Negative anomalous dimensions in $\mathcal{N}$=4 SYM

Ryo Suzuki (Oxford)
in collaboration with Yusuke Kimura (Okayama)

based on arXiv:1503.06210

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Spectral problem

What do we measure or compute?

Spectral problem

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle \sim \delta_{ij}\,|x|^{-2\Delta_i}$$

Two-point functions = Operator dimensions

Spectral problem

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\dots\mathcal{O}_n(x_n)\rangle$$

Correlation functions

Integrability in N=4 SYM

Study the dimensions of gauge-invariant operators of $\mathcal{N}$=4 SU(N_c) SYM

$$\mathcal{O}_1 = \mathrm{tr}(\Phi_{I_1}\Phi_{I_2}\ldots\Phi_{I_L})$$
$$\mathcal{O}_2 = \mathrm{tr}(\Phi_{I_1}\Phi_{I_2})\,\mathrm{tr}(\Phi_{I_3}\ldots\Phi_{I_L})$$
$$\mathcal{O}_3 = \ldots$$

Operators with the same charges mix under quantum corrections

The trace structure is conserved at planar limit

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Planar dilatation acting on the single-trace operators

= Hamiltonian of a quantum-integrable spin chain

$$\mathrm{tr}(ZZYZZY) \quad \leftrightarrow \quad |{\downarrow\downarrow\uparrow\downarrow\downarrow\uparrow}\rangle$$

[Minahan, Zarembo; hep-th/0212208]

Exact dimensions

Integrability methods (TBA, NLIE, QSC) are believed to predict the planar dimensions of $\mathcal{N}=4$ SU(N_c) SYM operators at any λ

[Bombardelli, Fioravanti, Tateo (2009)] [Gromov, Kazakov, Kozak, Vieira (2009)] [Arutyunov, Frolov (2009)] [Gromov, Kazakov, Leurent, Volin (2011-14)]

Do these nonlinear eqns always have regular or real solutions?

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Do these nonlinear eqns always have regular or real solutions?

The answer is subtle if taking the planar limit is subtle

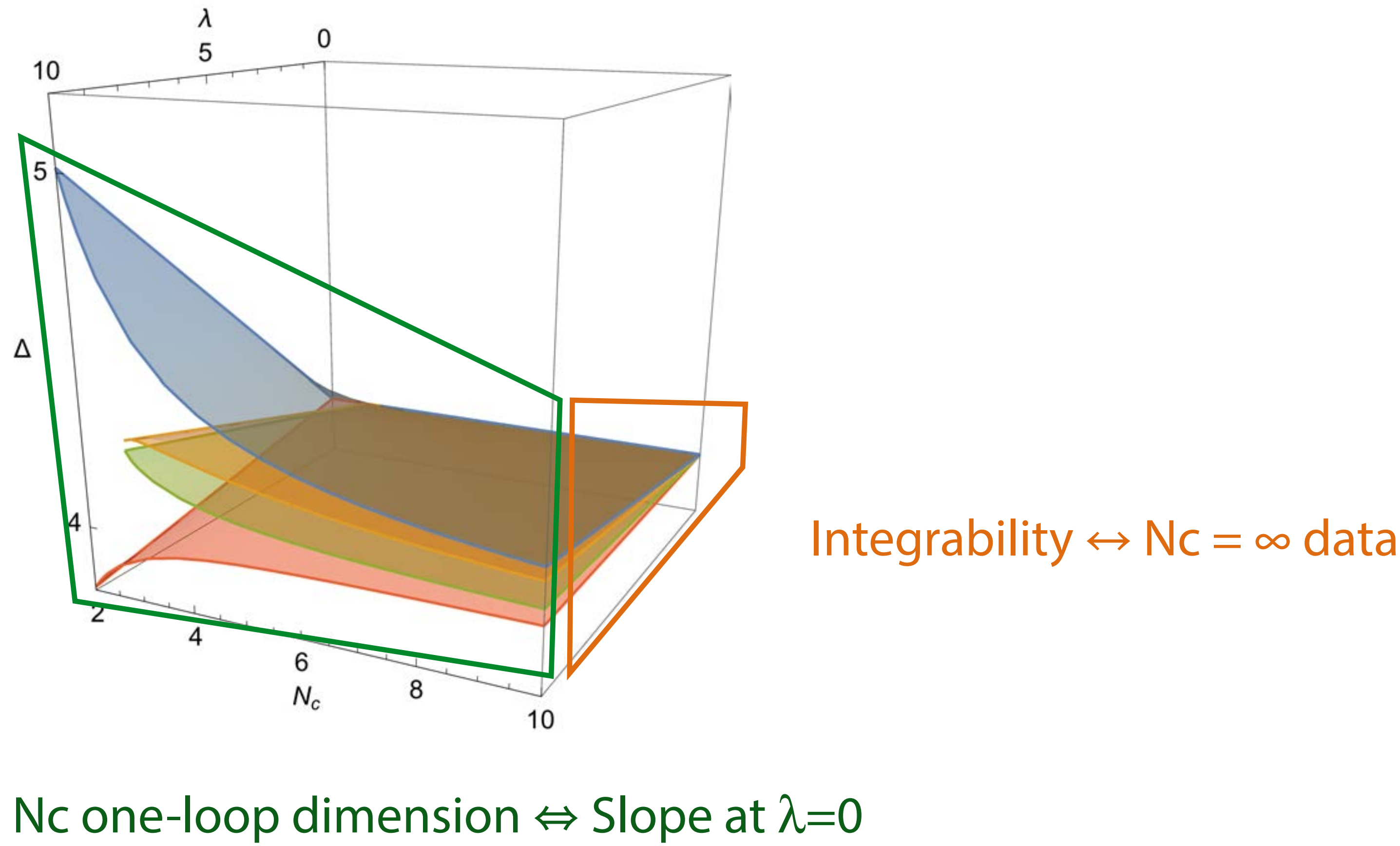
- Operators of O(N_c) length (multi-trace mixing)
- β deformation (SU(N_c) vs U(N_c))
- γ deformation (closed tachyons)
- D$\overline{\text{D}}$ branes (open tachyons)

[de Mello Koch et al. (2011-)]
[Ahn, Bajnok, Bombardelli, Nepomechie (2011)] [de Leeuw, van Tongeren (2012)] [Fokken, Sieg, Wilhelm (2013-14)]
[Bajnok, Drukker, Hegedus, Nepomechie, Palla, Sieg, RS (2013)] [Hegedus (2015)]

We revisit the one-loop operator mixing problem at finite N_c

A simple example

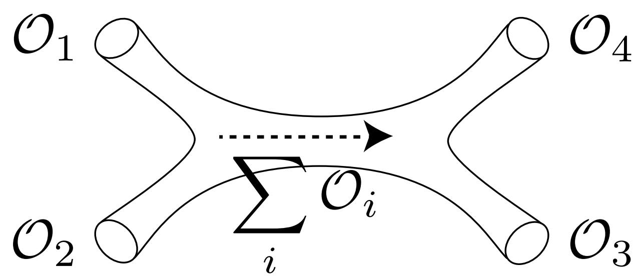
e.g. Tree + one-loop operator dimension surfaces in the (Δ, λ, N_c) space for some scalar operators

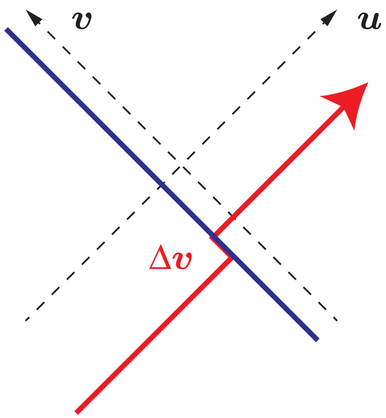


The 1/Nc corrections are (un)expectedly related to 4pt functions

Asymptotic Causality

Four-point functions in AdS/CFT in the Eikonal limit, $p_1 \sim p_3$, $p_2 \sim p_4$





Scattering of two light-like particles
~ Phase shift
~ Dimension of an intermediate operator

[Cornalba, Costa, Penedones, Schiappa, hep-th/0611122, 0611123, arXiv:0707.0120]

$$\mathcal{O}_{\text{double trace}} \sim :\mathcal{O}_1 \partial_{\mu_1} \ldots \partial_{\mu_j} (\partial^2)^n \mathcal{O}_2:$$

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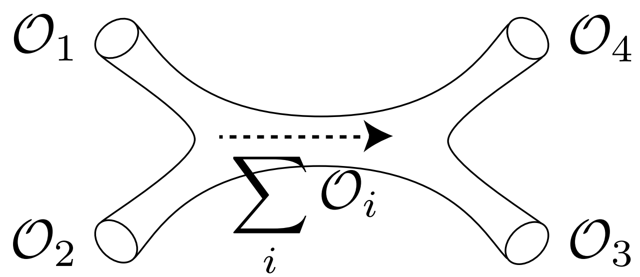

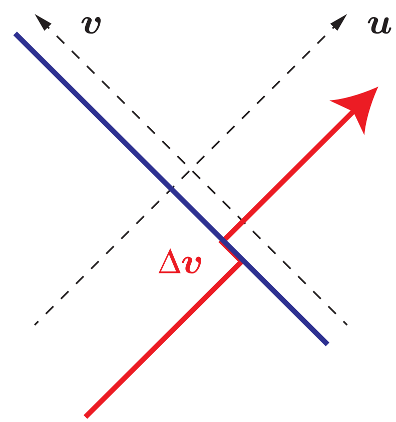

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$$\mathcal{O}_{\text{double trace}} \sim :\mathcal{O}_1 \partial_{\mu_1} \dots \partial_{\mu_j} (\partial^2)^n \mathcal{O}_2:$$

Phase shift ~ Time delay of gravitational shock wave
→ Positive from the asymptotic causality of gravity on AdS

Anomalous dimension of the double-trace operator must be negative

(non-planar effect)

[Camanho, Edelstein, Maldacena, Zhiboedov, arXiv:1407.5597]

Plan of Talk

✓ 1. Introduction

2. Operator Submixing Problem

3. Submixing from Correlators

4. Summary

Operator Submixing Problem

Large Nc degeneracy

Dilatation operator is hermitian w.r.t. 2pt function:

$$\left(-x\frac{\partial}{\partial x}\right)\{\langle\mathcal{O}_A(x)\mathcal{O}_B(0)\rangle - \langle\mathcal{O}_A(0)\mathcal{O}_B(-x)\rangle\} = 0$$

$$\Longrightarrow \quad \langle(\mathfrak{D}\mathcal{O}_A)\mathcal{O}_B\rangle = \langle\mathcal{O}_A(\mathfrak{D}\mathcal{O}_B)\rangle$$

$$\Longrightarrow \quad (\Delta_A - \Delta_B)\,\langle\mathcal{O}_A\mathcal{O}_B\rangle = 0$$

- The large Nc spectrum is highly degenerate owing to integrability
- Degenerate eigenstates can freely mix, even if they have different trace structure

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Operator submixing problem =
How to lift large Nc degeneracy by 1/Nc corrections

Interesting states : large Nc zero modes ($\Delta=O(1/Nc)$)

Lifting large Nc degeneracy

To lift the degeneracy, solve $\mathfrak{D}_{\text{one-loop}}\psi = \gamma\psi$ **perturbatively in** $1/N_c$

- -

Split $D_{\text{one-loop}}$ and ψ into planar and non-planar parts

$$\mathfrak{D}_{\text{one-loop}} = \mathfrak{D}_0 + N_c^{-1}\mathfrak{D}_1, \quad \psi = \sum_{i=0}^{\infty} N_c^{-i}\psi_i$$

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Consider large Nc zero modes, and regularize the planar dilatation as

$$\mathfrak{D}_\epsilon = \mathfrak{D}_0 + \epsilon, \quad \mathfrak{D}_\epsilon\psi_0 = \epsilon\psi_0$$

$$\Longrightarrow \quad \left(\mathfrak{D}_\epsilon + N_c^{-1}\mathfrak{D}_1\right)\left(\sum_{i=0}^{\infty} N_c^{-i}\psi_i\right) = \left(\epsilon + \sum_{i=0}^{\infty} N_c^{-i}\gamma_i\right)\left(\sum_{i=0}^{\infty} N_c^{-i}\psi_i\right)$$

Almost textbook problem of degenerate perturbation in QM

Lifting large Nc degeneracy

Almost textbook problem of degenerate perturbation in QM

$\psi^{(\alpha)} = u_I^{(\alpha)} O_I, \quad O_I$: monomial multi-trace operators like $\mathrm{tr}(\phi_a^2)^2$

$$E^{(\alpha\alpha')} = \delta_{\alpha\alpha'} E_0^{(\alpha)} + N_c^{-1} \langle \alpha | \mathfrak{D}_1 | \alpha' \rangle + N_c^{-2} \sum_{E_0^{(\beta)} \neq E_0^{(\alpha)}} \frac{\langle \alpha | \mathfrak{D}_1 | \beta \rangle \langle \beta | \mathfrak{D}_1 | \alpha' \rangle}{E_0^{(\alpha)} - E_0^{(\beta)}} + \ldots$$

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What is the bracket <α|β> ?

(a) Two-point functions: $S_{IJ} \equiv \langle O_I O_J \rangle$

✗ S_{IJ} depends on Nc; even the rank changes

(b) Dual basis: $\delta_{IJ} = \langle \check{O}_I | O_J \rangle$

✗ Matrix elements of the dilatation operator is not hermitian w.r.t. dual basis

Operator submixing equation

Observation

Degeneracy of the large Nc zero modes is lifted at the 2nd order

$$\gamma_{\mathbf{0}}^{(\alpha)} = \gamma_{\mathbf{1}}^{(\alpha)} = \mathbf{0}$$

$\Longleftarrow$ **Assumption:** ψ_0 **and** $\mathfrak{D}_1\psi_0$ **have different trace structure**

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$\Longleftarrow$ **Assumption:** ψ_0 **and** $\mathfrak{D}_1 \psi_0$ **have different trace structure**

After a little algebra, we find an eigenvalue problem:

$$\gamma_2^{(\alpha)} = \delta_{\alpha\alpha'} \frac{\left\langle \psi_0^{(\alpha')} \left(N_c \mathfrak{D}_1 - \mathfrak{D}_1 \mathfrak{D}_\epsilon^{-1} \mathfrak{D}_1 \right) \psi_0^{(\alpha)} \right\rangle}{\left\langle \psi_0^{(\alpha')} \psi_0^{(\alpha)} \right\rangle} \Rightarrow H_{\text{sm}}^\circ \psi_0^{(\alpha)} = \gamma_2^{(\alpha)} \psi_0^{(\alpha)}$$

H_{sm} = Operator Submixing Hamiltonian

$$H_{\text{sm}}^\circ = \lim_{\epsilon \to 0} P_\circ \left[N_c \mathfrak{D}_1 - \mathfrak{D}_1 \left(\mathfrak{D}_0 + \epsilon \right)^{-1} \mathfrak{D}_1 \right]$$

$$P_\circ = \lim_{\epsilon \to 0} \epsilon \left(\mathfrak{D}_0 + \epsilon \right)^{-1} \ : \ \textbf{Projector to Ker } \mathfrak{D}_0$$

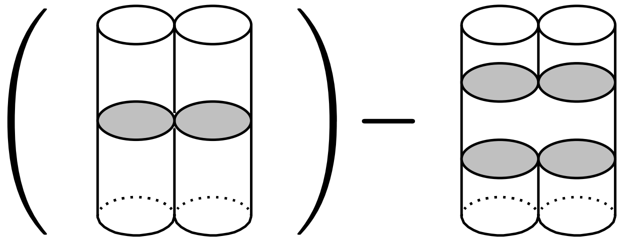
Submixing Hamiltonian

$$H^{\circ}_{\rm sm} = \lim_{\epsilon \to 0} P_{\circ} \left[N_c \, \mathfrak{D}_1 - \mathfrak{D}_1 \left(\mathfrak{D}_0 + \epsilon \right)^{-1} \mathfrak{D}_1 \right]$$

$$P_{\circ} = \lim_{\epsilon \to 0} \epsilon \left(\mathfrak{D}_0 + \epsilon \right)^{-1} \; : \; \textbf{Projector to Ker } \mathfrak{D}_0$$

Denoting a trace or a closed string (e.g. in pp-wave SFT) by a circle,

$$H^{\circ}_{\rm sm} \sim \; N_c$$



4-pt contact term (3-pt coupling)[2]

[Constable, Freedman, Headrick, Minwalla, hep-th/0209002] [Grignani, Orselli, Ramadanovic, Semenoff, Young; hep-th/0508126]

The 4pt contact term can be removed by redefining the state

$$\gamma_2 = - \lim_{\epsilon \to 0} \frac{\left\langle \Psi' \mathfrak{D}_{\epsilon}^{-1} \Psi' \right\rangle}{\langle \psi_0 \psi_0 \rangle} \,, \qquad \Psi' = \left(\frac{1}{2} \mathfrak{D}_{\epsilon} - \frac{1}{N_c} \mathfrak{D}_1 \right) \psi_0$$

Case study: so(6) singlets

Number of large Nc zero modes and all scalar so(6) singlets :

L	2	4	6	8	10	12
$\mathcal{Z}_L$	0	1	2	5	11	34
$\dim \mathcal{H}_L$	1	4	15	71	469	4477

Singlet large Nc zero modes = Products of 1/2-BPS single-traces

$$\mathcal{C}^{(\ell)} = \mathcal{C}_{i_1 i_2 \ldots i_\ell} = \mathrm{tr}(\Phi_{(i_1}\Phi_{i_2}\ldots\Phi_{i_\ell)}) \ : \ \textbf{symmetric traceless}$$

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$$\{\mathcal{C}_{ij}\,\mathcal{C}_{ij}\}_{L=4}, \quad \{\mathcal{C}_{ijk}\,\mathcal{C}_{ijk}\,,\;\mathcal{C}_{ij}\,\mathcal{C}_{jk}\,\mathcal{C}_{ki}\}_{L=6},$$

$$\{\mathcal{C}_{ijkl}\,\mathcal{C}_{ijkl}\,,\;\mathcal{C}_{ijkl}\,\mathcal{C}_{ij}\,\mathcal{C}_{kl}\,,\;\mathcal{C}_{ijk}\,\mathcal{C}_{ijl}\,\mathcal{C}_{kl}\,,\;\mathcal{C}_{ij}\,\mathcal{C}_{ji}\,\mathcal{C}_{kl}\,\mathcal{C}_{lk}\,,\;\mathcal{C}_{ij}\,\mathcal{C}_{jk}\,\mathcal{C}_{kl}\,\mathcal{C}_{li}\}_{L=8}$$

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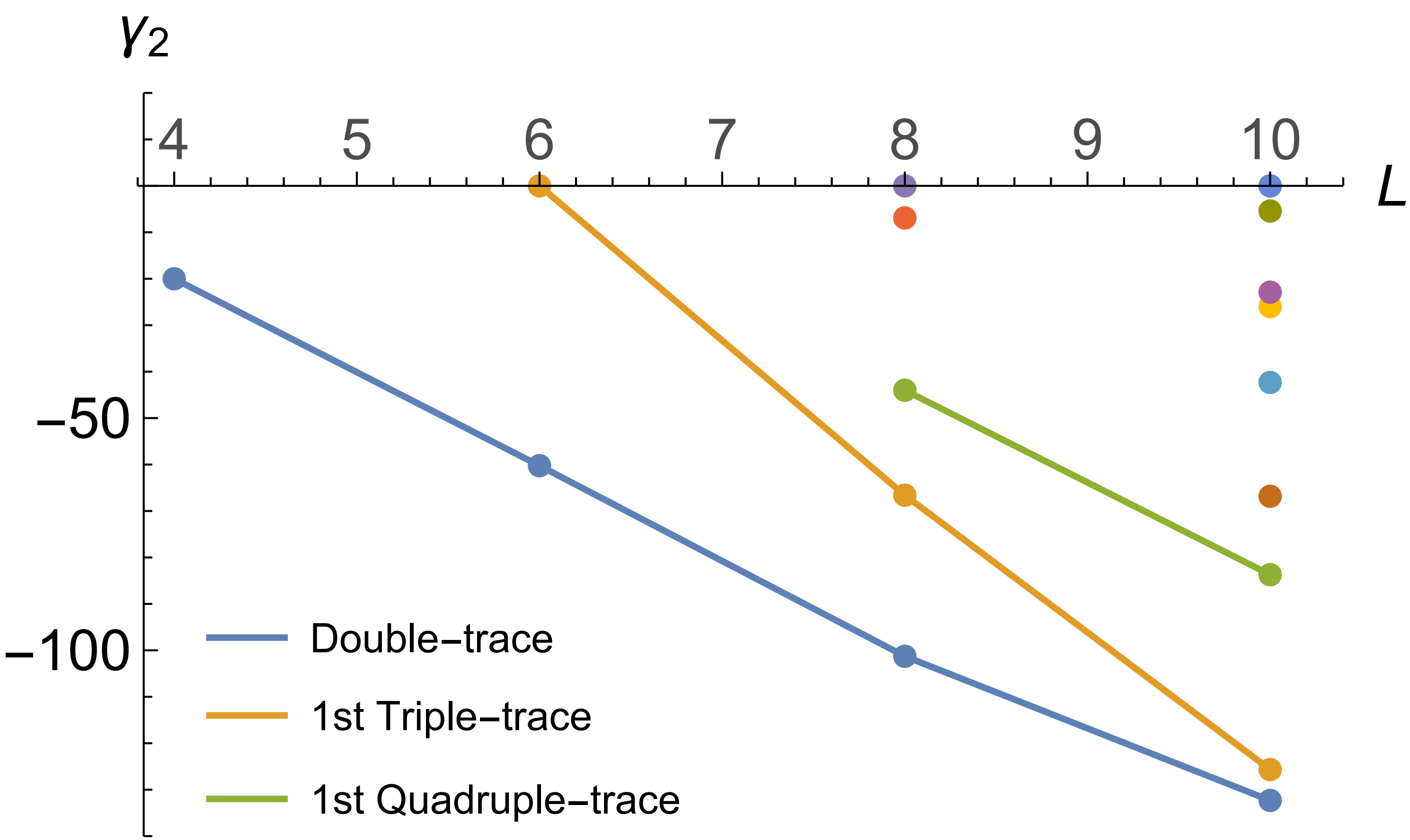
We computed the operator mixing and submixing matrices explicitly up to L=10 at one-loop by using Mathematica

Submixing Eigenvalues

γ_2 of so(6) singlet large Nc zero modes



Operators with fewer traces have more negative dimensions

Submixing patterns

Based on explicit results, we conjecture that

1. Degeneracies are lifted at $O(1/N_c^2)$ and all second-order corrections are non-positive
2. All submixing eigenstates have a definite number of traces
3. Submixing density projectively commute

Submixing patterns

Based on explicit results, we conjecture that

1. Degeneracies are lifted at O(1/Nc[2]) and all second-order corrections are non-positive
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3. Submixing density projectively commute

1. $\gamma = \sum_i N_c^{-i} \gamma_i, \quad \boxed{\gamma_0 = \gamma_1 = 0, \ \gamma_2 \leq 0,} \quad H_{\rm sm}^{\circ} \, \psi_0 = \gamma_2 \, \psi_0$

Not always true for non-zero modes or not N=4 SYM

Related to gravitational attraction and the causality of $AdS_5 \times S^5$

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2. Matrix elements of H_{sm} are block-diagonal according to #(traces)

Two triple-trace zero modes at L=8, 4+2+2 and 3+3+2, submix only among themselves
Anything may submix for non-zero modes

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3. Define the density by $H^{\circ}_{\rm sm} \sim \int_0^{\infty} dt \, h^{\circ}_{\rm sm}(t)$

$$\boxed{P^{\circ}_{\rm min} \ [h^{\circ}_{\rm sm}(t), h^{\circ}_{\rm sm}(t')] = [h^{\circ}_{\rm sm}(t), h^{\circ}_{\rm sm}(t')] \ P^{\circ}_{\rm min} = 0}$$

Non-planar integrability

$P^{\circ}_{\rm min} =$ **Projector to** $\mathcal{H}^{\circ}_{\rm min}$, **products of length-two traces in Ker** $\mathfrak{D}_0$

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0,1,2,5,11,34 Search Hints

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Sort: relevance | references | number | modified | created Format: long | short | data

A080068 Iterates of A080067.

0, 1, 2, 5, 11, 34, 82, 287, 923, 3016, 8664, 37407, 102983, 414050, 1488163

OFFSET 0,3
LINKS Table of n, a(n) for n=0..14.
FORMULA a(0) = 0, a(n) = A080067(a(n-1))
CROSSREFS Corresponding totally balanced binary sequences in A063171: A080070.
KEYWORD nonn
AUTHOR Antti Karttunen Jan 27 2003
STATUS approved

A226919 Asymptotic number of completely symmetric polynomials of degree n up to momentum conservation in the limit as the number of particles increases.

1, 0, 1, 2, 5, 11, 34, 87, 279

OFFSET 0,4
COMMENTS The values for n >= 6 are only conjectural.
LINKS Table of n, a(n) for n=0..8.
R. H. Boels, On the field theory expansion of superstring five point amplitudes, arXiv preprint arXiv:1304.7918, 2013
CROSSREFS These are the limiting values for the columns in the array A226920.
KEYWORD nonn,more
AUTHOR N. J. A. Sloane, Jul 12 2013
STATUS approved

A254342 Number of singlets of length 2n written as a product of so(N) traceless completely symmetric representations at large N.

0, 1, 2, 5, 11, 34

OFFSET 1,3
LINKS Table of n, a(n) for n=1..6.
Y. Kimura and R. Suzuki, Negative anomalous dimensions in N=4 SYM, Appendix D, arXiv:1503.06210 [hep-th], 2015.
CROSSREFS Cf. A226919.
KEYWORD nonn,more
AUTHOR Ryo Suzuki, Mar 27 2015
STATUS approved

Z_L coincides with the number of symmetric polynomials of Mandelstam variables at degree (L/2) under massless momentum conservation

Mandelstam variables

$\mathcal{Z}_{2K}$: **Number of $\mathfrak{so}(N_f)$ singlet large N_c zero modes at $N_f \gg 1$**

$\mathcal{Z}'_K$: **Number of symmetric polynomials of Mandelstam variables at degree K**

$$(\mathcal{Z}_2\,,\mathcal{Z}_4\,,\mathcal{Z}_6\,,\mathcal{Z}_8\,,\mathcal{Z}_{10}\,,\mathcal{Z}_{12}) = (0,1,2,5,11,34)$$
$$(\mathcal{Z}'_1\,,\mathcal{Z}'_2\,,\mathcal{Z}'_3\,,\mathcal{Z}'_4\,,\mathcal{Z}'_5\,,\mathcal{Z}'_6) = (0,1,2,5,11,34)$$

Mandelstam variables of massless particles

$$s_{ij} = \left(p_i^\mu + p_j^\mu\right)^2 = 2\,p_i\cdot p_j\,,\quad (i,j=1,2,\ldots,n),\quad n\gg 1$$

$$s_{ij} = s_{ji}\,,\quad \sum_{j=1}^{n} s_{ij} = 0,\quad \left(\sum_{i=1}^{n} p_i^\mu\right)^2 = \sum_{i<j}^{n} s_{ij} = 0$$

Symmetric polynomials

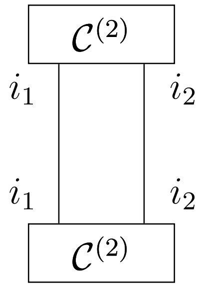
$$\sigma_2^{(1)} = s_{12}^2 + \mathcal{S}_n \text{ permutations},\quad \sigma_2^{(2)} = s_{12}s_{13} + \ldots,\quad \sigma_2^{(3)} = s_{12}s_{34} + \ldots.$$

4-pt string scattering amplitude on the flat spacetime

$$\mathcal{A}_4 = \sum_{a,b} M_{a,b}\sigma_2^a\sigma_3^b,\quad s+t+u=0,\quad \sigma_2 = st+tu+us,\quad \sigma_3 = stu,$$

Graphical connection

$\mathcal{C}_{i_1 i_2}\mathcal{C}_{i_1 i_2} =$



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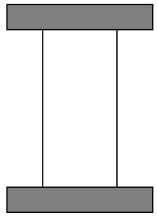

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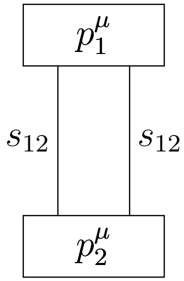
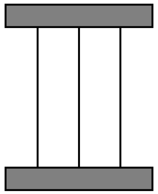


$= s_{12}^2 + \text{permutations},$

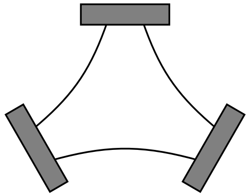
$$\boldsymbol{\mathcal{C}_{ij}\mathcal{C}_{ij} \leftrightarrow \sum_{\sigma\in\mathcal{S}_n} s^2_{\sigma(1)\sigma(2)}}$$

$\mathcal{C}_{i_1 i_2 i_3}\mathcal{C}_{i_1 i_2 i_3} =$



$$\boldsymbol{\mathcal{C}_{ijk}\mathcal{C}_{ijk} \leftrightarrow \sum_{\sigma\in\mathcal{S}_n} s^3_{\sigma(1)\sigma(2)}}$$

$\mathcal{C}_{i_1 i_2}\mathcal{C}_{i_2 i_3}\mathcal{C}_{i_3 i_1} =$



$$\boldsymbol{\mathcal{C}_{ij}\mathcal{C}_{jk}\mathcal{C}_{ki} \leftrightarrow \sum_{\sigma\in\mathcal{S}_n} s_{\sigma(1)\sigma(2)}s_{\sigma(2)\sigma(3)}s_{\sigma(3)\sigma(1)}}$$

Can construct the basis of independent symmetric polynomials!

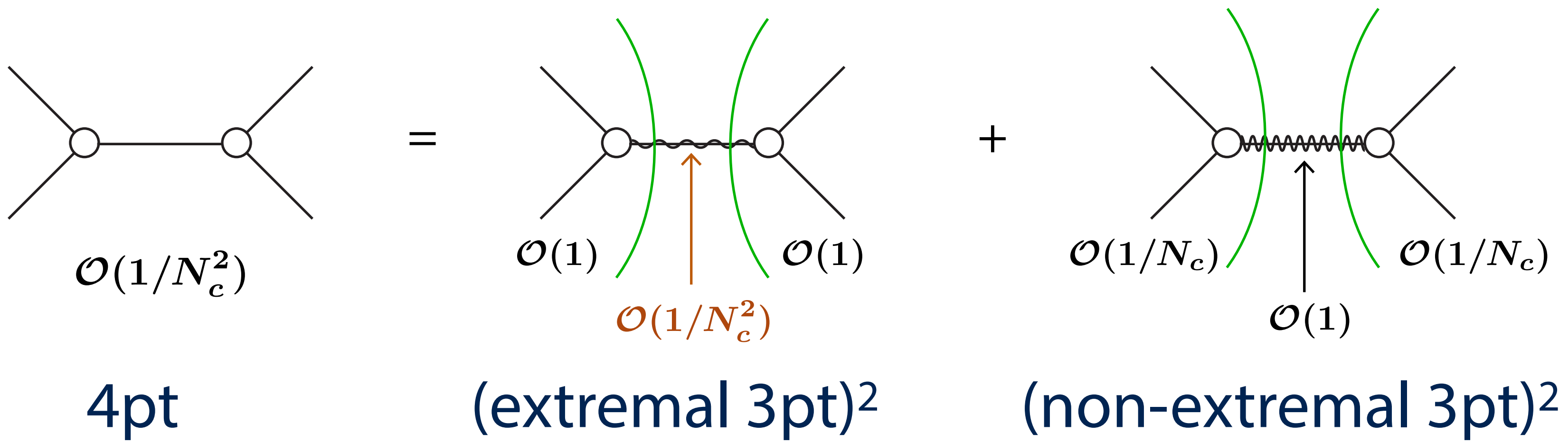
Submixing from Correlators

Alternative method

- Analytic computation of spectrum is difficult at finite Nc
- Planar higher-point correlators know 1/Nc corrections

Planar 4-pt function to non-planar 2pt:

[D'Hoker, Mathur, Matusis, Rastelli (1999)]
[Arutyunov, Frolov, Petkou (2000)] [Arutyunov, Dolan, Osborn, Sokatchev (2002)] [Dolan Osborn (2004)] and a lot more



$$\left\langle \prod_{i=1}^{4} \mathcal{C}^{(p)}(x_i) \right\rangle \sim \sum_I C^2_{p,p,I}\, u^{\frac{\Delta_I - \ell_I}{2}}\, G_{\Delta_I,\ell_I}(u,v), \quad (u,v) \equiv \left(\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \right)$$

Average anomalous dimensions

(Super)conformal partial-wave decomposition of 4pt functions

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Expand around $\lambda=0$ and take the limit $(u,v) \to (0,1)$

$$\left\langle \prod_{i=1}^{4} \mathcal{C}^{(p)}(x_i) \right\rangle \sim \frac{p^2}{N_c^2} \sum_{\rho=\text{irrep}} \sum_{r=0}^{\infty} \sum_{s=1}^{r} \lambda^r \log^s(u)\, \underline{f_{\rho,rs}(u,v)}$$

~ $(\text{3pt})^2 \times (\text{anomalous dim}) \times {}_2F_1$

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~ $(3\mathrm{pt})^2$ × (anomalous dim) × ${}_2F_1$

Perturbative 4pt functions are explicitly known at small p, which are given by {log, polylog, multiple polylog, …}

By comparing them, we obtain a weighted sum of anomalous dimensions, where the averaging weight is given by the 3pt coupling between two external and one internal states

Average anomalous dimensions

If the operators propagate the internal line are scalar so(6) singlets at twist-four,

$$\mathcal{H} = \left\{ \mathrm{tr}(\phi_{i_1}\phi_{i_2}\phi_{i_1}\phi_{i_2}), \mathrm{tr}(\phi_{i_1}\phi_{i_1}\phi_{i_2}\phi_{i_2}), \mathrm{tr}(\phi_{i_1}\phi_{i_2})\,\mathrm{tr}(\phi_{i_2}\phi_{i_1}), \mathrm{tr}(\phi_{i_1}\phi_{i_1})\,\mathrm{tr}(\phi_{i_2}\phi_{i_2}) \right\}$$

$$\sum_{I=1}^{4} a_I^2 = 1, \quad \sum_{I=1}^{4} a_I^2\, \gamma^{(I)} = -\lambda \frac{4}{N_c^2}, \quad \sum_{I=1}^{4} a_I^2\, (\gamma^{(I)})^2 = \lambda^2 \frac{18}{N_c^2}$$

Negative!

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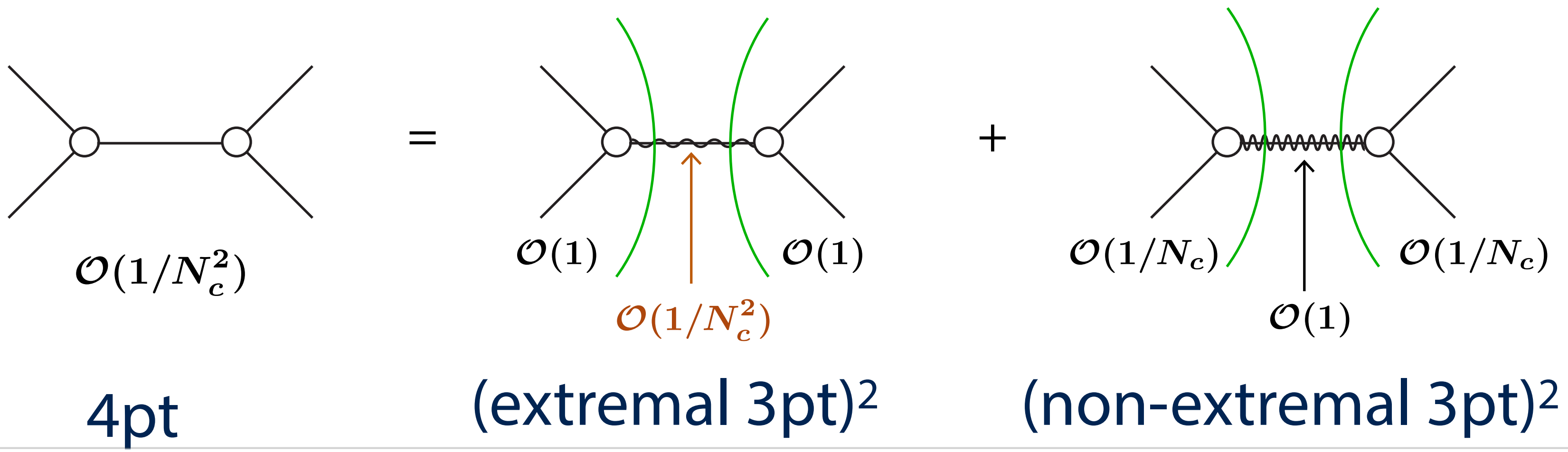


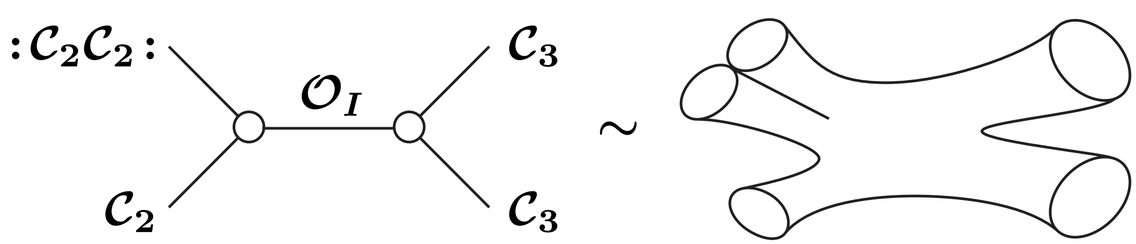
Multi-trace 4pt functions

4pt functions of products of BPS operators probe internal operators which are more than double traces

$$\Big\langle :\mathcal{C}^{(2)}\mathcal{C}^{(2)}:(x_1)\,\mathcal{C}^{(2)}(x_2)\,\mathcal{C}^{(2)}(x_3)\,:\mathcal{C}^{(2)}\mathcal{C}^{(2)}:(x_4)\Big\rangle \quad \to \quad \mathcal{O}_I \sim :\mathcal{C}^{(2)}\mathcal{C}^{(2)}\mathcal{C}^{(2)}:$$

Some multi-trace 4pt functions are smaller than O(1/Nc²)

$$F \equiv \Big\langle :\mathcal{C}^{(2)}\mathcal{C}^{(2)}:(x_1)\,\mathcal{C}^{(2)}(x_2)\,\mathcal{C}^{(3)}(x_3)\,\mathcal{C}^{(3)}(x_4)\Big\rangle \sim \sum_I C_{(2,2),2,I}\,C_{3,3,I}\,u^{\frac{\Delta_I-\ell_I}{2}}\,G_{\Delta_I,\ell_I}(u,v)$$



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Some multi-trace 4pt functions are smaller than $O(1/Nc^2)$

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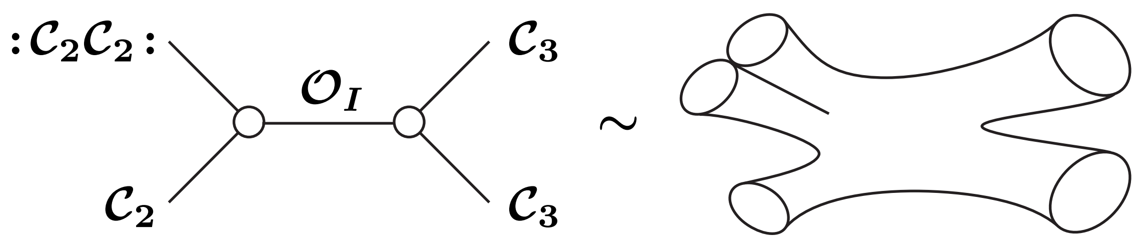
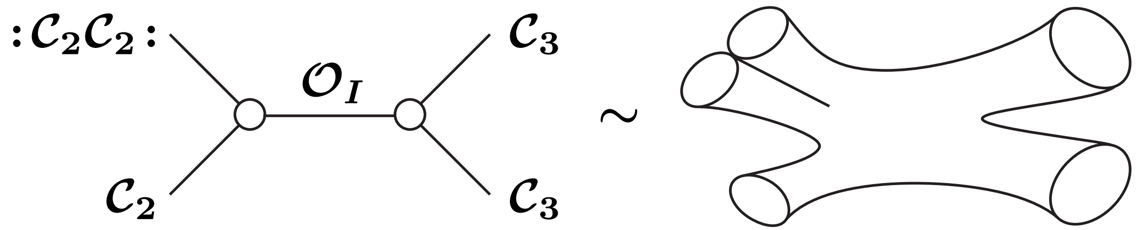


$$F\Big|_{\mathcal{O}(\lambda^n)} \sim \sum_I C_{(2,2),2,I}\,C_{3,3,I}\,(\gamma^{(I)})^n \sim \mathcal{O}(1/N_c^3) \quad (\forall n \geq 1)$$

$$C_{(2,2),2,I}\,C_{3,3,I}\,\gamma^{(I)} \sim \mathcal{O}(1/N_c^3) \quad (\forall I)$$

The anomalous dimension or 3pt coupling should vanish

Constraints on submixing



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Can the following large Nc zero mode be an internal operator?

$$\mathcal{O}_I^{(\mathrm{hyp})} \sim c_1 :\mathcal{C}^{(2)}\mathcal{C}^{(2)}\mathcal{C}^{(2)}: + \, c_2 :\mathcal{C}^{(3)}\mathcal{C}^{(3)}:$$

Submixing eigenvectors should have the same number of traces, unless their anomalous dimensions are O(1/Nc^3)

(linear combination of (4,2,2) and (3,3,2) is still allowed)

Consistent with our computation, and generalizable for any L

Summary

Discussion

- Revisited one-loop spectrum of $\mathcal{N}$=4 SYM at finite Nc
- Studied operator submixing problem;
  how to lift the degeneracy of the large Nc zero modes
- Agreed with multi-trace 4pt functions

Future Directions

- Integrable model for operator submixing?
- Beyond scalar so(6) singlets; e.g. sl(2) sector
- Analytic computation;
  using amplitude/bootstrap, group-theory methods
- New interpolating function of AdS/CFT at $O(1/Nc^2)$

Thank you for attention

Appendix

Integrable deformations

γ deformation of $\mathcal{N}$=4 SYM

$$[X,Y] \;\to\; [X,Y]_* = e^{iq_x \wedge q_y} XY - e^{iq_y \wedge q_x} YX, \quad q_x \wedge q_y = -\epsilon^{abc}\gamma_a (q_x)_b (q_y)_c$$

= TsT transformation of $AdS_5 \times S^5$ background

= Twisting Bethe Ansatz equations by a constant phase

$$\phi_a(2\pi) - \phi_a(0) = 2\pi \left(n_a - \epsilon_{abc}\gamma_b J_c\right), \quad n_a \in \mathbb{Z}$$

[Frolov, Roiban, Tseytlin; hep-th/0507021] [Beisert, Roiban; hep-th/0505187]

γ deformed theories are UV finite

[Ananth, Kovacs, Shimada; hep-th/0609149, 0702020]

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β deformation $\gamma_i = \beta \in \mathbb{R}$

- Exactly marginal deformation
- 𝒩=1 SCFT

η deformation $\mathfrak{psu}(2|2)^2 \to \mathfrak{psu}_q(2|2)^2, \quad q = \exp\left(-\frac{1}{g}\frac{2\eta}{1+\eta^2}\right)$

[Klimcik, arXiv:0802.3518][Delduc, Magro, Vicedo; arXiv:1309.5850][Arutyunov, Borsato, Frolov; arXiv:1312.3542] and others

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The *-product prescription is not always well-defined

$$\mathrm{tr}\,(\phi_i * \phi_j) \neq \mathrm{tr}\,(\phi_j * \phi_i)$$

Running double-trace coupling in γ-deformed, sensitive to U(N) vs SU(N)

$$\delta\mathcal{L} = -s\,\frac{g_{\rm YM}^2}{N}\,Q_{lk}^{ij}\,\mathrm{tr}\left(\bar\phi_i\bar\phi_j\right)\mathrm{tr}\left(\phi^k\phi^l\right), \quad s = \delta_{G,SU(N)}$$

Non-conformal gauge theory ~ Spacetime with closed tachyons

[Spradlin, Takayanagi, Volovich; hep-th/0509036][Dymarsky, Klebanov, Roiban; hep-th/0509132]

Singular wrapping effects

Double-trace coupling in β-deformed, sensitive to U(N) vs SU(N)

$$\delta\mathcal{L} = -s\,\frac{g_{\mathrm{YM}}^2}{2N}\,\mathrm{tr}\left([\bar\phi_j,\bar\phi_k]_*\right)\mathrm{tr}\left([\phi^j,\phi^k]_*\right)$$

The shortest scalar operator is protected for the SU(N) theory

$$\Delta\left[\mathrm{tr}(XZ)\right] = \begin{cases} 0 & \text{for } SU(N_c) \\ \dfrac{\lambda}{2\pi}\,\sin^2(\pi\beta) & \text{for } U(N_c) \end{cases}$$

[Freedman, Gursoy; hep-th/0506128][Penati, Santambrogio, Zanon; hep-th/0506150]

Integrability methods cannot reproduce the SU(N) result for tr(XZ)

- Twisted Asymptotic Bethe Ansatz reproduces U(N) results
- Wrapping corrections diverge both for β and γ

$$\delta\Delta_\beta\left[\mathrm{tr}(XZ^L)\right] = g^{2L+2}\sum_{n=1}^{\lfloor\frac{L+1}{2}\rfloor} b(n,L)\,\zeta(2L-2n+1)$$

[Fokkwn, Sieg, Wilhelm; arXiv:1308.4420, 1312.2959, 1405.6712]
[Ahn, Bajnok, Bombardelli, Nepomechie; arXiv:1108.4914] [de Leeuw, van Tongeren; arXiv:1201.1451]

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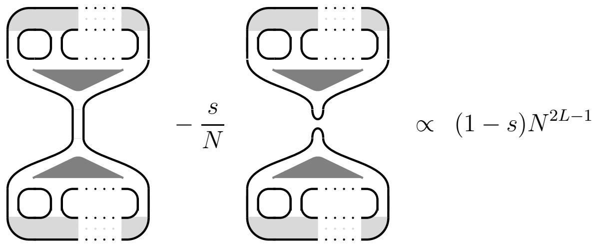
- Wrapping corrections diverge both for β and γ
- Prewrapping effects start at one-loop for tr(XZ)



Finite Nc Spectrum

Finite Nc constraints

Tensors vanish if more than Nc indices are anti-symmetrized

$$\mathbf{0 = T_{[i_1 i_2 \ldots i_{N_c+1}]} \quad (i_k = 1, 2, \ldots, N_c)}$$

Well studied by group-theoretical bases at tree-level:

Sum of multi-trace operators ↔ Set of Young diagrams of height ≦ Nc

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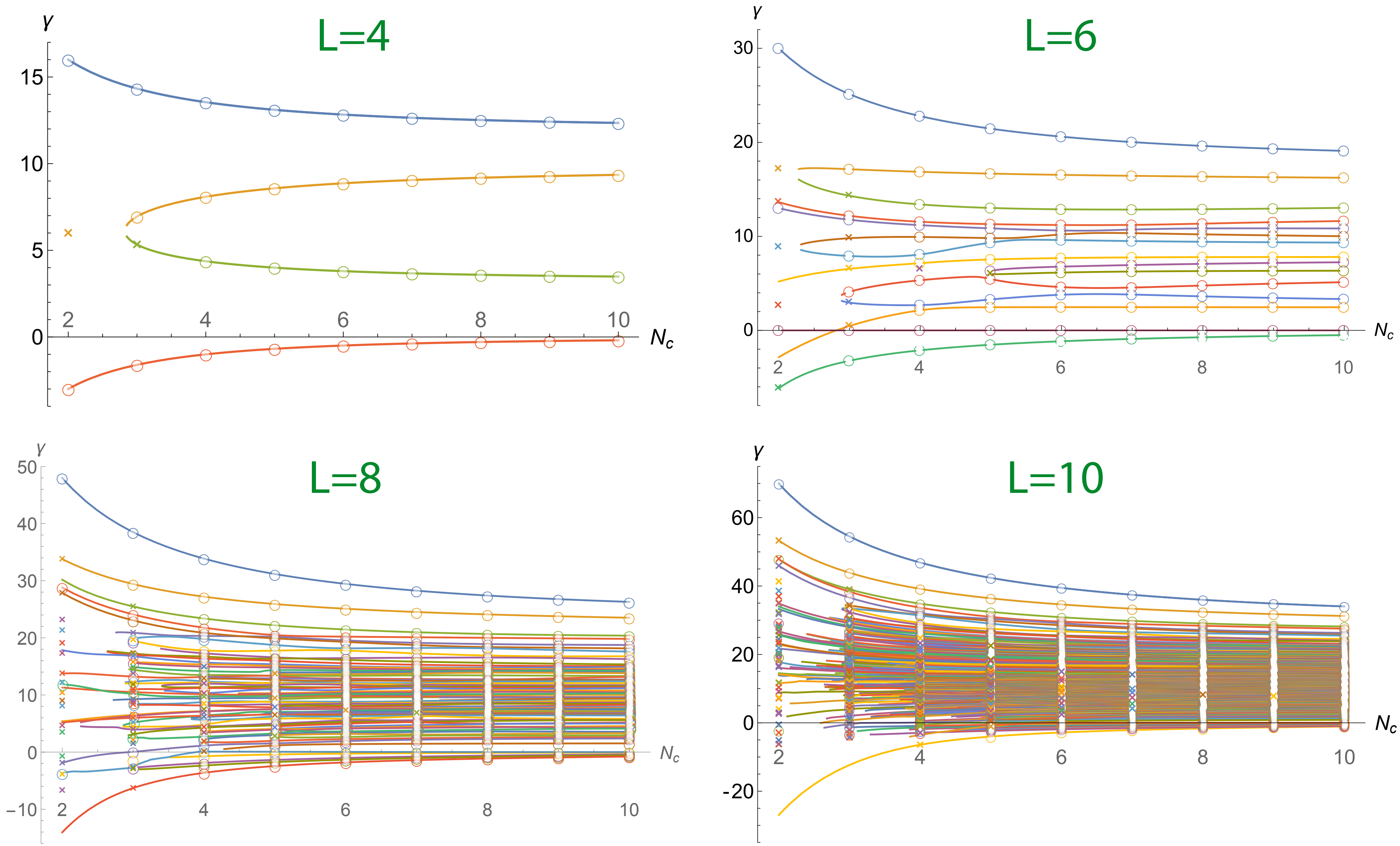
At one-loop, dilatation eigenstates provide a orthonormal basis for any Nc

At finite Nc :	(1) the eigenvalue remains unchanged (2) the eigenvector may become null

$$\mathfrak{D}_{\text{one-loop}}\, \psi_I = \gamma_I\, \psi_I \quad \overset{\text{finite } N_c}{\Longrightarrow} \quad \left\{ \mathfrak{D}_{\text{one-loop}}\, \psi'_I = \gamma_I\, \psi'_I \quad \text{or} \quad \psi'_I = \vec{0} \right\}$$

- Eigenvalues can be complex since the mixing matrix is not Hermitian
- Eigenvector of a complex eigenvalue must be null

Spectral data:

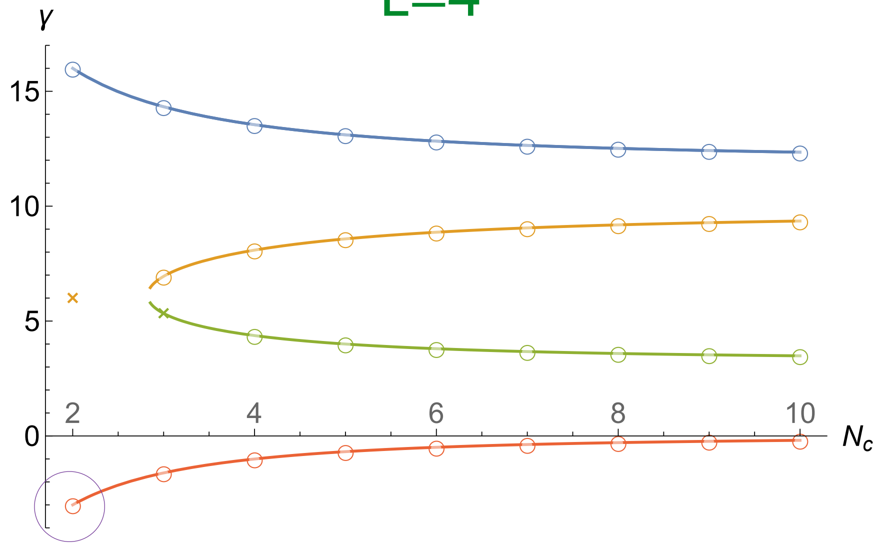


All eigenvalues are real, excluding null eigenvectors denoted by ×

Lowest-dimension operator

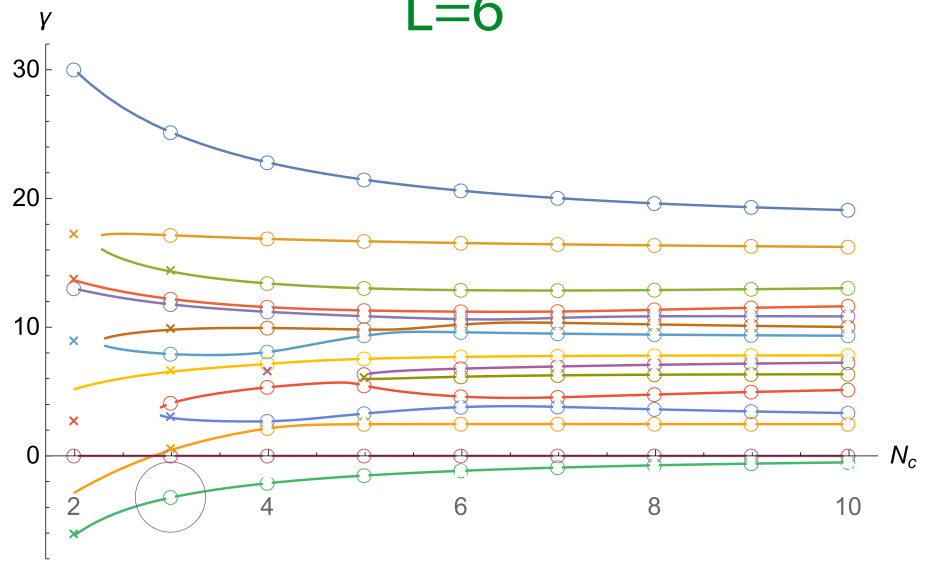
The lowest dimension of so(6) singlets with L=2Nc is negative

L=4



Nc=L/2

L=6



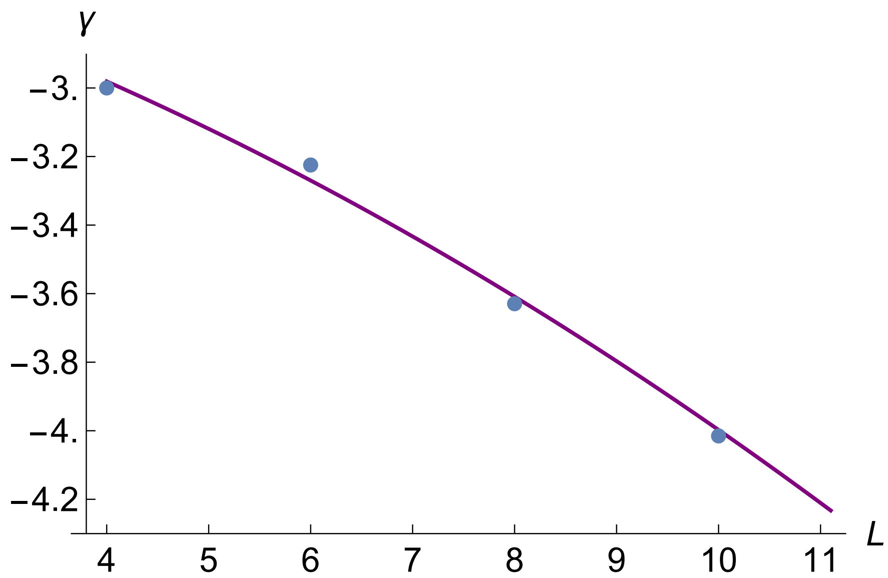
Nc=L/2

~ "double-determinant" operator

Lowest-dimension operator

The lowest dimension of so(6) singlets with L=2Nc is negative

and it seems to diverge as Nc → ∞



$$\Delta = 2N_c - \frac{N_c\, g_{\rm YM}^2}{8\pi^2}\left(2.70242 + 0.015834\, N_c + 0.002515\, N_c^2 + \dots\right) + \dots,$$

- Such operator cannot be studied in the 't Hooft limit
- Dual to a "bound state" of Witten's baryon vertices?

(D5 brane wrapping on S^5) [Witten; hep- th/9805112]

(No) level-crossing

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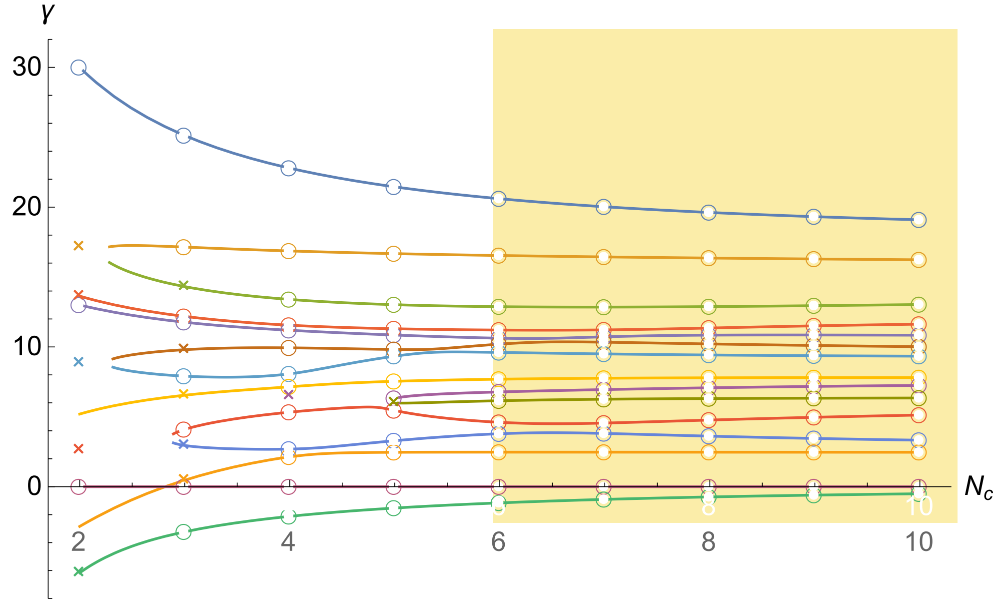
No two energy levels cross for Nc ≥ L

L=6



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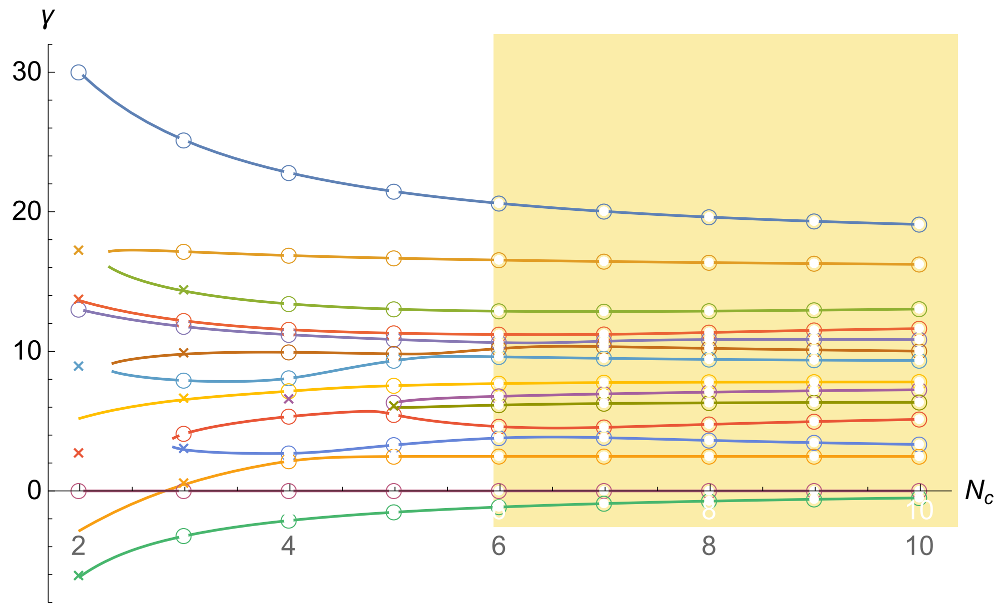

Consistent with non-integrablity according to von Neumann - Wigner

$$\textbf{Diagonalize}\ \ H = \begin{pmatrix} \frac{1}{2}\epsilon + V_{11} & V_{12} \\ V_{21} & \frac{1}{2}\epsilon + V_{22} \end{pmatrix}$$

$$\Rightarrow \quad \gamma_{\pm} = \frac{1}{2}\left(\epsilon + V_{11} + V_{22} + \boxed{\sqrt{(\epsilon + V_{11} - V_{22})^2 + 4\,V_{12}V_{21}}\,}\right) = 0?$$

$$\Rightarrow \quad \textbf{Requires}\ \ \epsilon + V_{11} - V_{22} = 0 \ \ \textbf{and}\ \ V_{12} = V_{21}^{*} = 0 \ \ \textbf{if}\ V\ \textbf{is Hermite}$$

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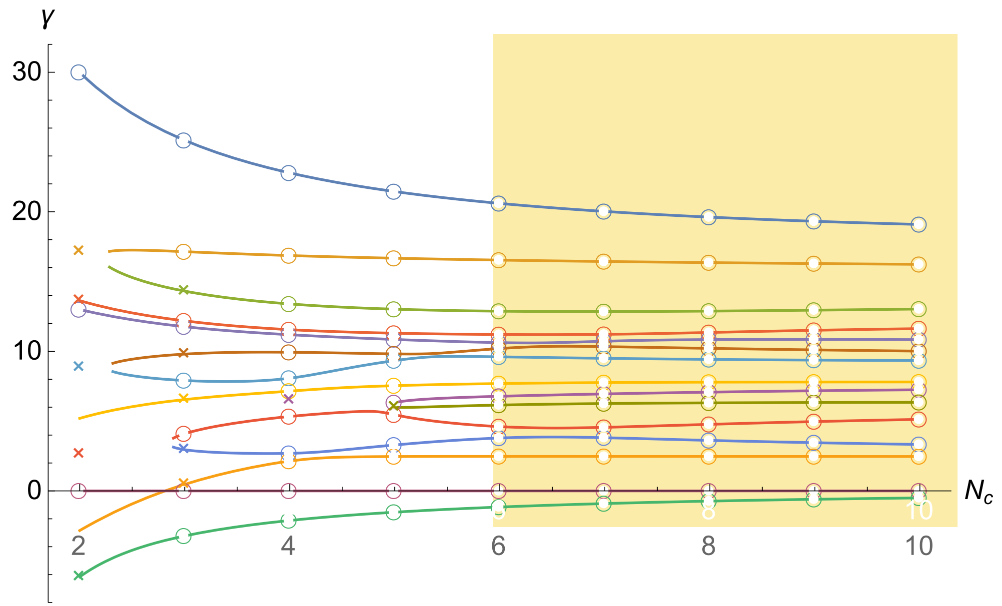

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Energy become complex after collision, and the eigenstates must be null

Eigenvalue curves

How are the small Nc states connected to the large Nc states?

1. Embed the one-loop spectrum of SU(Nc) $\mathcal{N}$=4 SYM to that of (PS)U(Nc+k|k) $\mathcal{N}$=4 SYM at large k

- Non-unitary AdS/CFT via ghost D-branes
- No finite Nc constraints, analytically continuation to Nc<0

[Vafa; arXiv:1409.1603]
[Okuda, Takayanagi, hep-th/0601024]

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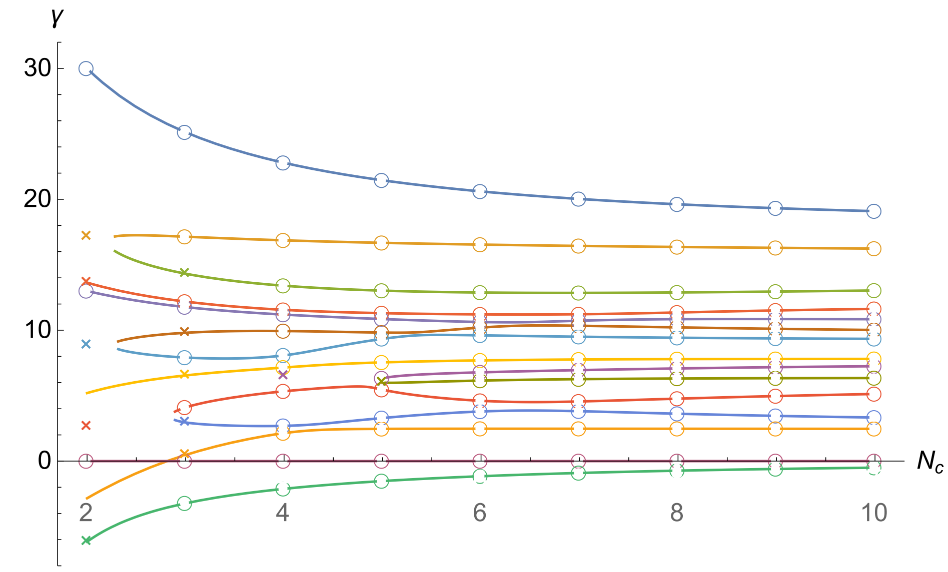

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2. Rescale the one-loop dimension

$$\gamma = \tilde{\gamma}/N_c, \quad \Delta = L + \left(\frac{N_c\, g_{\rm YM}^2}{8\pi^2}\right)\gamma + \dots$$

to represent the symmetry of the characteristic polynomial

$$\mathfrak{P}(\gamma, N_c) = \mathfrak{P}(\gamma, -N_c) \quad \Leftrightarrow \quad \mathfrak{P}\left(\tilde{\gamma}\, N_c, N_c\right) = \mathfrak{P}\left((-\tilde{\gamma})(-N_c), -N_c\right).$$

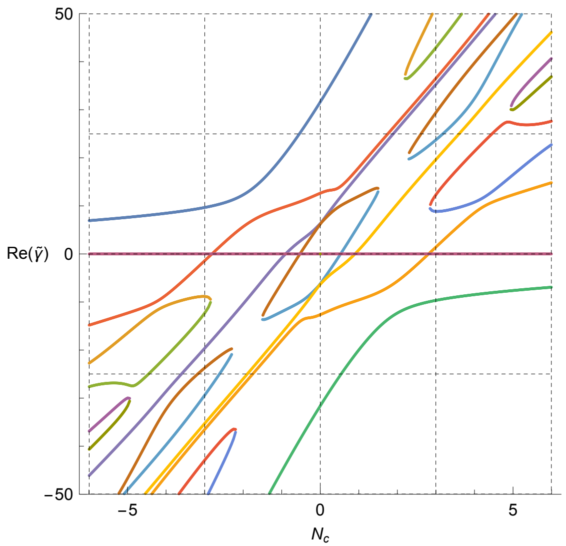
$$\mathfrak{D}_{\rm one-loop} \cdot \mathcal{O}_I = M_{IJ}\, \mathcal{O}_J, \quad \mathfrak{P}(\gamma) \equiv \det(M_{IJ} - \gamma\, \delta_{IJ}) = \prod_a \mathfrak{P}_a(\gamma)$$

Symmetry through non-unitarity

Eigenvalue curves specify an operator/branch modulo ι

$$\iota \; : \; (\tilde{\gamma}, N_c) \; \mapsto \; (-\tilde{\gamma}, -N_c)$$

L=6



Product of Konishi operators, tr($\phi_i \phi_i$)

$$\tilde{\gamma} = 3LN_c + N_c^{-1} L(L-2)/8 + \ldots$$

Large Nc zero-modes of double-trace type

$$\tilde{\gamma} = N_c^{-1}\gamma_2 + \ldots, \quad \gamma_2 < 0$$

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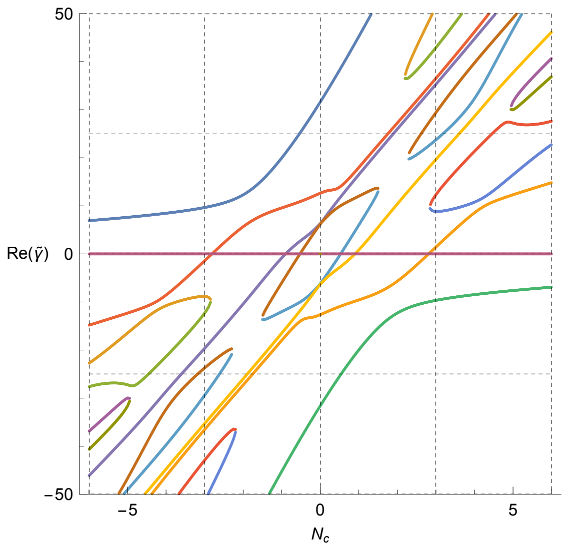

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- Two ways to unitarize the U(K|M) theory: K → 0 or M → 0;
  both reduce to U(Nc) $\mathcal{N}$=4 SYM, but in different ways
- High-energy states are paired with low-energy states,
  “duality” among multi-trace or multi-string states