] \right\rangle$$

3pt function in the res. Schur basis

Substitute everything

$$\tilde{C}_{123} = \frac{1}{\prod_{i=1}^{3} L_i!\,(\overline{L}_i!)^2}\,\frac{1}{L!} \sum_{\{U_i\}\in S_L^{\otimes 3}} \left(\prod_{p=1}^{L} h^{\hat{A}^{(1)}_{U_1(p)}\hat{A}^{(2)}_{U_2(p)}\hat{A}^{(3)}_{U_3(p)}} \right) \times$$

$$\sum_{\{t_i \vdash \overline{L}_i\}} \left(\prod_{i=1}^{3} d_{t_i} \right) \sum_{\{\hat{\alpha}_i \in S_{L_i} \times 1_{\overline{L}_i}\}} \left(\prod_{i=1}^{3} \chi^{R_i \otimes t_i}(\hat{\alpha}_i) \right) N_c^{C(U_1^{-1}\,\hat{\alpha}_1\,U_1 U_2^{-1}\,\hat{\alpha}_2\,U_2 U_3^{-1}\,\hat{\alpha}_3\,U_3)}$$

This equation looks awfully complicated, but can be simplified by using the representation theory of finite groups and other methods. Let me briefly explain how.

3pt function in the res. Schur basis

Substitute everything

$$\tilde{C}_{123} = \frac{1}{\prod_{i=1}^{3} L_i!\,(\overline{L}_i!)^2}\,\frac{1}{L!} \sum_{\{U_i\}\in S_L^{\otimes 3}} \left(\prod_{p=1}^{L} h^{\hat{A}^{(1)}_{U_1(p)}\hat{A}^{(2)}_{U_2(p)}\hat{A}^{(3)}_{U_3(p)}} \right) \times$$

$$\sum_{\{t_i \vdash \overline{L}_i\}} \left(\prod_{i=1}^{3} d_{t_i} \right) \sum_{\{\hat{\alpha}_i \in S_{L_i}\times 1_{\overline{L}_i}\}} \left(\prod_{i=1}^{3} \chi^{R_i \otimes t_i}(\hat{\alpha}_i) \right) N_c^{C(U_1^{-1}\,\hat{\alpha}_1\,U_1 U_2^{-1}\,\hat{\alpha}_2\,U_2 U_3^{-1}\,\hat{\alpha}_3\,U_3)}$$

From Schur-Weyl duality

$$N^{C(\sigma)} = \sum_{\hat{R}\vdash L} \mathrm{Dim}_N(\hat{R})\,\chi^{\hat{R}}(\sigma)$$

3pt function in the res. Schur basis

Substitute everything (v2)

$$\tilde{C}_{123} = \frac{1}{\prod_{i=1}^{3} L_i!\,(\overline{L}_i!)^2}\,\frac{1}{L!}\sum_{\{U_i\}\in S_L^{\otimes 3}}\left(\prod_{p=1}^{L} h^{\hat{A}^{(1)}_{U_1(p)}\hat{A}^{(2)}_{U_2(p)}\hat{A}^{(3)}_{U_3(p)}}\right)\times$$

$$\sum_{\{t_i\vdash\overline{L}_i\}}\left(\prod_{i=1}^{3} d_{t_i}\right)\sum_{\{\hat{\alpha}_i\in S_{L_i}\times 1_{\overline{L}_i}\}}\sum_{\hat{R}\vdash L}\mathrm{Dim}_{N_c}(\hat{R})\times$$

$$\left(\prod_{i=1}^{3}\chi^{R_i\otimes t_i}(\hat{\alpha}_i)D^{\hat{R}}_{\hat{I}_i\hat{J}_i}(\hat{\alpha}_i)\right)D^{\hat{R}}_{\hat{J}_1\hat{I}_2}(U_1U_2^{-1})D^{\hat{R}}_{\hat{J}_2\hat{I}_3}(U_2U_3^{-1})D^{\hat{R}}_{\hat{J}_3\hat{I}_1}(U_3U_1^{-1})$$

3pt function in the res. Schur basis

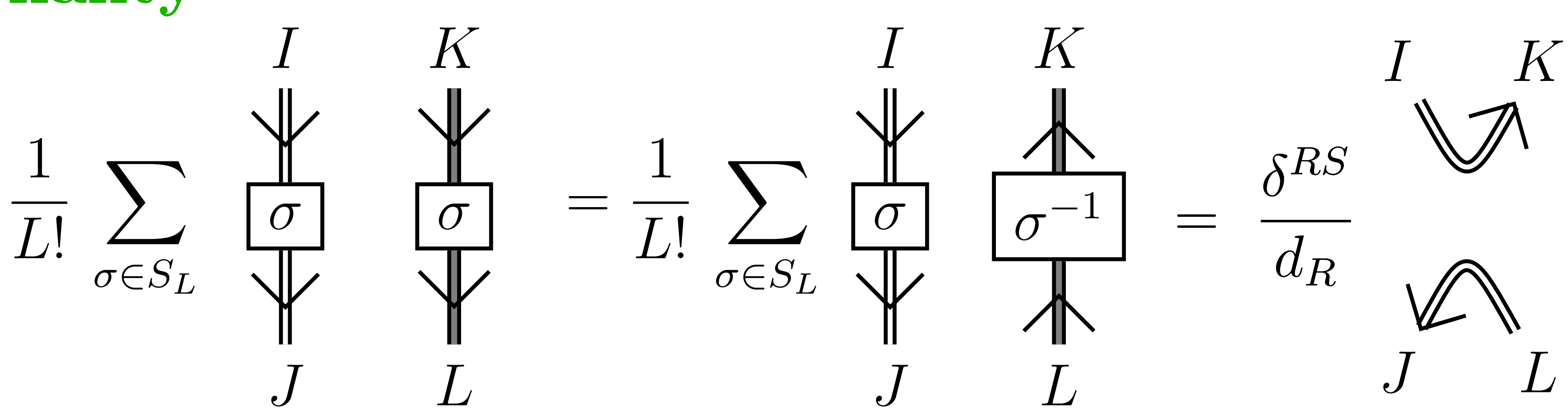
Substitute everything (v2)

$$\tilde{C}_{123} = \frac{1}{\prod_{i=1}^{3} L_i!\,(\overline{L}_i!)^2}\,\frac{1}{L!}\sum_{\{U_i\}\in S_L^{\otimes 3}}\left(\prod_{p=1}^{L} h^{\hat{A}^{(1)}_{U_1(p)}\hat{A}^{(2)}_{U_2(p)}\hat{A}^{(3)}_{U_3(p)}}\right)\times$$

$$\sum_{\{t_i\vdash\overline{L}_i\}}\left(\prod_{i=1}^{3} d_{t_i}\right)\boxed{\sum_{\{\hat{\alpha}_i\in S_{L_i}\times 1_{\overline{L}_i}\}}}\sum_{\hat{R}\vdash L}\mathrm{Dim}_{N_c}(\hat{R})\times$$

$$\left(\prod_{i=1}^{3}\chi^{R_i\otimes t_i}\boxed{(\hat{\alpha}_i)}D^{\hat{R}}_{\hat{I}_i\hat{J}_i}\boxed{(\hat{\alpha}_i)}\right)D^{\hat{R}}_{\hat{J}_1\hat{I}_2}(U_1U_2^{-1})\,D^{\hat{R}}_{\hat{J}_2\hat{I}_3}(U_2U_3^{-1})\,D^{\hat{R}}_{\hat{J}_3\hat{I}_1}(U_3U_1^{-1})$$

Grand orthogonality



3pt function in the res. Schur basis

Substitute everything (v2)

$$\tilde{C}_{123} = \frac{1}{\prod_{i=1}^{3} L_i!\,(\overline{L}_i!)^2}\frac{1}{L!}\sum_{\{U_i\}\in S_L^{\otimes 3}}\left(\prod_{p=1}^{L} h^{\hat{A}^{(1)}_{U_1(p)}\hat{A}^{(2)}_{U_2(p)}\hat{A}^{(3)}_{U_3(p)}}\right)\times$$

$$\sum_{\{t_i\vdash\overline{L}_i\}}\left(\prod_{i=1}^{3} d_{t_i}\right)\sum_{\{\hat{\alpha}_i\in S_{L_i}\times 1_{\overline{L}_i}\}}\sum_{\hat{R}\vdash L}\mathrm{Dim}_{N_c}(\hat{R})\times$$

$$\left(\prod_{i=1}^{3}\chi^{R_i\otimes t_i}(\hat{\alpha}_i)D^{\hat{R}}_{\hat{I}_i\hat{J}_i}(\hat{\alpha}_i)\right)D^{\hat{R}}_{\hat{J}_1\hat{I}_2}(U_1U_2^{-1})D^{\hat{R}}_{\hat{J}_2\hat{I}_3}(U_2U_3^{-1})D^{\hat{R}}_{\hat{J}_3\hat{I}_1}(U_3U_1^{-1})$$

Projector $\mathscr{P}^{\hat{R}\to\,\mathrm{sub}}_{\hat{I}_i\hat{J}_i}$ from irrep $\hat{R}$ to subrepresentation $R_i\otimes t_i$

$$S_L\downarrow(S_{L_i}\otimes S_{\overline{L}_i}),\qquad \hat{R}=\bigoplus_{R_i'\vdash L_i}\bigoplus_{T_i\vdash\overline{L}_i}\bigoplus_{\mu_i=1}^{g(R_i',t_i';\hat{R})}(R_i'\otimes T_i)_{\mu_i}$$

3pt function in the res. Schur basis

Substitute everything (v3)

$$\tilde{C}_{123} = \left(\prod_{i=1}^{3} \frac{1}{\overline{\overline{L}}_i!}\right) \frac{1}{L!} \sum_{\{U_i\} \in S_L^{\otimes 3}} \left(\prod_{p=1}^{L} h^{\hat{A}^{(1)}_{U_1(p)} \hat{A}^{(2)}_{U_2(p)} \hat{A}^{(3)}_{U_3(p)}}\right) \times$$

$$\sum_{\hat{R} \vdash L} \frac{\mathrm{Dim}_{N_c}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \times$$

$$\sum_{\{T_i, \mu_i\}} \left(\prod_{i=1}^{3} \mathscr{P}^{\hat{R} \to \mathrm{sub}}_{\hat{I}_i \hat{J}_i}\right) D^{\hat{R}}_{\hat{J}_1 \hat{I}_2}(U_1 U_2^{-1})\, D^{\hat{R}}_{\hat{J}_2 \hat{I}_3}(U_2 U_3^{-1})\, D^{\hat{R}}_{\hat{J}_3 \hat{I}_1}(U_3 U_1^{-1})$$

3pt function in the res. Schur basis

Substitute everything (v3)

$$\tilde{C}_{123} = \left(\prod_{i=1}^{3} \frac{1}{\overline{\overline{L}}_i!} \right) \frac{1}{L!} \sum_{\{U_i\} \in S_L^{\otimes 3}} \left(\prod_{p=1}^{L} h^{\hat{A}^{(1)}_{U_1(p)} \hat{A}^{(2)}_{U_2(p)} \hat{A}^{(3)}_{U_3(p)}} \right) \times$$

$$\sum_{\hat{R} \vdash L} \frac{\mathrm{Dim}_{N_c}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \times$$

Wick contraction factor reduces the sum range of U's

$$\sum_{\{T_i, \mu_i\}} \left(\prod_{i=1}^{3} \mathscr{P}^{\hat{R} \to \mathrm{sub}}_{\hat{I}_i \hat{J}_i} \right) D^{\hat{R}}_{\hat{J}_1 \hat{I}_2}(U_1 U_2^{-1}) \, D^{\hat{R}}_{\hat{J}_2 \hat{I}_3}(U_2 U_3^{-1}) \, D^{\hat{R}}_{\hat{J}_3 \hat{I}_1}(U_3 U_1^{-1})$$

3pt function in the res. Schur basis

Substitute everything (v3)

$$\tilde{C}_{123} = \left(\prod_{i=1}^{3} \frac{1}{\overline{L}_i!}\right) \frac{1}{L!} \boxed{\sum_{\{U_i\} \in S_L^{\otimes 3}}} \left(\prod_{p=1}^{L} \boxed{h^{\hat{A}^{(1)}_{U_1(p)} \hat{A}^{(2)}_{U_2(p)} \hat{A}^{(3)}_{U_3(p)}}}\right) \times$$

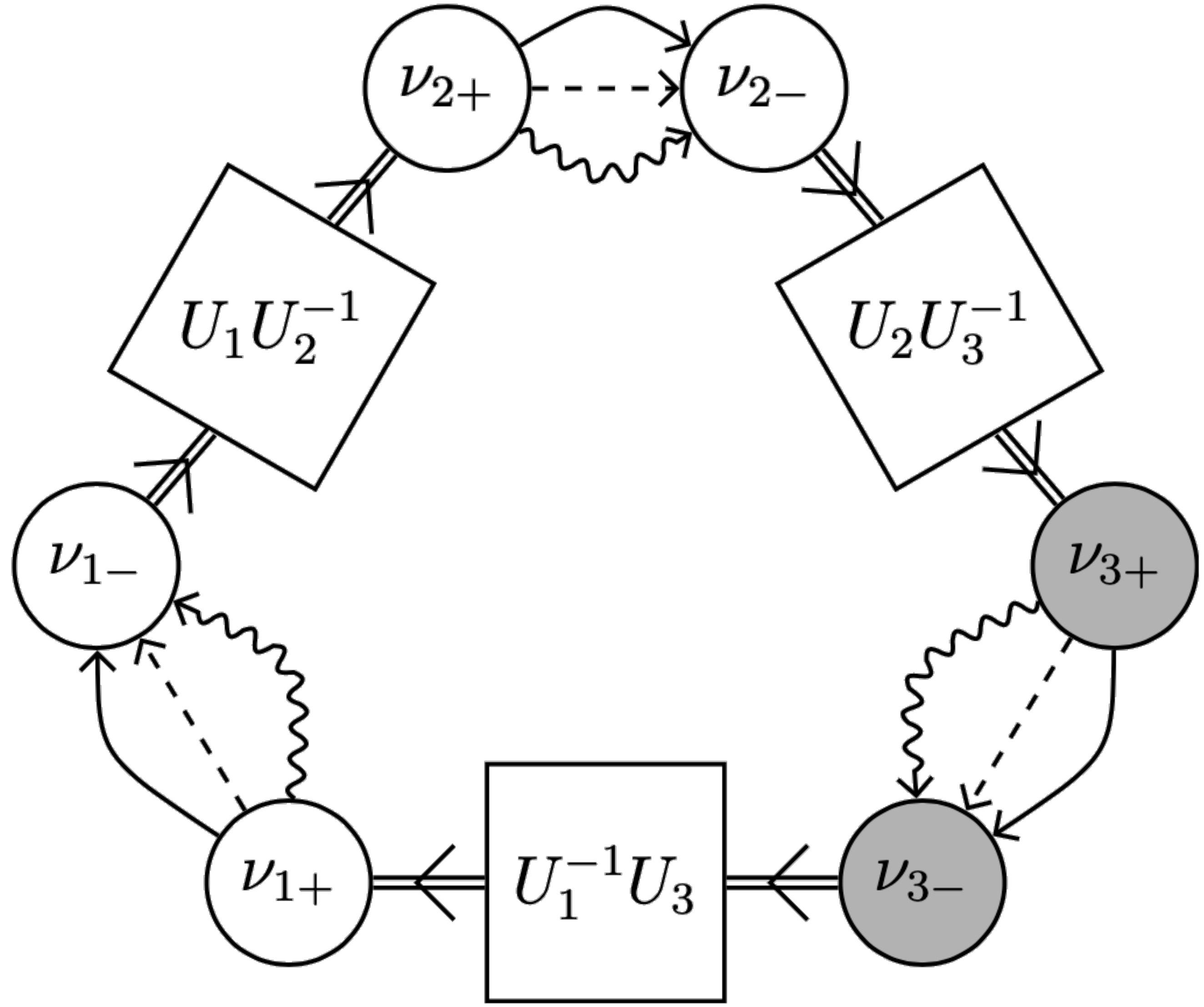
$$U_i \in S_{L_i} \otimes 1_{\overline{L}_i} \equiv \mathcal{S}_i$$

$$\sum_{\hat{R} \vdash L} \frac{\mathrm{Dim}_{N_c}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \times$$

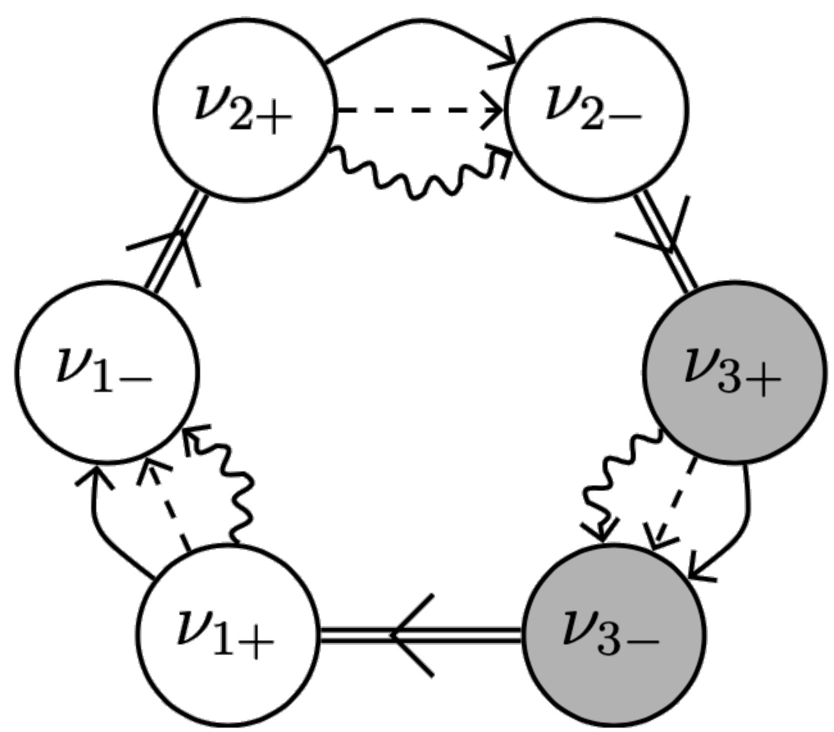
$$\sum_{\{T_i, \mu_i\}} \left(\prod_{i=1}^{3} \boxed{\mathscr{P}^{\hat{R} \to \mathrm{sub}}_{\hat{I}_i \hat{J}_i}}\right) \boxed{D^{\hat{R}}_{\hat{J}_1 \hat{I}_2}(U_1 U_2^{-1})\, D^{\hat{R}}_{\hat{J}_2 \hat{I}_3}(U_2 U_3^{-1})\, D^{\hat{R}}_{\hat{J}_3 \hat{I}_1}(U_3 U_1^{-1})}$$

The restricted U's commute with the projector, and annihilate with each other

$$\tilde{C}_{123} \sim \sum_{\hat{R} \vdash L} \frac{\mathrm{Dim}_{N_c}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \sum_{\{U_i \in \mathcal{S}_i\}}$$



$$\sim \sum_{\hat{R} \vdash L} \frac{\mathrm{Dim}_{N_c}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \; |\mathrm{Wick}|$$



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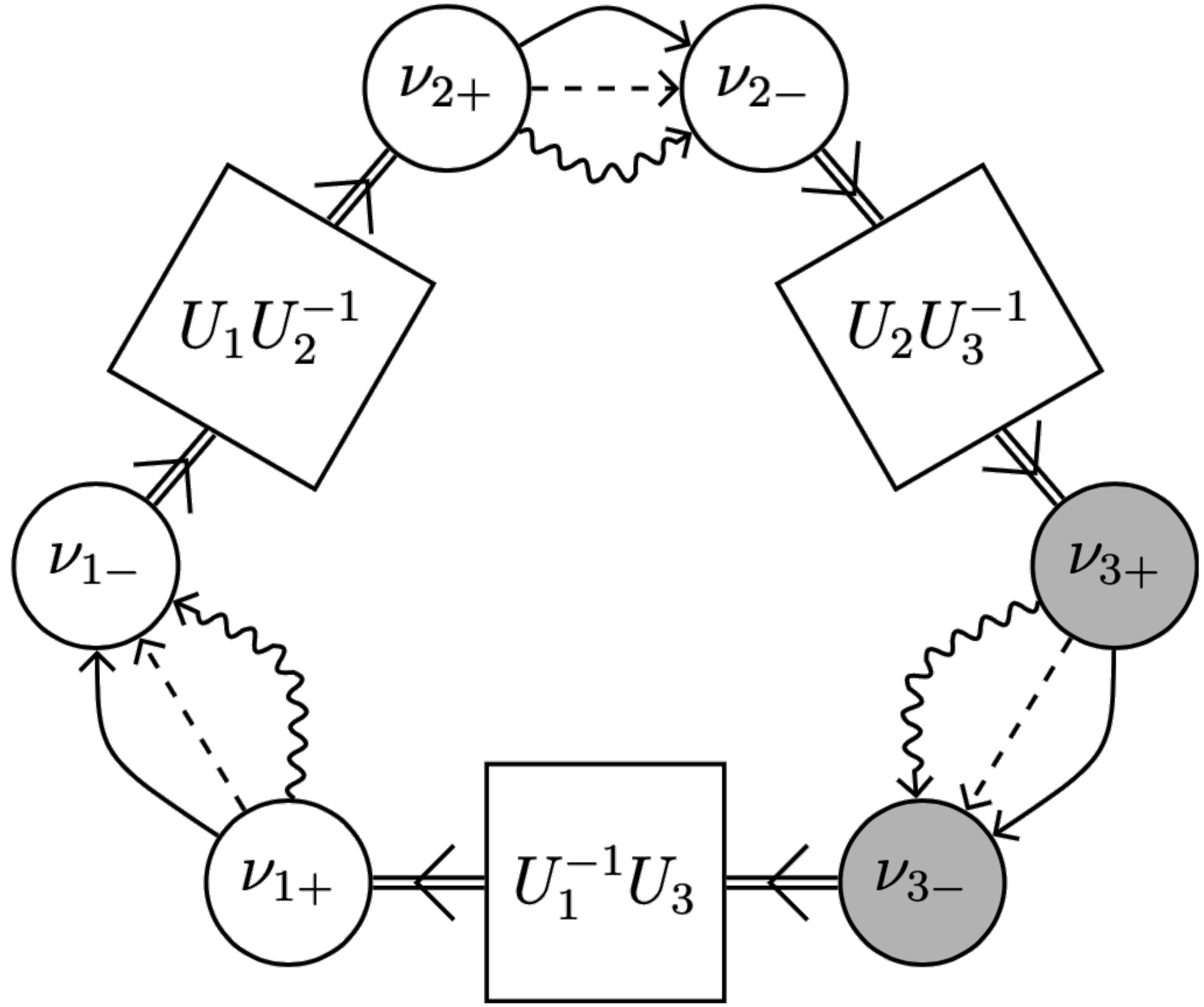
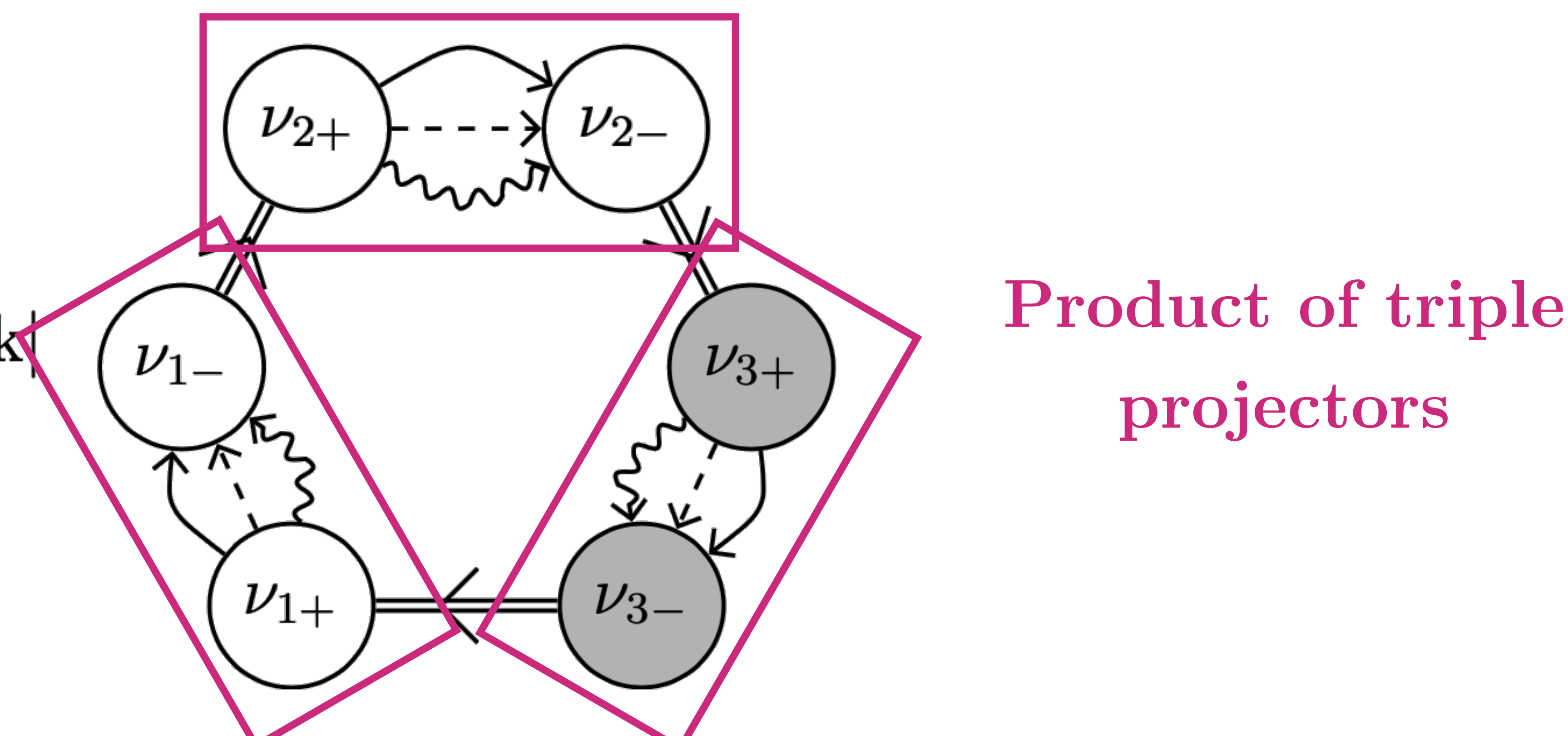


$$\sim \sum_{\hat{R}\vdash L} \frac{\mathrm{Dim}_{N_c}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \; |\mathrm{Wick}|$$



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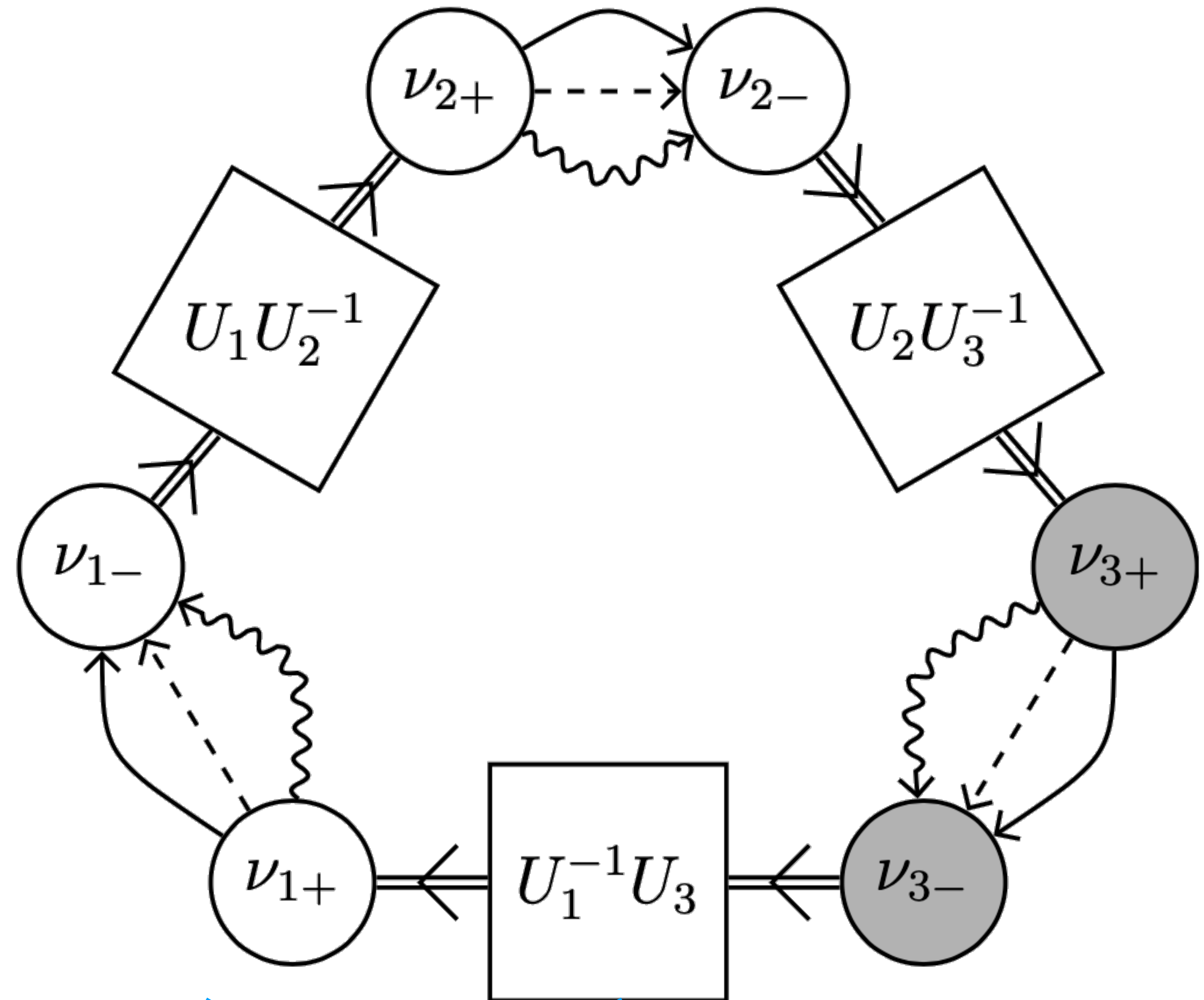
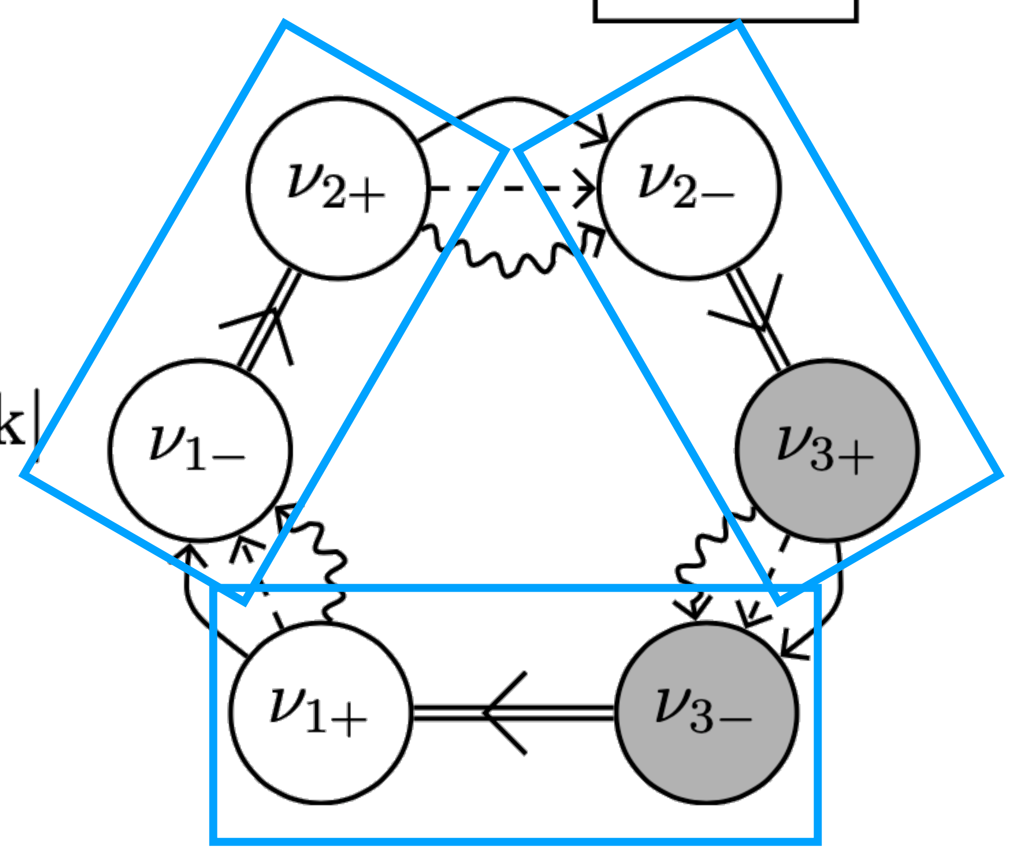


$$\sim \sum_{\hat{R}\vdash L} \frac{\mathrm{Dim}_{N_c}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \; |\mathrm{Wick}|$$



Product of (generalized) Wigner's $6j$ symbols

Generalized Racah-Wigner tensors

- **Wigner's 6j symbol ~ Associativity of triple tensor product**

$$\begin{Bmatrix} j_1 & j_2 & j_{1+2} \\ j_3 & J & j_{2+3} \end{Bmatrix} : \mathrm{Hom}\Big((j_1 \otimes j_2) \otimes j_3, J\Big) \to \mathrm{Hom}\Big(j_1 \otimes (j_2 \otimes j_3), J\Big)$$

Generalized Racah-Wigner tensors

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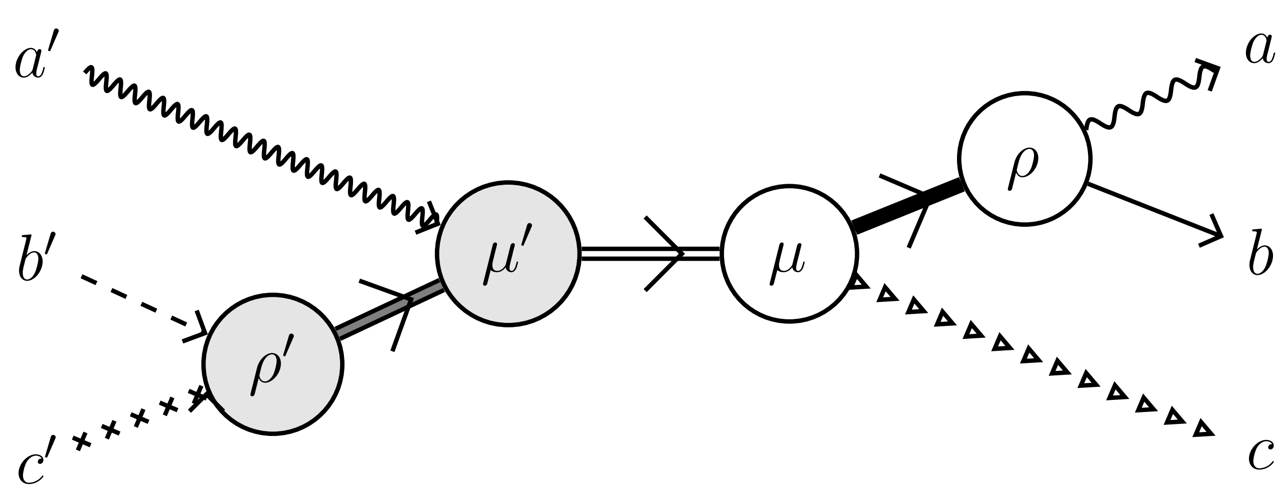
- **Similarly, consider two ways of double restriction**

$$S_L \downarrow (S_{L_1+L_2} \otimes S_{L_3}) \downarrow (S_{L_1} \otimes S_{L_2} \otimes S_{L_3})$$

$$S_L \downarrow (S_{L_1} \otimes S_{L_2+L_3}) \downarrow (S_{L_1} \otimes S_{L_2} \otimes S_{L_3})$$

- **Define generalized Racah-Wigner tensor by**



Generalized Racah-Wigner tensors

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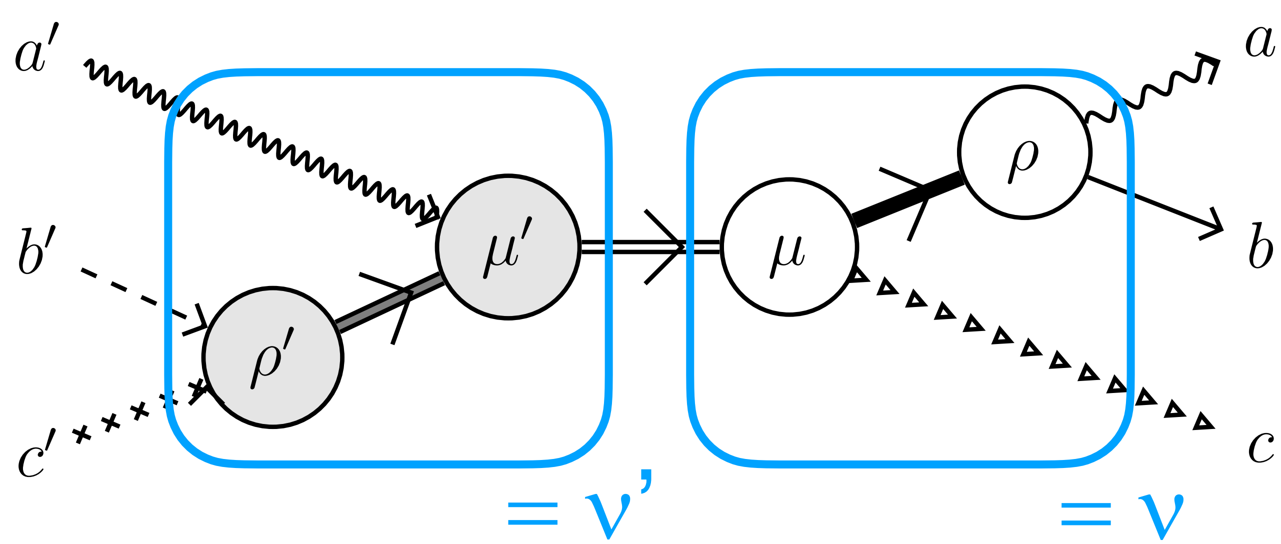


Main results

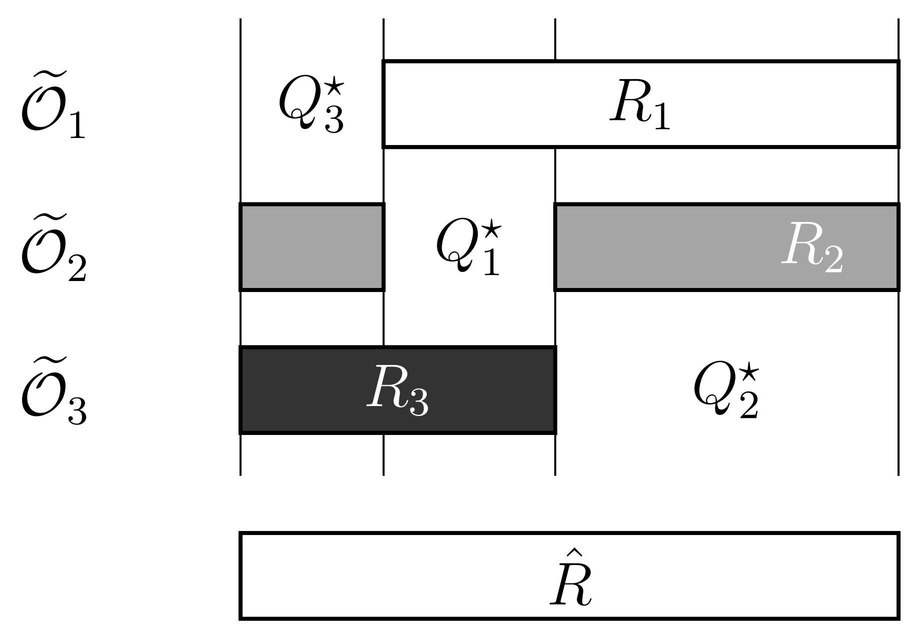
- 3pt function of half-BPS operators in the restricted Schur basis

$$\tilde{C}_{\circ\circ\circ} = \left(\prod_{i=1}^{3} \frac{L_i!}{\overline{\overline{L}}_i!} \right) \sum_{\hat{R} \vdash L} \frac{\mathrm{Dim}_{N_c}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \sum_{Q_1 \vdash \overline{L}_2} \sum_{Q_2 \vdash \overline{L}_3} \sum_{Q_3 \vdash \overline{L}_1} \left(\prod_{i=1}^{3} d_{Q_i} \right) \mathcal{G}_{123}$$

- Sum over all possible Q's satisfying

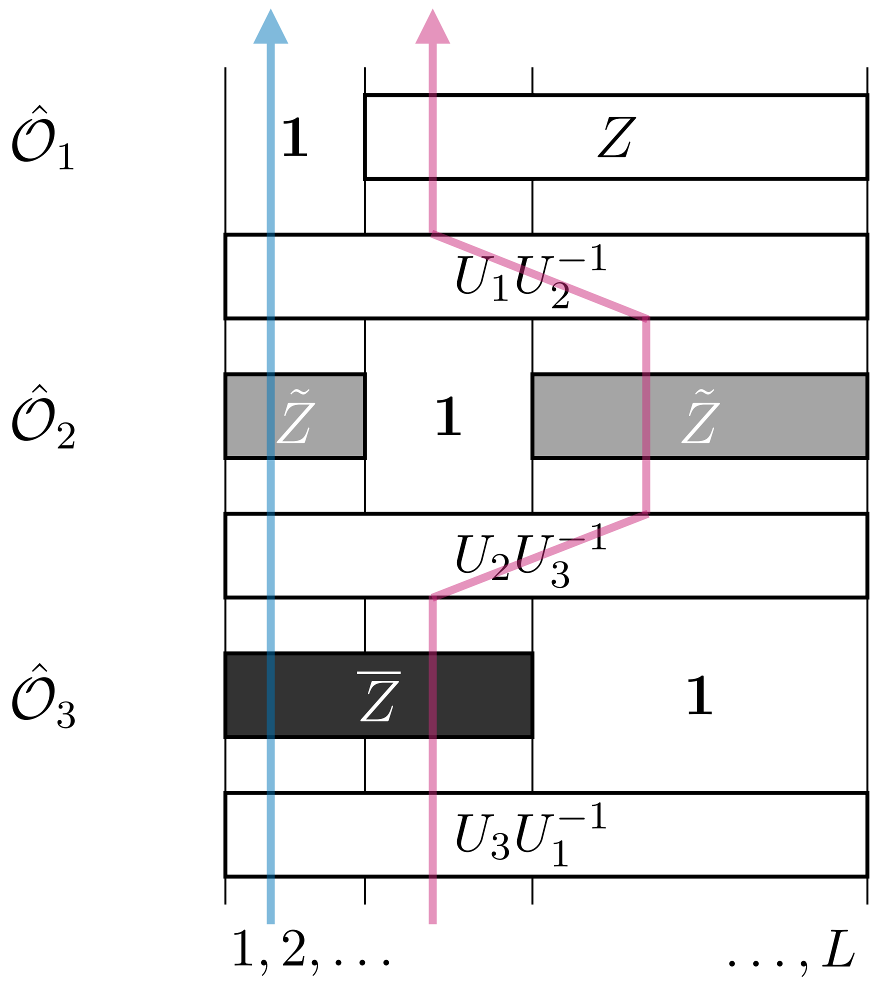
$$R_1 = Q_1^\star \otimes Q_2^\star \,, \; R_2 = Q_2^\star \otimes Q_3^\star \,, \; R_3 = Q_3^\star \otimes Q_1^\star \,, \; \hat{R} = Q_1^\star \otimes Q_2^\star \otimes Q_3^\star$$

- $\mathcal{G}_{123}$ is given by the overlap (triple product of RW tensors)

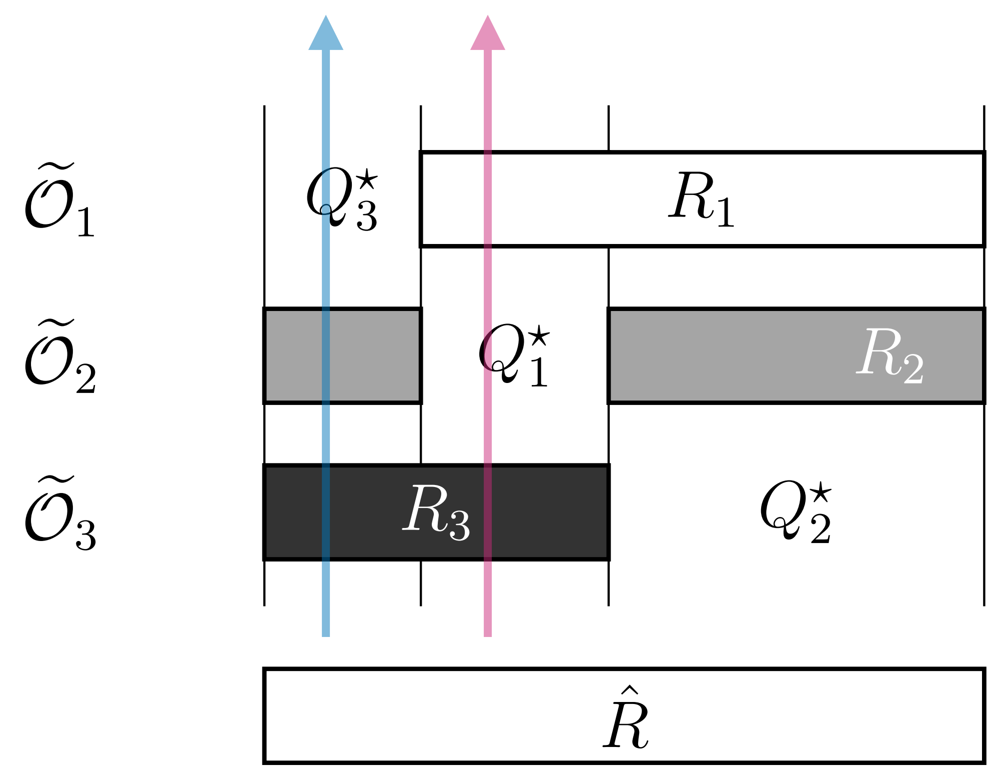


Main results

Permutation basis



Restricted Schur basis



We conjecture that $\mathcal{G}_{123}$ is written by LR coefficients

$$\mathcal{G}_{123} = \frac{g(Q_1, Q_2; R_1)\, g(R_1, Q_3; \hat{R})\, g(Q_2, Q_3; R_2)\, g(R_2, Q_1; \hat{R})\, g(Q_3, Q_1; R_3)\, g(R_3, Q_2; \hat{R})}{g(Q_1, Q_2, Q_3; \hat{R})^2}$$

Another example

Fourier transform of $\left\langle \mathrm{tr}_{L_1}\left(\alpha_1\,\overline{X}^{\otimes(\ell_{31}-h_2)}\,\overline{Y}^{\otimes h_3}\,Z^{\otimes(\ell_{12}-h_3+h_2)}\right)\times\right.$

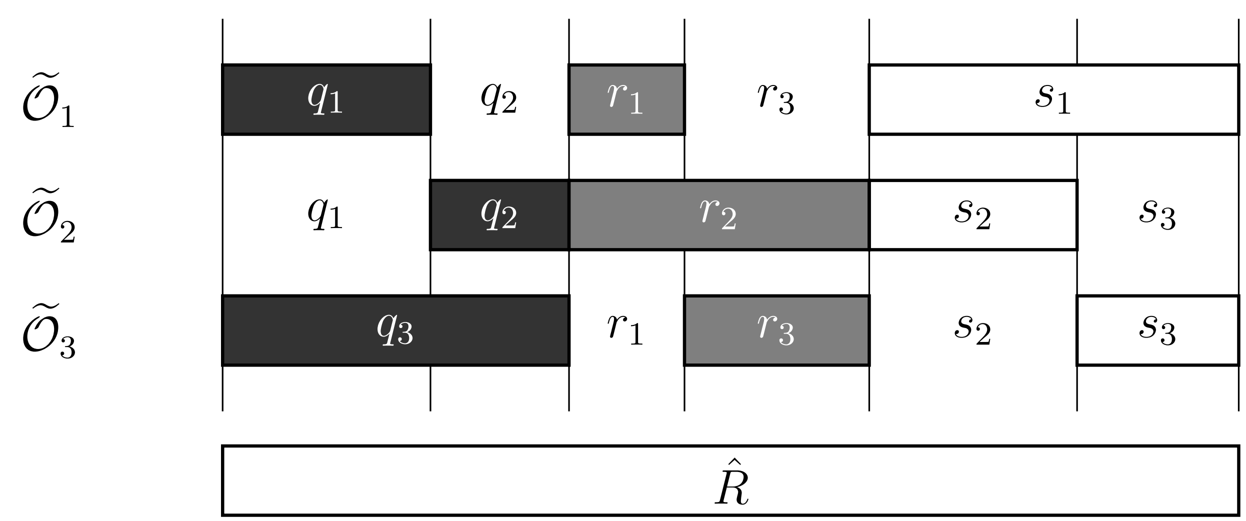
$$\mathrm{tr}_{L_2}\left(\alpha_2\,\overline{X}^{\otimes h_1}\,Y^{\otimes(\ell_{23}-h_1+h_3)}\,\overline{Z}^{\otimes(\ell_{12}-h_3)}\right)\times$$

$$\left.\mathrm{tr}_{L_3}\left(\alpha_3\,X^{\otimes(\ell_{31}-h_2+h_1)}\,\overline{Y}^{\otimes(\ell_{23}-h_1)}\,\overline{Z}^{\otimes h_2}\right)\right\rangle$$

$$=\left(\prod_{i=1}^{3}\frac{L_i!}{\overline{\overline{L}}_i!}\right)\sum_{\hat{R}\vdash L}\frac{\mathrm{Dim}_{N_c}(\hat{R})}{d_{R_1}d_{R_2}d_{R_3}}\left(d_{q_1}\,d_{q_2}\,d_{r_1}\,d_{r_3}\,d_{s_2}\,d_{s_3}\right)\times$$

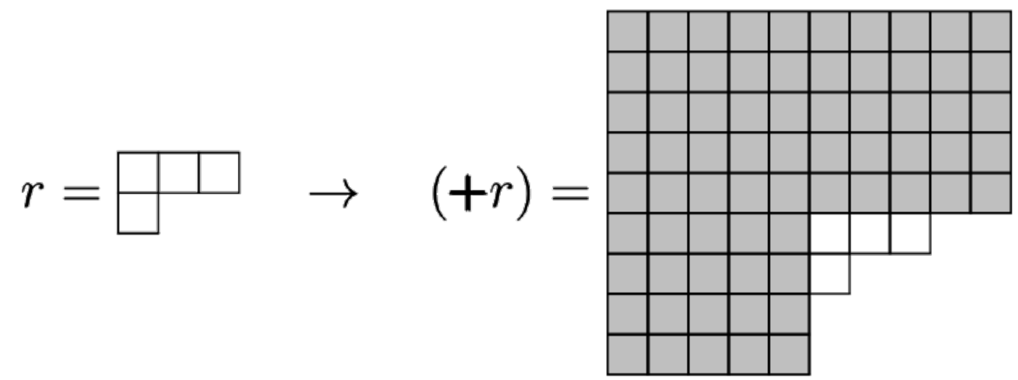
$$\overline{\delta}^{\,\nu_1-\,\nu_2+}\,\overline{\delta}^{\,\nu_2-\,\nu_3+}\,\overline{\delta}^{\,\nu_3-\,\nu_1+}\,\underline{\mathcal{G}'_{123}}$$

$=$ overlap given by



Background independence of OPE coefficients

- Recall that LLM operators can be constructed by attaching a Young diagram (for Z's) to the background $\mathcal{B}$



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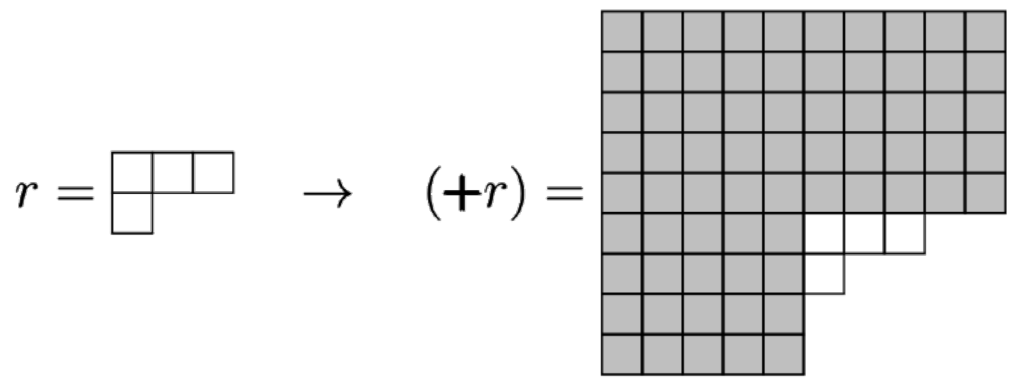


- LR coefficients satisfies the useful identity at large N_c

$$g(r, s; R) \simeq g(+r, s; +R), \quad (N_c \gg 1)$$

- This suggests background independence of 3pt at large N_c

$$\tilde{C}^{\mathrm{LLM}}_{\circ\circ\circ} \simeq (\mathcal{B}\ \text{dependent numerical factor})\, \tilde{C}^{\mathrm{LLM}}_{\circ\circ\circ}\Big|_{N_c \to N_c'}$$

also depends on $\mathcal{B}$

5. Summary

Conclusions and Outlook

- **Studied free *n-point functions* of $\mathcal{N}$=4 SYM at finite N_c using finite-group methods**
- **Non-trivial large N_c limit of $\mathcal{N}$=4 SYM should be dual to LLM geometry**
- **Showed that 3pt functions are background independent in the new large N_c limit (with a conjecture on $\mathcal{G}_{123}$)**

Conclusions and Outlook

- Studied free n-point functions of $\mathcal{N}=4$ SYM at finite Nc using finite-group methods
- Non-trivial large Nc limit of $\mathcal{N}=4$ SYM should be dual to LLM geometry
- Showed that 3pt functions are background independent in the new large Nc limit (with a conjecture on $\mathcal{G}_{123}$)

- 4pt functions at finite Nc? Bootstrap?

[Lin (2020)]

- How to prove the conjecture on Racah-Wigner tensor? (c.f. knots)

[Itoyama, Mironov, Morozov, Morozov (2012)] [Nawata, Ramadevi, Zodinmawia (2013)] [Morozov, Sleptsov (2019)]

- Application of finite-group methods? (e.g. octagon frame)

[Coronado (2018)] [Bargheer, Coronado, Vieira (2019)] [Belitsky, Korchemsky (2019,2020)]

Thank you for listening

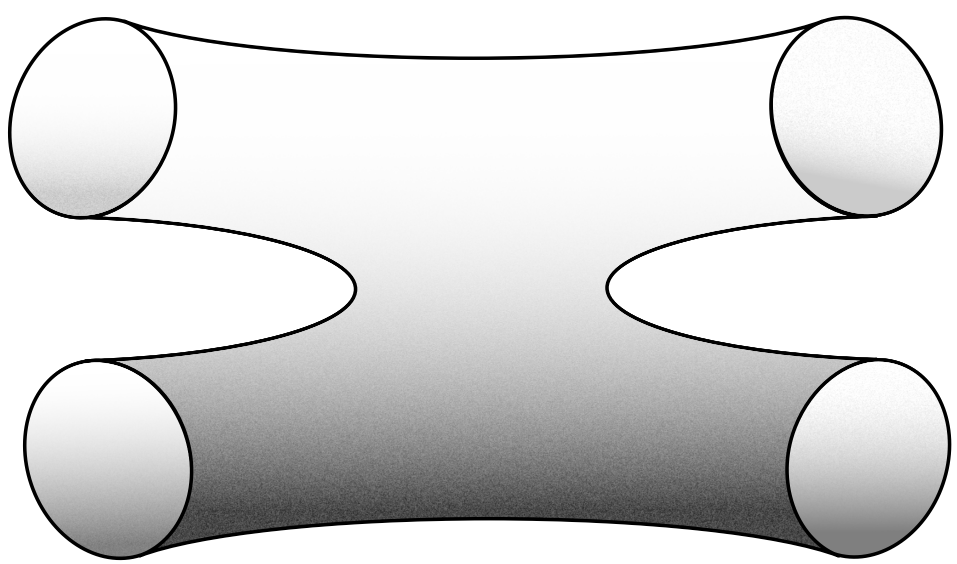
Bootstrap in large Nc CFT

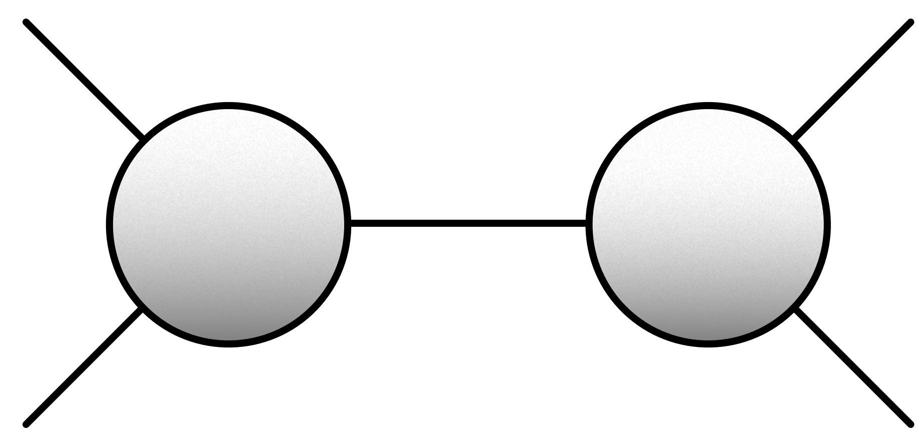
- **In CFT,** $$\mathbf{2pt + 3pt \to n\text{-}pt}$$
- **In large Nc CFT,** $$\Big(\mathbf{2pt + 3pt}\Big)_{N_c=\infty} \overset{?}{\to} \Big(\mathbf{n\text{-}pt}\Big)_{N_c=\infty}$$

Bootstrap in large Nc CFT

- **In CFT,** $\quad \mathbf{2pt + 3pt \rightarrow n\text{-}pt}$
- **In large Nc CFT,** $\quad \left(\mathbf{2pt + 3pt}\right)_{N_c=\infty} \overset{?}{\rightarrow} \left(\mathbf{n\text{-}pt}\right)_{N_c=\infty}$

Take 4pt function of $\mathcal{N}$=4 SYM, expand in OPE limit

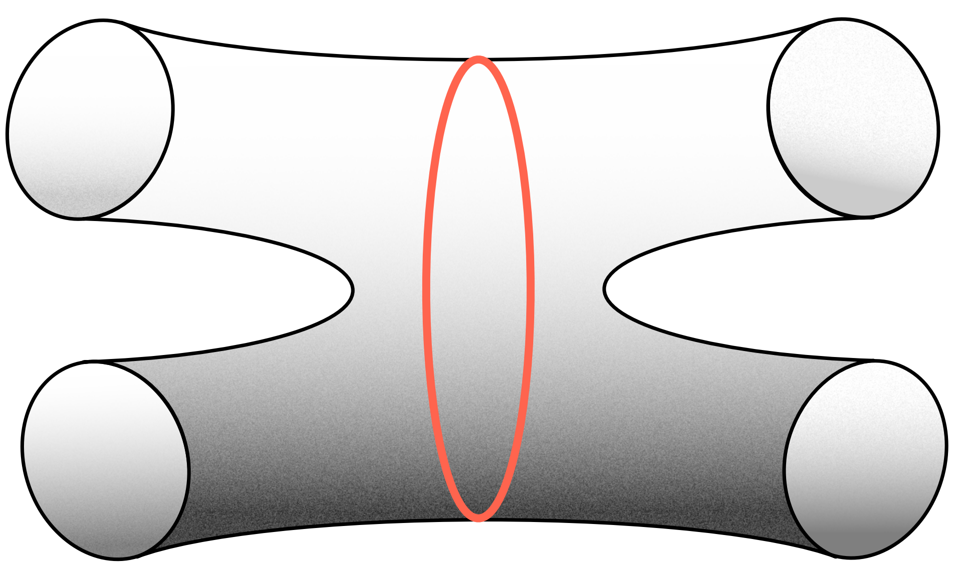


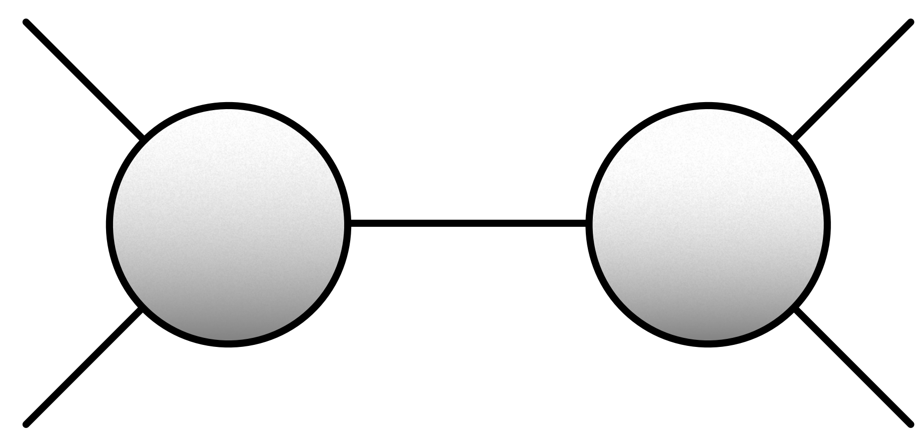


Bootstrap in large Nc CFT

- In CFT, $\qquad \text{2pt} + \text{3pt} \to \text{n-pt}$
- In large Nc CFT, $\qquad \left(\text{2pt} + \text{3pt}\right)_{N_c=\infty} \overset{?}{\to} \left(\text{n-pt}\right)_{N_c=\infty}$

Take 4pt function of $\mathcal{N}$=4 SYM, expand in OPE limit





intermediate state: single-trace 3pt interaction: non-extremal

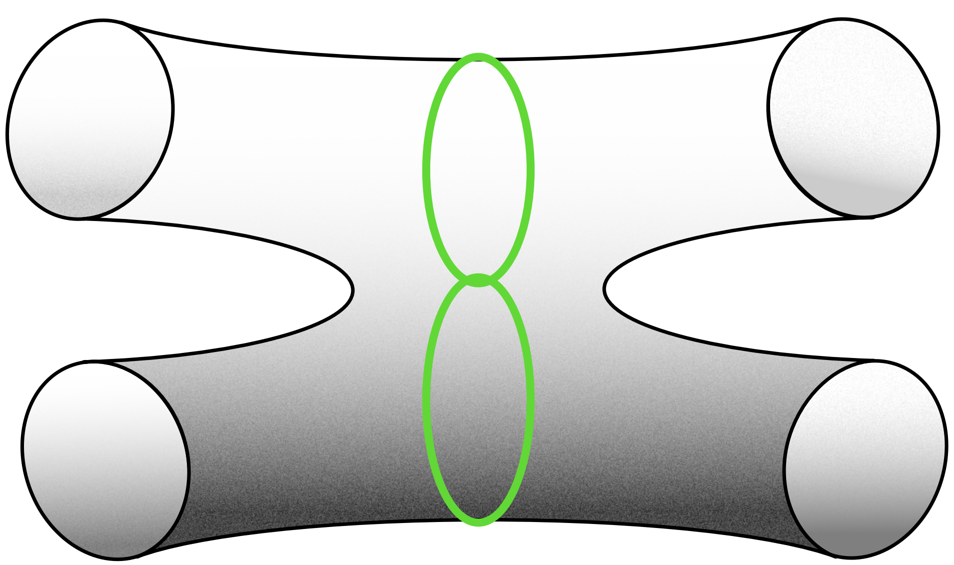
Consistent with the planar 2 & 3pt data

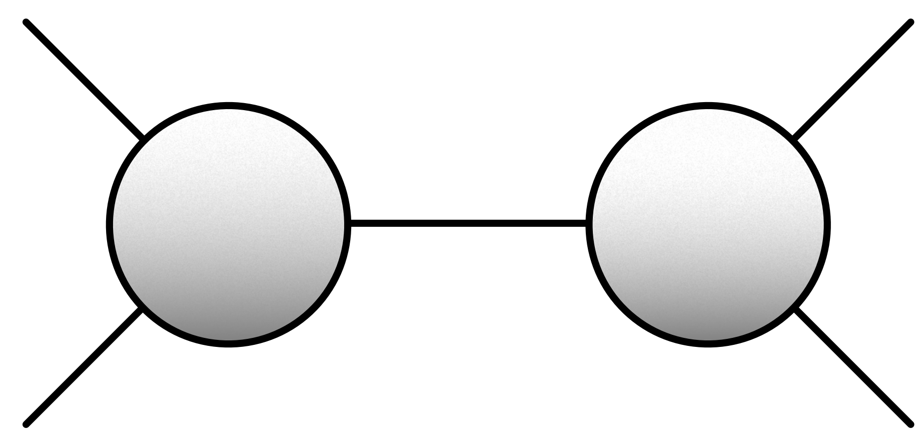
[Basso, Coronado, Komatsu, Lam, Vieira, Zhong (2017)]

Bootstrap in large Nc CFT

- In CFT, $\mathbf{2pt} + \mathbf{3pt} \to \mathbf{n\text{-}pt}$
- In large Nc CFT, $\left(\mathbf{2pt} + \mathbf{3pt}\right)_{N_c=\infty} \overset{?}{\to} \left(\mathbf{n\text{-}pt}\right)_{N_c=\infty}$

Take 4pt function of $\mathcal{N}=4$ SYM, expand in OPE limit





intermediate state: double-trace **3-pt interaction: extremal**

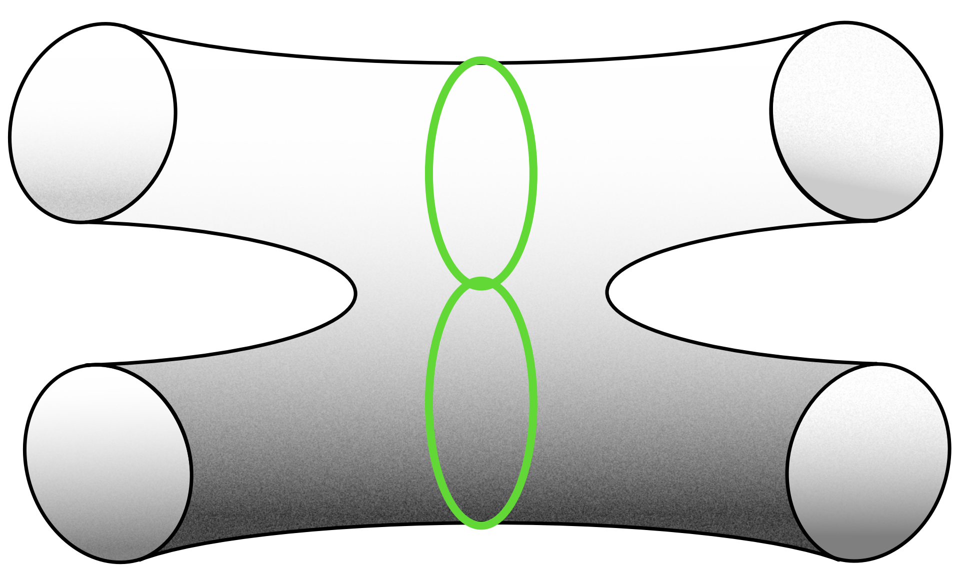
Extremal 3pt = Multi-trace mixing coefficient (non-planar 2pt)

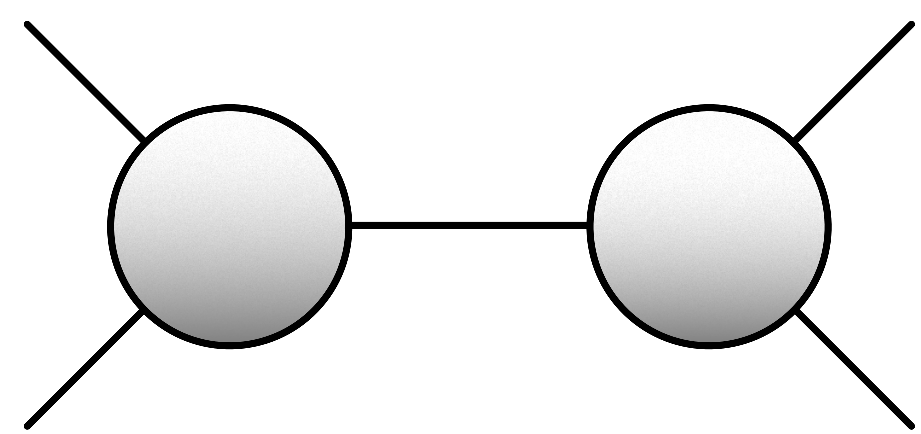
Bootstrap in large Nc CFT

- In CFT, $\quad \text{2pt} + \text{3pt} \rightarrow \text{n-pt}$
- In large Nc CFT, $\quad \Big(\text{2pt} + \text{3pt}\Big)_{N_c=\infty} \overset{?}{\rightarrow} \Big(\text{n-pt}\Big)_{N_c=\infty}$

this does not always work

Take 4pt function of $\mathcal{N}$=4 SYM, expand in OPE limit





intermediate state: double-trace   3-pt interaction: extremal

Extremal 3pt = Multi-trace mixing coefficient (non-planar 2pt)

Hexagonization of n-point

The "simplest" 4pt functions were computed more explicitly

$$\mathcal{O}_\circ = \mathrm{tr}\,(Z^{2k}),\ \mathrm{tr}\,(W^{2k}),\ \mathrm{tr}\,(\overline{W}^k \overline{Z}^k) + (\text{permutations}),\ \ldots$$

$$\langle \mathcal{O}_\circ(x_1)\, \mathcal{O}_\circ(x_2)\, \mathcal{O}_\circ(x_3)\, \mathcal{O}_\circ(x_4) \rangle = (\text{tree-level}) \times \underline{\mathbb{O}(\lambda, N_c)^2}$$

Octagon function can be computed if $k \sim \sqrt{N_c} \gg 1$

[Coronado (2018)] [Bargheer, Coronado, Vieira (2019)] [Belitsky, Korchemsky (2019,2020)]

動機の補足

Finite Nc には２つの独立な研究方向がある

AdS/CFT

BPS 演算子の対角化

[Corley, Jevicki, Ramgoolam (2001)]

非 BPS 演算子の対角化

't Hooft ループ補正

"non-planar integrability"

[Carlson, de Mello Koch, Lin (2011)]

LLM geometry

"integrable subsector"

[de Mello Kim, Zyl (2018)]

Instantons, S-dualities

[Beem, Rastelli, van Rees (2013)]

$\mathcal{N}$=2 SYM の局所化

(non-BPS data for $\mathcal{N}$=4 SYM)

[Binder, Chester, Pufu, Wang (2019)]
[Binder, Chester, Pufu, Wang, Wen (2020)]

Consistent with (analytic bootstrap of) $AdS_5 \times S^5$

[Alday, Bissi (2017)] [Alday (2018)]