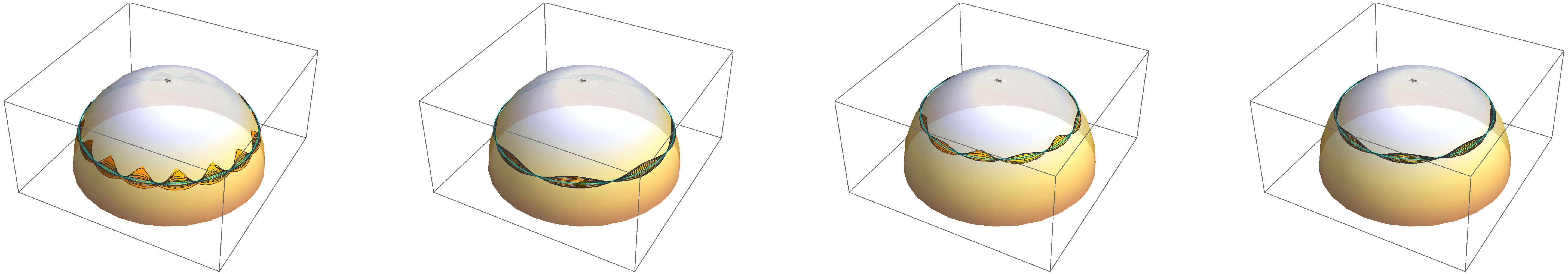
# Oscillating Multiple Giants




Ryo Suzuki

(Shing-Tung Yau Center of Southeast University)



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# Plan of Talk

1. Motivation

2. Finite group methods

3. Weak coupling

4. Strong coupling

5. Summary and Outlook

# Motivation

# AdS/CFT and integrability

AdS/CFT is usually a conjecture in the planar large $\boldsymbol{Nc}$ limit

Maximally supersymmetric theories in this limit are "integrable"

**$\mathcal{N}=4$ super Yang-Mills in $D$=4**

$$\lambda = N_c\, g_{\rm YM}^2$$

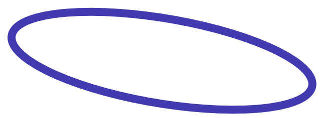
**Single-trace operators with R-charge $L$**

$$\mathrm{tr}\,(Z^{L-M} Y^M) + \dots$$

**Superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$**

$$\lambda = \frac{R^4}{\alpha'^2} = \frac{N_c\, g_s}{4\pi}$$

**Strings with angular momentum $L$**



Planar large $N_c$ limit : $N_c \gg L \gg 1$

Computations reformulated as integrable system, predicting

**Asymptotic term + Wrapping corrections (usually negligible if $L \gg 1$ or $\lambda \ll 1$)**

# AdS/CFT and integrability

The integrability prediction is exact in $\lambda$, but has limitations due to the planarity:

- Integrability seems lost on the non-planar level — [Beisert, Kristjansen, Staudacher (2003)] + many more
- Hard to solve the multi-trace mixing problem — [Bellucci, Casteill, Morales, Sochichiu (2004))] + many more
- Even the $1/N_c$ corrections to the BPS 4pt are complicated — [Bargheer, Caetano, Fleury, Komatsu, Vieira (2017,2018)]

Difficult because non-planar effects ~ string coupling (quantum gravity) corrections

Possible directions:

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Possible directions:

- ☆ Refine perturbative computations of $\mathcal{N}$=4 SYM, or $AdS_5 \times S^5$ string
- ☆ Look for hints from other approaches (bootstrap, localization, …)
- ☆ Study non-planar large $N_c$ limits

# AdS/CFT at non-planar large $N_c$

Non-planar large $N_c$ limit : $L \gtrsim N_c \gg 1$

**Operators with huge dimensions**

$L = O(N_c^1)$

$L \geq O(N_c^2)$

**Deforming $\mathrm{AdS}_5 \times \mathrm{S}^5$ background**

**Giant gravitons (D-branes)**

**LLM geometry**

This setup is generally not integrable. An exception is

Single trace + determinant-like operator ⟷ Open strings ending on a single D-brane

↘ ↙

Spin chain with integrable boundaries

[Hofman, Maldacena] (2007)
and many more

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Generalize these objects by using finite group methods (not integrability)

# AdS/CFT at non-planar large $N_c$

Non-planar large $\boldsymbol{N_c}$ limit : $\boldsymbol{L \gtrsim N_c \gg 1}$

| Operators with huge dimensions | Deforming $\mathbf{AdS_5 \times S^5}$ background |
| --- | --- |
| $L = O(N_c^1)$ | Giant gravitons (D-branes) |
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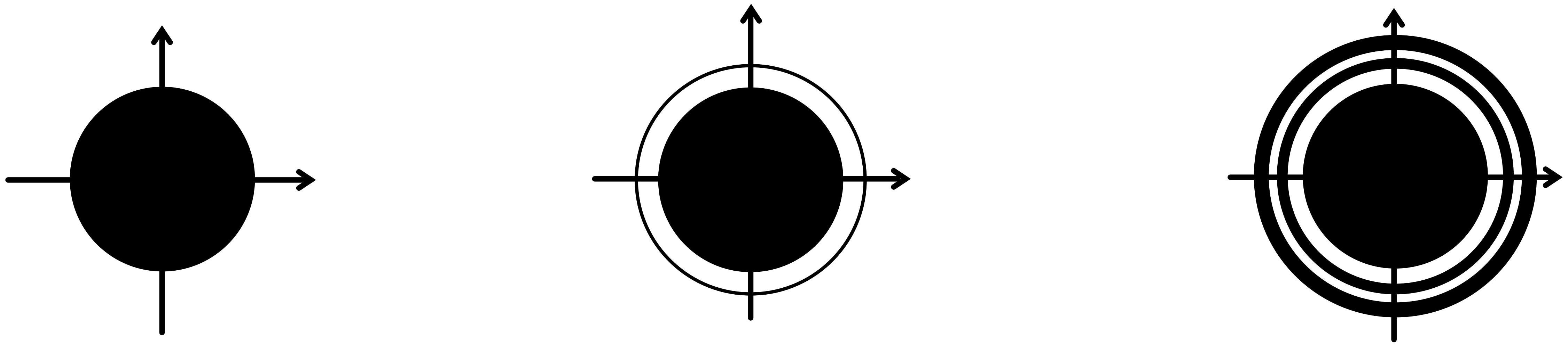
# LLM geometry

General half-BPS regular solutions of IIB supergravity with the residual symmetry $\mathfrak{psu}(2|2)^2$

$$ds^2 = -2y\cosh G\left(dt + V_i\, dx^i\right)^2 + \frac{dy^2 + (dx^i)^2}{2y\cosh G} + ye^{G} d\Omega_{S^3} + ye^{-G} d\Omega_{\tilde{S}^3}$$

Droplet pattern: $\tanh G\,(x^1, x^2, y = 0) = \pm 1,$ either $S^3$ or $\tilde{S}^3$ collapses



$AdS_5 \times S^5$ Giant graviton Concentric circles

# Schur operators

- General multi-trace half-BPS operators of $\mathcal{N}$=4 SYM $$\prod_i \mathrm{tr}\, Z^{n_i} = \mathrm{tr}_L(\alpha Z^{\otimes L})$$

$$\text{Multi-trace structure of } \underline{\mathrm{tr}_L(\alpha Z^{\otimes L})} \leftrightarrow \text{Cycle type of } \alpha \in S_L$$

- Organize multi-trace operators into the basis labeled by a Young diagram $R$

$$\mathcal{O}^R(Z) = \frac{1}{L!} \sum_{\alpha \in S_L} \chi^R(\alpha)\, \underline{\mathrm{tr}_L\left(\alpha Z^{\otimes L}\right)}$$

$$\chi^R(\alpha) = S_L \text{ character of irrep } R$$

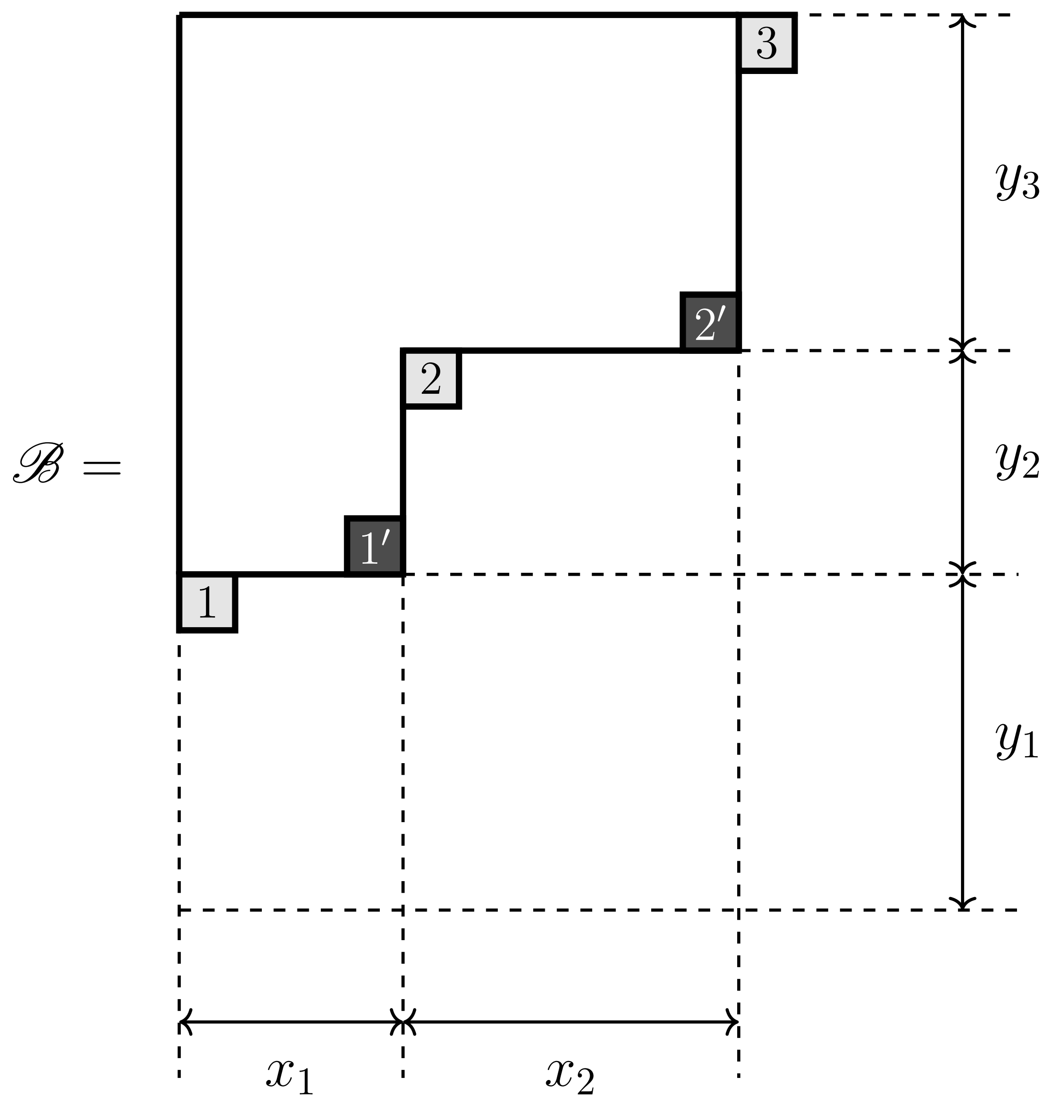
$$Z = \mathrm{diag}(z_1, z_2, \ldots, z_{N_c}) \Rightarrow \mathcal{O}^R(Z) = \text{Schur polynomial of } \{z_i\}$$

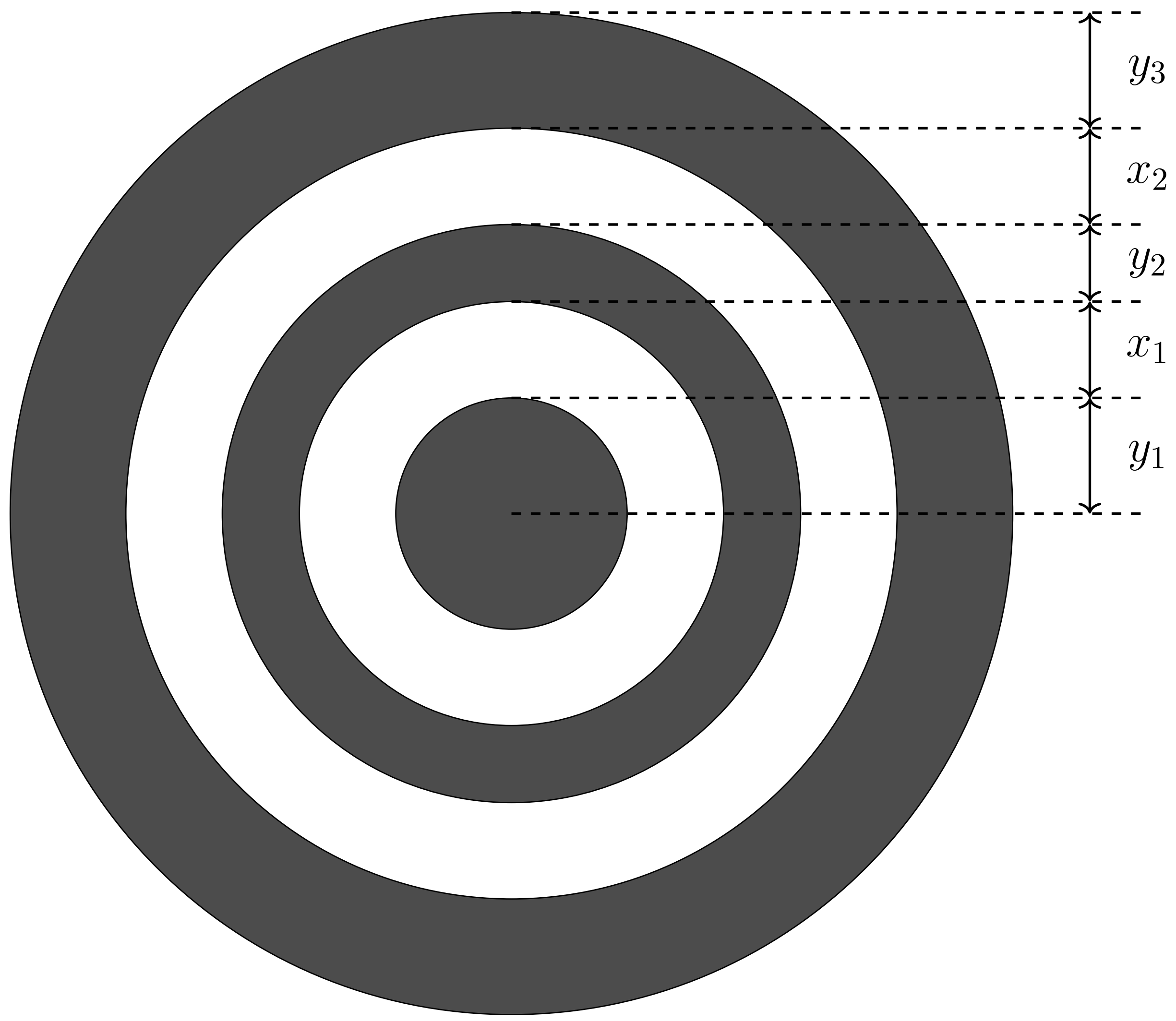
[Corley, Jevicki, Ramgoolam (2001)]

# LLM/Schur as AdS/CFT

[Lin, Lunin, Maldacena] (2004)

Half-BPS states labeled by Young diagram $\mathcal{B}$ = Concentric droplets in supergravity




Young diagram is huge : edge lengths $\boldsymbol{x_i}$ , $\boldsymbol{y_i}$ are order $\boldsymbol{N_c}$

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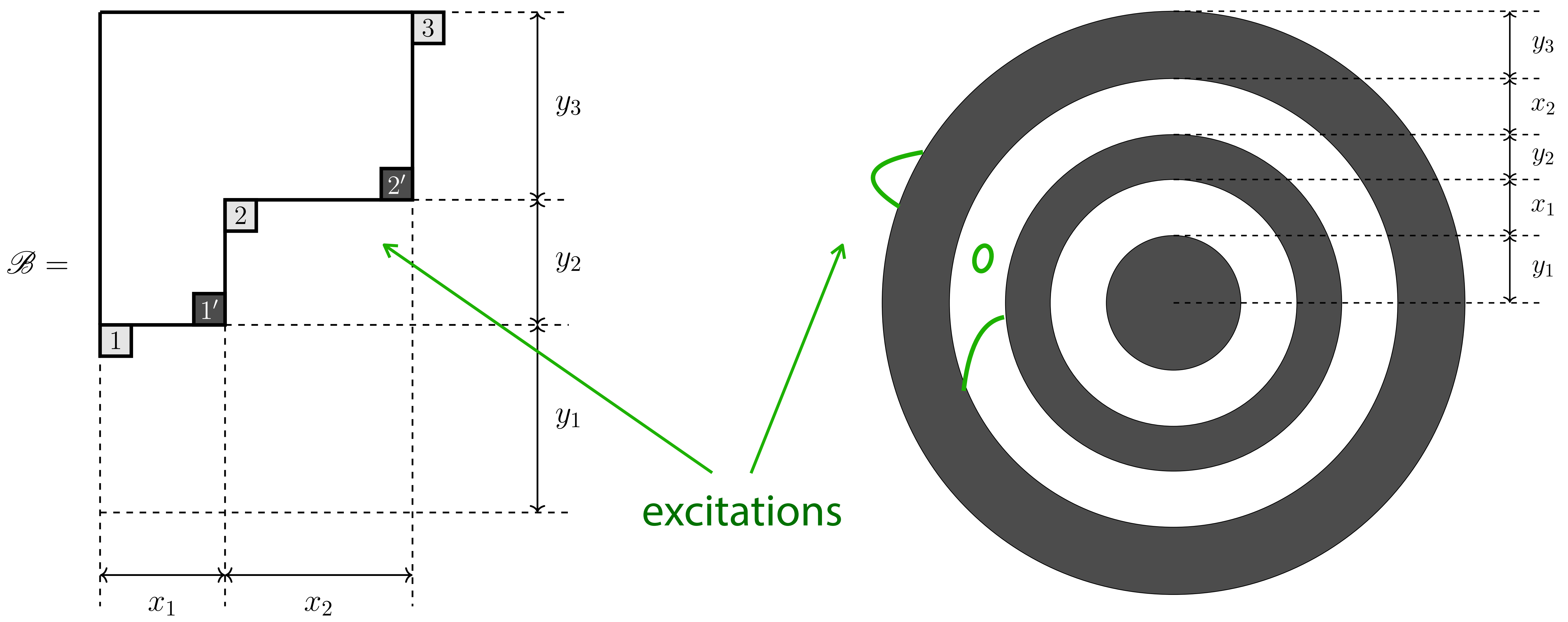
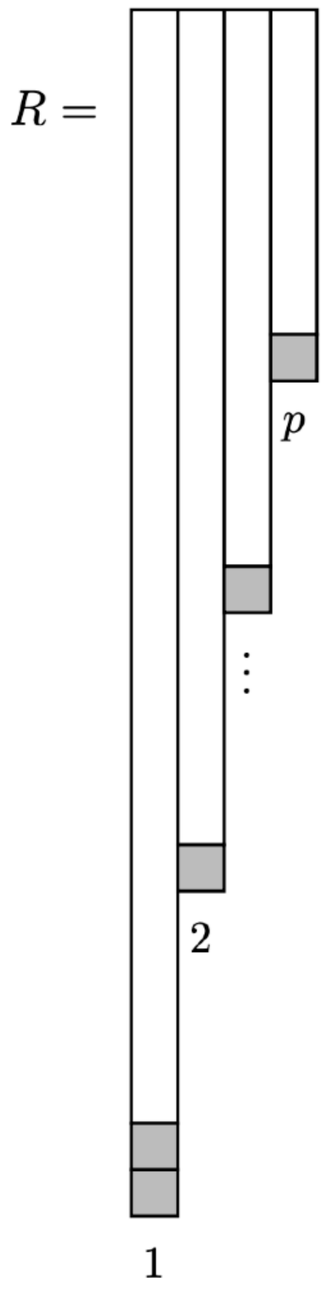


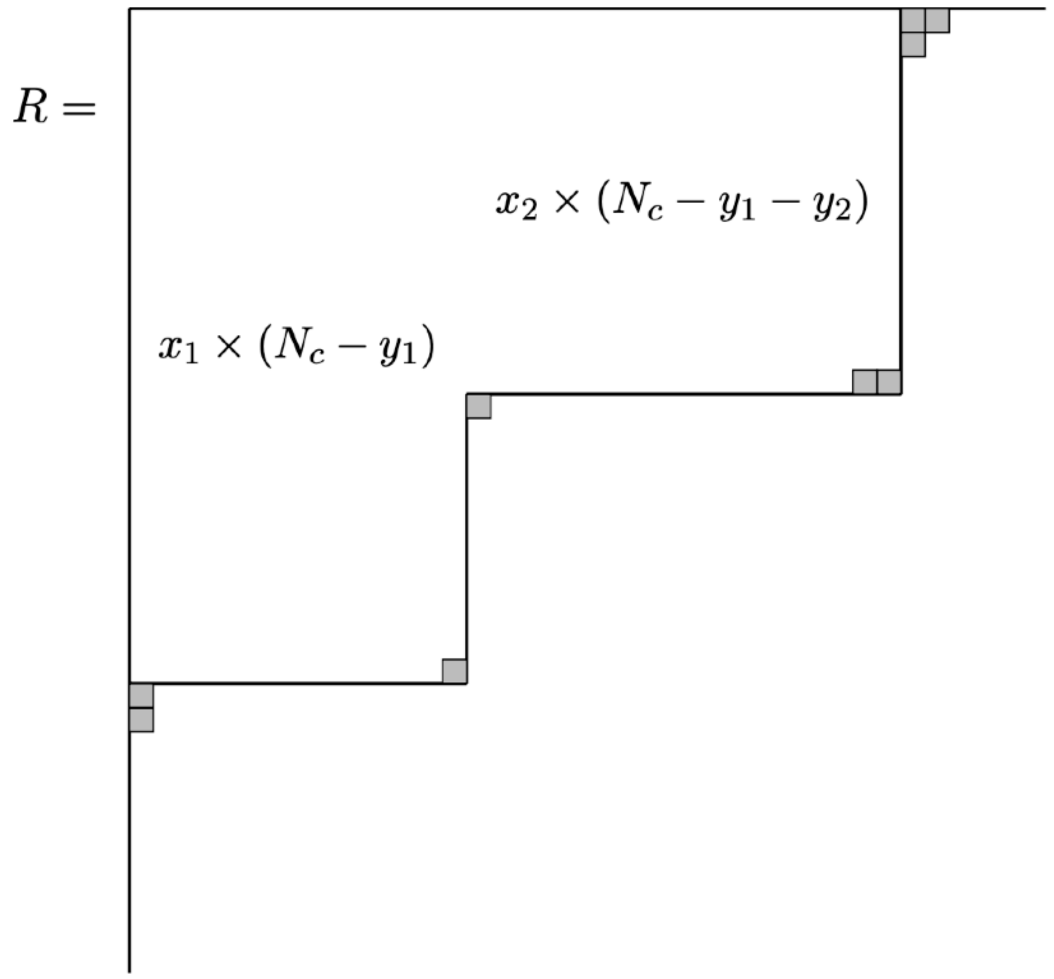

Young diagram is huge : edge lengths $x_i$ , $y_i$ are order $N_c$

# Two types of non-BPS huge operators

Operators dual to multi giant gravitons
~ Young diagram $R$ with $p$ long columns



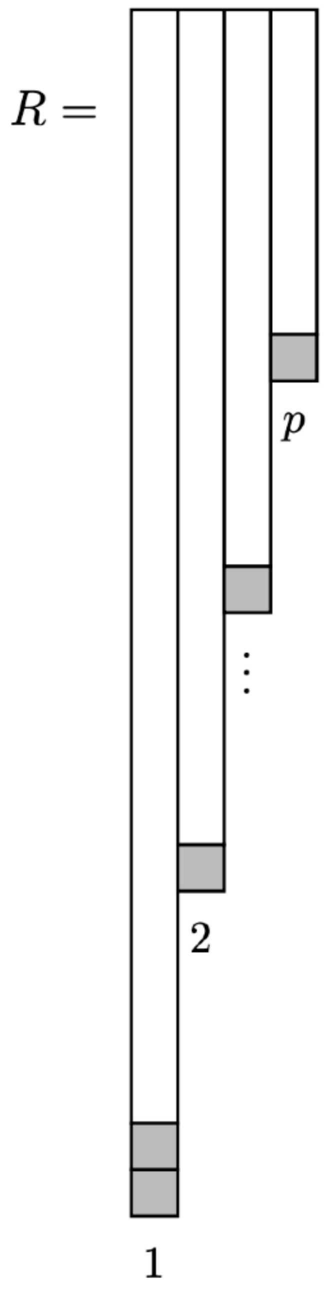

Operators dual to LLM geometry
~ Young diagram $R$ with big blocks




# (white / gray boxes) = # ($Z$ / $Y$)
for $O_R$ = Sum of multi-traces ($ZZZ \ldots YY \ldots$)

# Two types of non-BPS huge operators

Operators dual to multi giant gravitons
~ Young diagram $\boldsymbol{R}$ with $\boldsymbol{p}$ long columns




- Operators having different Young diagrams $\boldsymbol{R}$ mix under renormalization
- One-loop mixing at large $\boldsymbol{Nc}$ takes a simple form
- Mixing of gray boxes (excitations) is diagonalized by Gauss graph basis [de Mello Koch, Ramgoolam] (2012)
- Mixing of white boxes (background) gives a set of coupled harmonic oscillators

$$\boldsymbol{D - J \sim - \sum_{\substack{i,j=1 \\ i \neq j}}^{p} \underbrace{n_{ij}(\sigma)}_{\text{Non-negative integers}} \underbrace{\Delta_{ij}}_{\text{Difference operators}}}$$

# An AdS/CFT proposal

We *propose* an all-loop ansatz for the $p = 2$ excited (spherical) giant gravitons

$$\Delta - J = f_1(\lambda)\, n_{12}\, \frac{m}{N_c}\,, \quad \left(1 \leq m \leq \left\lceil N_c - \frac{n_Z}{2} + 1 \right\rceil\right), \quad f_1(\lambda) = \frac{\lambda}{\pi^2} + O(\lambda^2)$$

[RS (2021)]

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- The dispersion relation is gapless at large $Nc$
- Dual to the classical motion of oscillating D-branes (not open strings)
- … but there are critical assumptions / caveats which need to be justified

[RS (2021)]

# Why important?

1. Possible "non-planar integrability" at large $N_c$

| Planar large $N_c$ | Non-planar large $N_c$ |
| --- | --- |
| Spectrum of string motions | Spectrum of D-brane motions |
| Yangian (quantum) | Finite group (classical) |

Is the $psu(2|2)^2$ symmetry centrally extended again?

2. Possible relation to the "TTbar deformation" of $\mathcal{N}$=4 SYM / $AdS_5 \times S^5$

- TTbar deformation in $D$=2 is exactly solvable, giving a square-root dispersion
- TTbar deformation in $D$=4 also preserves the $psu(2|2)^2$ symmetry

[Zamolodchikov, (2004)] [Caetano, Peelaers, Rastelli (2020)]

# Finite group methods

# Schur-Weyl duality

Let $V$ be a fundamental representation of $\mathbf{U(N)}$

$$V^{\otimes L} = \bigoplus_{R \vdash L} V_R^{U(N)} \otimes V_R^{S_L}$$

= duality between the Lie group $\mathbf{U(N)}$ and the permutation group $S_L$

Examples at $L = 2$ : $\quad$ Assume $\psi^i \in V, \; (i = 1, 2, \ldots, N)$

$$\psi_1^i \psi_2^j = \psi_1^{(i} \psi_2^{j)} + \psi_1^{[i} \psi_2^{j]}$$

$$N^2 = \frac{N(N+1)}{2} + \frac{N(N-1)}{2}$$

## **Example at $L = 3$ :**

$$\phi_{i_1}(x_1)\,\phi_{i_2}(x_2)\,\ldots\,\phi_{i_L}(x_L) \equiv |1,2,\ldots,L\rangle\,, \quad (i_k = 1,2,\ldots,N, N \geq L)$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} = \frac{1}{\sqrt{6}}\left(|123\rangle + |231\rangle + |312\rangle + |132\rangle + |321\rangle + |213\rangle\right)$$

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} = \frac{1}{\sqrt{6}}\left(|123\rangle + |231\rangle + |312\rangle - |132\rangle - |321\rangle - |213\rangle\right)$$

$$\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\right)_1 = \frac{1}{\sqrt{12}}\left(2\,|123\rangle + 2\,|213\rangle - |321\rangle - |312\rangle - |132\rangle - |231\rangle\right)$$

$$\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\right)_1 = \frac{1}{2}\left(|132\rangle + |231\rangle - |321\rangle - |312\rangle\right)$$

irrep

$$\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\right)_2 = \frac{1}{2}\left(|132\rangle + |312\rangle - |321\rangle - |231\rangle\right)$$

$$\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\right)_2 = \frac{1}{\sqrt{12}}\left(2\,|123\rangle - 2\,|213\rangle + |321\rangle - |312\rangle + |132\rangle - |231\rangle\right)$$

irrep

multiplicity two

# Schur-Weyl duality

Counting the dimensions from the Schur-Weyl duality

$$\dim V^{\otimes L} = \bigoplus_{R \vdash L} \dim\left(V_R^{U(N)} \otimes V_R^{S_L}\right)$$

$$\Rightarrow \quad N^L = \sum_{R \vdash L} \underline{\mathrm{Dim}_N(R)}\, \underline{d_R}$$

dimension as $\mathrm{U(N)}$ rep    dimension as $S_L$ rep

General formula to count powers of $N$

$$\underline{N^{C(\alpha)}} = \sum_{R \vdash L} \mathrm{Dim}_N(R)\, \underline{\chi^R(\alpha)}$$

number of cycles in $\alpha$    $S_L$ character

# Application: multi-trace 2pt

Denote a multi-trace operator by

$$\mathrm{tr}_L\left(\alpha Z^{\otimes L}\right) \equiv \sum_{i_1,i_2,\ldots,i_L=1}^{N_c} Z^{i_1}_{i_{\alpha(1)}} Z^{i_2}_{i_{\alpha(2)}} \ldots Z^{i_L}_{i_{\alpha(L)}}$$

The tree-level two-point function of $\mathbf{U(N)}$ $\mathcal{N}$=4 SYM is

$$\left\langle \mathrm{tr}_L\left(\alpha Z^{\otimes L}\right)(x)\;\mathrm{tr}_L\left(\beta \overline{Z}^{\otimes L}\right)(0)\right\rangle = |x|^{-2L} \sum_{\sigma\in S_L} N_c^{C(\alpha\sigma\beta\sigma^{-1})}$$

$$= |x|^{-2L} \sum_{R\vdash L} \sum_{\sigma\in S_L} \mathrm{Dim}_{N_c}(R)\,\chi^R(\alpha\sigma\beta\sigma^{-1})$$

$\mathbf{U(N)}$ ←- - - - - - - - - - - - - - - - -→ $S_L$

*duality*

# Representation matrices

Young tableaux = Numbers filled in a Young diagram in the standard way

$$R = \begin{array}{|c|c|c|}\hline \phantom{1} & \phantom{1} & \phantom{1} \\ \hline \phantom{1} & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad \Rightarrow \quad \left\{ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \right\}, \quad (i = 1, 2, \ldots, d_R)$$

Determine the matrix elements of irreducible representations: $D^R_{ij}(\sigma)$ for $\sigma \in S_L$

Decompose the permutation as a product of transpositions: $\{(1,2),\ (2,3), \ldots (L-1, L)\}$

Young-Yamanouchi form

$$D^R\Big((a, a+1)\Big)\ |R, i\rangle = \frac{1}{d_{a,a+1}}\ |R, i\rangle + \sqrt{1 - \frac{1}{d^2_{a,a+1}}}\ |R, (a, a+1)i\rangle$$

$$d_{a,a+1} = c_i(a+1) - c_i(a), \quad c_i(a) = N_c + x - y, \quad a\text{-th box sits at } (x, y) \text{ of } |R, i\rangle$$

# Restricted Schur operators

Multi-trace operators of length $L$ in the su(2) sector of $\mathcal{N}$=4 SYM

$$\mathrm{tr}_L\left(\alpha\cdot Z^{\otimes n_Z}Y^{\otimes n_Y}\right)\equiv\sum_{i_1,i_2,\ldots,i_L=1}^{N_c}Z^{i_1}_{i_{\alpha(1)}}Z^{i_2}_{i_{\alpha(2)}}\ldots Z^{i_{n_Z}}_{i_{\alpha(n_Z)}}Y^{i_{n_Z+1}}_{i_{\alpha(n_Z+1)}}Y^{i_{n_Z+2}}_{i_{\alpha(n_Z+2)}}\ldots Y^{i_L}_{i_{\alpha(L)}}$$

$$\alpha\in S_L\quad(L=n_Y+n_Z)$$

The restricted Schur operators give a basis of multi-trace operators (diagonal at tree-level)

$$\mathcal{O}^{R,(r,s),\nu_+,\nu_-}=\frac{1}{n_Z!\,n_Y!}\sum_{\alpha\in S_L}\underbrace{\chi^{R,(r,s),\nu_+,\nu_-}(\alpha)}_{\text{Restricted Schur character}}\mathrm{tr}_L\left(\alpha\cdot Z^{\otimes n_Z}Y^{\otimes n_Y}\right)$$

Restricted Schur character

$r$ ~ irrep for $Z$, $s$ ~ irrep for $Y$, $R$ ~ product of $(r, s)$

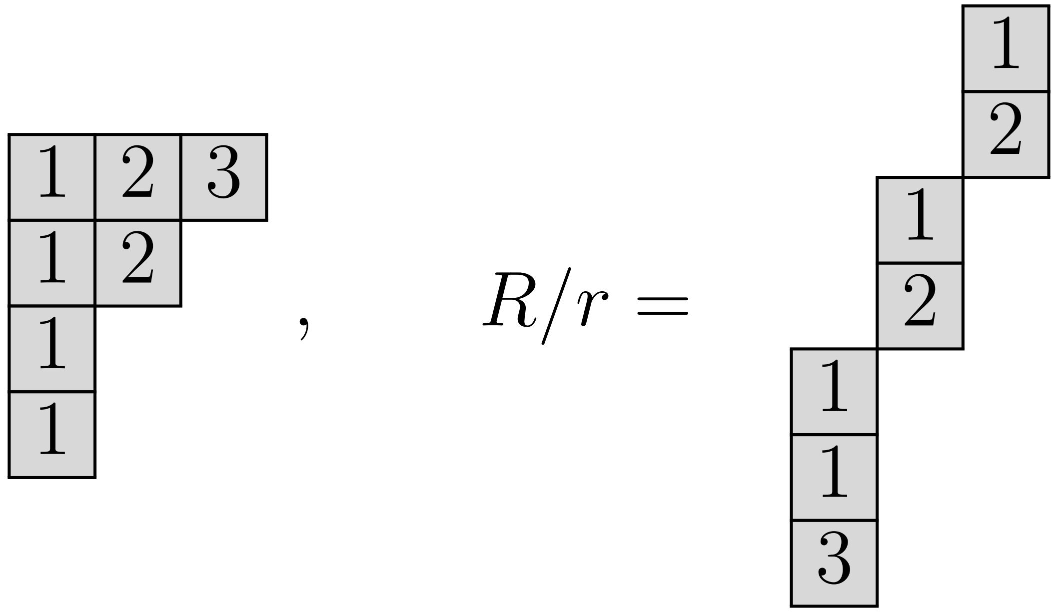
# Excitation labels

- Two ways to specify the representation of $\boldsymbol{Y}$'s

  An example at $\boldsymbol{p=3}$,

  $$\boldsymbol{s = (4,2,1), \quad R/r = (3,2,2)}$$

$s =$ , $R/r =$




- Adjacency matrix

  $$\boldsymbol{n_{i\to j} = (\text{how many } i\text{'s appear in the } j\text{-th column of } R/r)}$$

$$\{\boldsymbol{n_{i\to j}}\} = \begin{pmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

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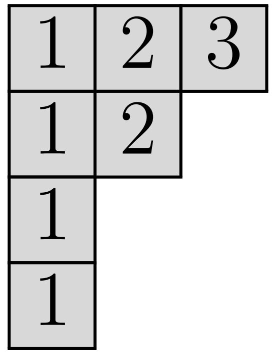
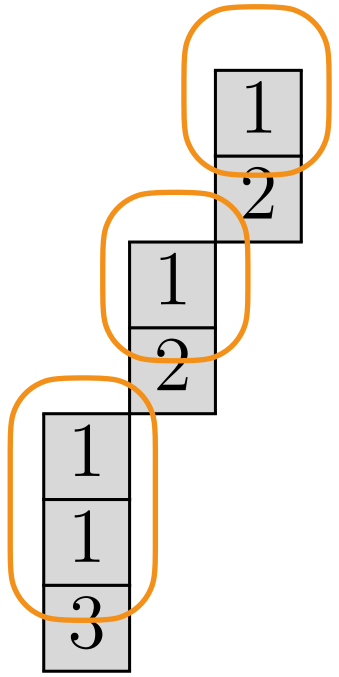
$s =$  , $R/r =$ 

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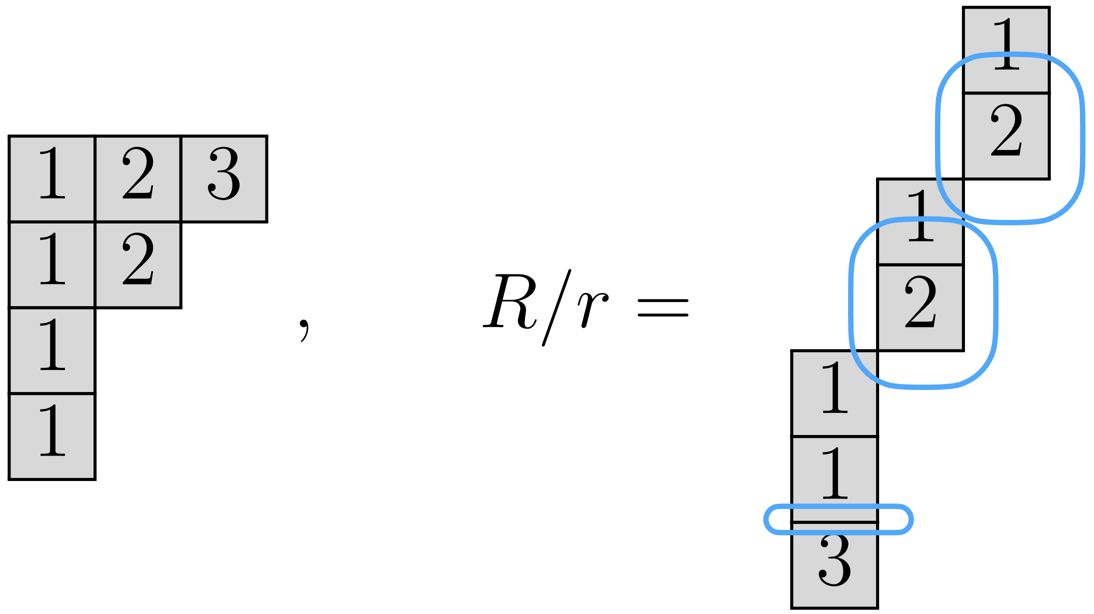

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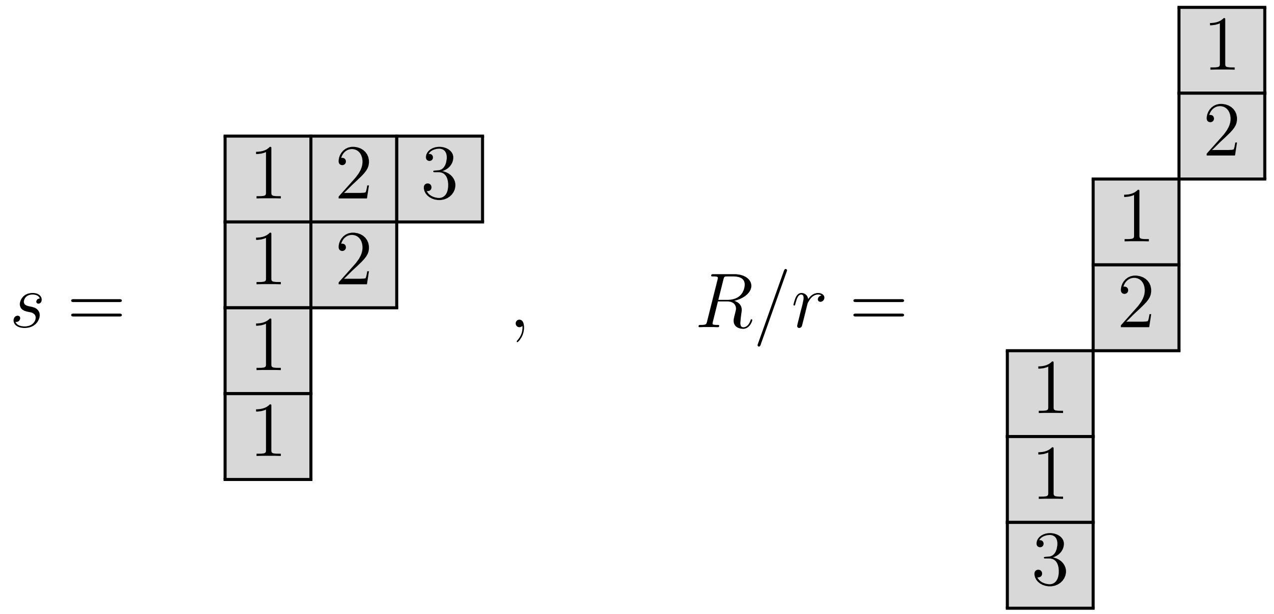


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The column label ($\mathbf{1,2}, \ldots, \boldsymbol{p}$) becomes important

# Gauss graph basis

- Take a "good" linear combination of the restricted Schur operators

$$O^{R,r}(\sigma) = |H|\sqrt{n_Y!}\sum_{j,k}\sum_{s\vdash n_Y}\sum_{\nu_-,\nu_+} D^s_{jk}(\sigma)\, \underbrace{B_j^{s\to 1_H,\nu_-}\; B_k^{s\to 1_H,\nu_+}}_{\text{to symmetrize } Y\text{'s within the same column}} \mathcal{O}^{R,(r,s),\nu_+,\nu_-}$$

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to symmetrize $Y$'s within the same column

- Simplified notation for a fixed σ

$$\mathcal{O}(\vec{\ell}) \quad \text{for} \quad r = (l_1, l_2, \ldots, l_p) \vdash n_Z\,, \quad n_{ij}(\sigma) = n_{i\to j} + n_{j\to i}$$

adjacency matrix

- Adjacency matrix satisfies Gauss graph constraints

  (= conservation of the number of arrowheads, "in" and "out")

$$\sum_{j=1}^{p} n_{i\to j} = \sum_{i=1}^{p} n_{i\to j}$$

# Weak coupling

# Dilatation operators (1)

Expand the dilatation operator of $\mathcal{N}$=4 SYM at weak coupling

$$D(g_{\mathrm{YM}}) = \sum_{\ell=0}^{\infty} \left( \frac{g_{\mathrm{YM}}}{4\pi} \right)^{2\ell} D_\ell$$

Introduce three different expressions of the dilatation in the su(2) sector

1. In terms of $\mathcal{N}$=4 SYM fields,

$$D_1 = -2 : \mathrm{Tr}\, [Y, Z][\check{Y}, \check{Z}] :$$

satisfying $\mathrm{U}(N_c)$ Wick contraction rule:

$$\mathrm{Tr}\,(A\check{\Phi}B\Phi) = \mathrm{Tr}\,(A)\,\mathrm{Tr}\,(B), \quad \mathrm{Tr}\,(A\check{\Phi})\,\mathrm{Tr}\,(B\Phi) = \mathrm{Tr}\,(AB), \quad \mathrm{Tr}\,(1) = N_c$$

# Dilatation operators (2)

2. In terms of the restricted Schur basis (properly normalized), with $(n, m) = (n_Z, n_Y)$

$$D_1\, O^{R,(r,s)jk}(Z,Y) = -2 \sum_{T,(t,u)lq} N^{R,(r,s)jk}_{T,(t,u)lq}\, O^{T,(t,u)lq}(Z,Y)$$

$$N^{R,(r,s)jk}_{T,(t,u)lq} = \sum_{R'} \sqrt{\frac{f_R\, f_T}{f_{R'}\, f_{T'}}}\, \frac{n\, m}{(n+m)} \sqrt{\frac{d_R\, d_T}{d_r\, d_s\, d_t\, d_u}} \times$$
$$\frac{1}{d_{R'}} \mathrm{Tr}\Big( \big[ D^R((1, m+1)), P_{R\to(r,s)jk} \big] I_{R'T'} \big[ D^T((1, m+1)), P_{T\to(t,u)ql} \big] I_{T'R'} \Big)$$

- This expression is very complicated, but exact in $N_c$
- The $k$-loop dilatation $D_k$ moves $k$ boxes of the Young diagrams

# Dilatation operators (2)

2. In terms of the restricted Schur basis (properly normalized), with $(n, m) = (n_Z, n_Y)$

$$D_1\, O^{R,(r,s)jk}(Z,Y) = -2 \sum_{T,(t,u)lq} N^{R,(r,s)jk}_{T,(t,u)lq}\, O^{T,(t,u)lq}(Z,Y)$$

Product of box weights ($N_c$ dependent)

$$N^{R,(r,s)jk}_{T,(t,u)lq} = \sum_{R'} \sqrt{\boxed{\frac{f_R\, f_T}{f_{R'}\, f_{T'}}}}\, \frac{n\, m}{(n+m)} \sqrt{\frac{d_R\, d_T}{d_r\, d_s\, d_t\, d_u}} \times$$

$$\frac{1}{d_{R'}} \mathrm{Tr}\Big( \big[ \underbrace{D^R((1, m+1))}_{\text{permute } Y \text{ and } Z}, \underbrace{P_{R\to(r,s)jk}}_{\text{projector}} \big] \underbrace{I_{R'T'}}_{\text{intertwine } R' \text{ and } T'} \big[ D^T((1, m+1)), P_{T\to(t,u)ql} \big] I_{T'R'} \Big)$$

permute $Y$ and $Z$ — projector — intertwine $R'$ and $T'$

$R'$ = (one box removed from $R$) = $T'$

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$$\frac{1}{d_{R'}} \mathrm{Tr}\Big(\Big[D^R((1,m+1)), P_{R\to(r,s)jk}\Big] I_{R'T'} \Big[D^T((1,m+1)), P_{T\to(t,u)ql}\Big] I_{T'R'}\Big)$$

$\sim \mathrm{O}(1)$

- When $R \sim T \sim r \sim t$ are "big" and $s \sim u$ are "small", most factors cancel
- If we move a box over a "long" distance, the commutator terms almost vanish

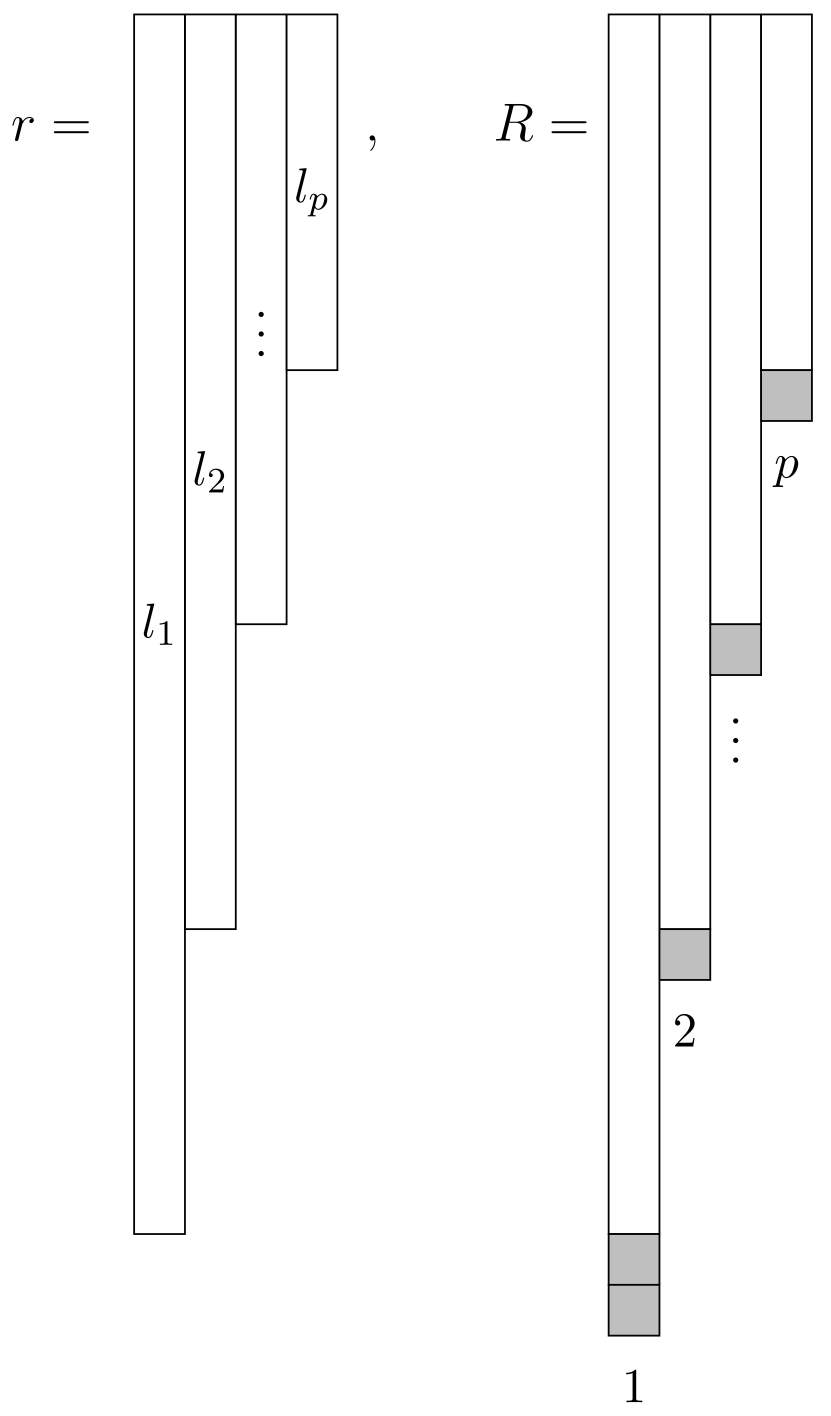
# Distant corners approximation

Consider the case $\boldsymbol{n_Z}$ = O($\boldsymbol{N_c}$) ≫ 1 and $\boldsymbol{n_Y}$ = O(1);

$\boldsymbol{r}$ has $\boldsymbol{p}$ long columns, $\boldsymbol{l_i}$ = length of the $\boldsymbol{i}$-th column;

The corners of $\boldsymbol{r}$ are separated by long distances

$$\boldsymbol{n_Z = \sum_{i=1}^{p} l_i = O(N_c) \gg 1, \quad l_i - l_{i+1} \gg 1}$$



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$$n_Z = \sum_{i=1}^{p} l_i = O(N_c) \gg 1, \quad l_i - l_{i+1} \gg 1$$

Approximation has two consequences:

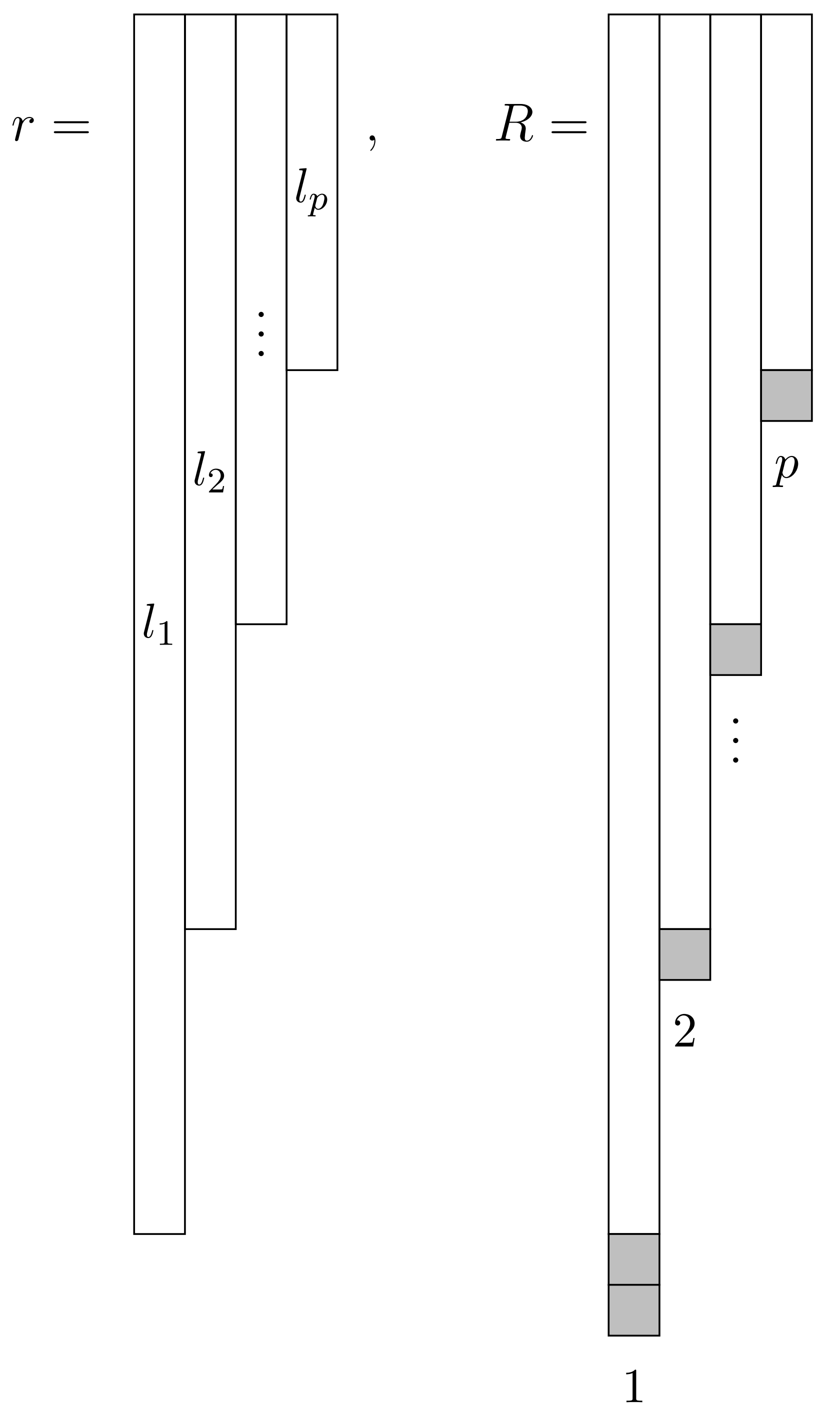
1. Truncation of mixing matrix

   Moving a box to the $(p+1)$-th column is negligible

   → Hamiltonian of an effective $\mathrm{U}(p)$ theory

2. Large $Nc$ continuum limit

   The Hamiltonian becomes a differential operator



# Dilatation operators (3)

3. Acting on of the Gauss graph basis, after distant corners approximation (before continuum limit)

$$D^G(g_{\mathrm{YM}}) = \sum_{\ell=0}^{\infty} \left(\frac{g_{\mathrm{YM}}}{4\pi}\right)^{2\ell} D^G_\ell$$

$$D^G_1 = -\sum_{i \neq j=1}^{p} \underline{n_{ij}(\sigma)}\, \underline{\Delta^{(1)}_{ij}}$$

$$D^G_2 = -\sum_{i \neq j=1}^{p} \underline{n_{ij}(\sigma)} \left\{ (L - 2N_c)\, \underline{\Delta^{(1)}_{ij} + \Delta^{(2)}_{ij}} \right\}$$

adjacency matrix related to $Y$

difference operators related to $Z$

We call $D^G$ the Hamiltonian of an effective $\mathrm{U}(p)$ theory

# Effective $\mathrm{U}(p)$ theory

- To see the $\mathbf{U}(\boldsymbol{p})$ symmetry, introduce a set of harmonic oscillators

$$d_i^- \, O(\vec{l}) = \sqrt{h(i,l_i)}\, O(\ldots, l_i - 1, \ldots)$$

$$d_i^+ \, O(\vec{l}) = \sqrt{h(i,l_i+1)}\, O(\ldots, l_i + 1, \ldots)$$

$$\hat{h}_i \, O(\vec{l}) = \underline{h(i,l_i)}\, O(\vec{l}) = (N_c + i - l_i)\; O(\vec{l})$$

= Weight of a box at the end of the $\boldsymbol{i}$-th column in $\boldsymbol{R}$

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= Weight of a box at the end of the $i$-th column in $R$

- Commutation relations $\quad d_i^+ d_i^- = \hat{h}_i\,, \quad [d_i^+, d_j^-] = \delta_{ij}$

- $\mathbf{GL}(\boldsymbol{p})$ generators $\quad E_{ij} \equiv d_i^+ \, d_j^-\,, \quad [E_{ij}\,, E_{kl}] = \delta_{jk}\, E_{il} - \delta_{il}\, E_{kj}$

- Hamiltonians

$$\Delta_{ij}^{(1)} = -\left(d_i^+ - d_j^+\right)\left(d_i^- - d_j^-\right)$$

$$\Delta_{ij}^{(2)} = -\left(d_i^+ - d_j^+\right)\left(1 + d_i^+ d_j^- + d_j^+ d_i^-\right)\left(d_i^- - d_j^-\right)$$

# Commutation relations (2-loop)

- Rewrite the dilatation operators (here $\mathcal{H}$ denotes $\Delta$)

$$D_1^G \sim \sum_{i \neq j} n_{ij}(\sigma) \mathcal{H}_{1,ij}, \quad D_2^G \sim \sum_{i \neq j} n_{ij}(\sigma) \left\{ (L - 2N_c) \mathcal{H}_{1,ij} + \mathcal{H}_{2,ij} \right\}$$

- It turns out that

$$[\mathcal{H}_{1,ij}, \mathcal{H}_{2,ij}] = 0$$

- When $p > 2$, we want to check

$$[D_1^G, D_2^G] = 0 \quad \Leftrightarrow \quad [\mathcal{H}_{1,ij}, \mathcal{H}_{2,ik}] + [\mathcal{H}_{1,ik}, \mathcal{H}_{2,ij}] = 0$$

which is true if we take the continuum limit (large $N_c$ in the distant corners approximation)

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which is true if we take the continuum limit (large $N_c$ in the distant corners approximation)

- The commutation implies that the eigenstates (for the mixing of $Z$'s) are one-loop exact

- What if $\quad [D^G(g), D^G(g')] = 0 \quad \Leftrightarrow \quad [D_\ell^G, D_m^G] = 0, \quad (\forall l, m)$

# Commutation relations (all-loop)

- Ansatz for higher-loop dilatations

$$D_\ell^G = \sum_{k=1}^{\ell} N_c^{\ell-k} x_{\ell,k}\, \mathcal{H}_k, \quad \mathcal{H}_\ell = \sum_{i \neq j}^{p} n_{ij}(\sigma)\, \mathcal{H}_{\ell,ij}$$

- At $p$ =2, the commutation for any values of $n_{ij}$ requires that

$$[\mathcal{H}_{\ell,ij}\,, \mathcal{H}_{\ell',ij}] = 0 \quad (\forall \ell, \ell')$$

- Commuting charges

$$\mathcal{Q}_{ab,ij} \equiv (d_i^\dagger - d_j^\dagger)^a : \left(d_i^\dagger d_j + d_j^\dagger d_i\right)^b : (d_i - d_j)^a\,, \quad (a = \ell - b = 0, 1, \ldots, \ell)$$

- Large $N_c$ continuum limit

$$\mathcal{Q}_{km,ij} \simeq (2N_c)^m \;: \left(\frac{\alpha}{4}\, y_{ij}^2 - \frac{1}{\alpha}\, \frac{\partial^2}{\partial y_{ij}^2}\right)^k :\,, \quad h(i, l_i) = N_c + i - y_i \sqrt{\alpha N_c}$$

$x_{l,k} = 0$ at $k = 0$, and the $k = 1$ terms are proportional to the one-loop dilatation

# All-loop ansatz

- One-loop dimensions for $p = 2$ (= spectrum of a finite oscillator)

$$\Delta - J = \frac{\lambda}{\pi^2} n_{12} \frac{m}{N_c} + O(\lambda^2), \qquad \left(m = 1, 2, \ldots, \left\lceil N_c - \frac{n_Z}{2} + 1 \right\rceil\right)$$

- All-loop dimensions for $p = 2$

$$\Delta - J = f_1(\lambda)\, n_{12} \frac{m}{N_c}, \quad f_1(\lambda) = \frac{\lambda}{\pi^2} + O(\lambda^2)$$

We guessed possible forms of $D_l$ based on perturbative data, solve the commutation relations.
In the large $Nc$ continuum limit, all remaining terms are proportional to $D_1$

- Critical assumption

$$[D^G(g), D^G(g')] = 0 \quad \Leftrightarrow \quad [D^G_\ell, D^G_m] = 0, \quad (\forall l, m)$$

# Finite oscillator

- One-loop mixing is solvable at $p$=2

$$D_1^G = -2n_{12}\,\mathcal{H}_{1,12}$$

- Introduce the coordinate $x$ through the ansatz

$$\mathcal{O}_f = \sum_{x=-l_2}^{\lceil (l_1-l_2)/2 \rceil} f(x) O(l_1 - x, l_2 + x)$$

- Hamiltonian of the finite oscillator

$$\mathcal{H}_{1,12} = h(1,l_1) + h(2,l_2) - \sqrt{h(1,l_1)\,h(2,l_2+1)}\,e^{-\partial_x} - \sqrt{h(1,l_1+1)\,h(2,l_2)}\,e^{+\partial_x}$$

- The eigenfunction is related to the matrix elements of the $\mathbf{su}(2)$ basis rotation

$$J_k\,|j,j_k\rangle_k = j_k\,|j,j_k\rangle_k\,, \quad (j_k = -j, -j+1, \ldots, j)$$

$${}_1\langle j, j_1 \,|\, j, j_3\rangle_3 = \frac{(-1)^{j+j_3}}{2^j}\sqrt{\binom{2j}{j+j_3}\binom{2j}{j+j_1}}\;{}_2F_1(-j-j_3, -j-j_1; -2j; 2)$$

Young diagram constraints → the wave-functions must be parity-odd

[N. Atakishiyev, G. Pogosyan, and K. Wolf (2005)] [Carlson, de Mello Koch, Lin] (2011)

# Strong coupling

# Single spherical giant graviton

- A spherical giant graviton is a classical solution of the D3-brane action on $\mathbf{AdS}_5 \times \mathbf{S}^5$

$$S = \frac{N_c}{2\pi^2 R^4}\left(-\int_{\Sigma_4} d^4\xi\, e^{-\varphi}\sqrt{-\det G_{ab}} + \int_{\Sigma_4} C^{(4)}\right)$$

- $\mathbf{S}^5$ coordinates; $X_3$ — $X_6$ wraps $\mathbf{S}^3$ inside $\mathbf{S}^5$

$$X_1 = R/\sqrt{\rho}\,\cos\eta\,\cos\theta_1 \qquad X_2 = R/\sqrt{\rho}\,\cos\eta\,\sin\theta_1 \qquad X_3 = R/\sqrt{\rho}\,\sin\eta\,\cos\theta_2$$

$$X_4 = R/\sqrt{\rho}\,\sin\eta\,\sin\theta_2 \qquad X_5 = R\sqrt{1-1/\rho}\,\cos\phi \qquad X_6 = R\sqrt{1-1/\rho}\,\sin\phi$$

[McGreevy, Susskind, Toumbas (2000)]

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- Static gauge and 2D ansatz:

$$(t,\theta_1,\theta_2,\eta) = (\xi^0,\xi^1,\xi^2,\xi^3), \qquad \rho = \rho(t,\eta), \quad \phi = \phi(t,\eta)$$

- BPS solution: $\rho = N_c/(g_s J) \equiv 1/j \quad \Rightarrow \quad E = J$

[McGreevy, Susskind, Toumbas (2000)]

# KK mode analysis

∵) su(2) sector in $\mathcal{N}$=4 SYM

- Perturb around the BPS solution, expand the fluctuations by spherical harmonics on $\mathbf{S}^3$

$$\rho = \frac{1}{j} + \epsilon\, \tilde{\rho}_1(t)\, \Phi_{k,0,0}(\eta), \quad \phi = t + \epsilon\, \tilde{\phi}_1(t)\, \Phi_{k,0,0}(\eta)$$

$$\Delta_{\mathbf{S}^3} \Phi_{k,m_1,m_2}(\eta,\theta_1,\theta_2) = -k(k+2)\, \Phi_{k,m_1,m_2}(\eta,\theta_1,\theta_2)$$

- No perturbed solutions exist when the giant graviton is maximal ($j = 0, 1$)

cf. [Das, Jevicki, Mathur (2000)] [Sadri, Sheikh-Jabbari (2003)]

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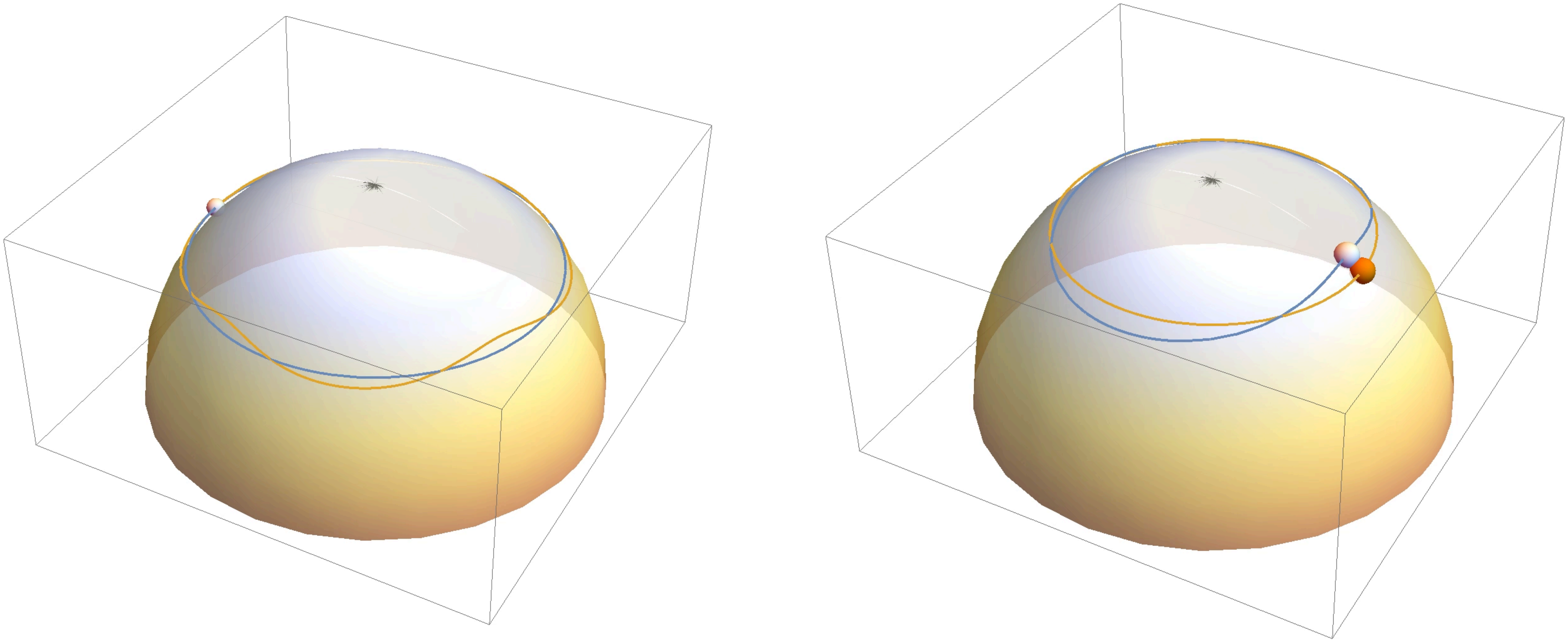
- The solutions to the linearized equations of motion are, for $0 < j < 1$,

$$E - J = \begin{cases} \dfrac{N_c}{g_s} \dfrac{\epsilon^2\, c_k^2\, (k+1)^2}{8(1-j)(k+2)} & (k > 0) \quad \text{Expanded} \\ \dfrac{N_c}{g_s} \dfrac{\epsilon^2\, \left(c_0^2 + \tilde{c}_0^2\right)}{32\pi^2(1-j)} & (k = 0) \quad \text{Point-like} \end{cases}$$

cf. [Das, Jevicki, Mathur (2000)] [Sadri, Sheikh-Jabbari (2003)]

$\boldsymbol{k} > 0$ (non-zero KK mode on $\mathbf{S}^3$)

$\boldsymbol{k} = 0$ (point-like)



Orange: BPS giant   Blue: excited giant

# AdS/CFT proposal

The finite oscillators (with $\boldsymbol{p}$ =2) should correspond to oscillating giant solutions at large $\boldsymbol{k}$

$$\boldsymbol{E - J \simeq \frac{N_c^2\, \epsilon^2}{\lambda}\, \frac{\pi\, c_k^2}{2(1-j)}\, k \qquad \leftrightarrow \qquad \Delta - J = \frac{\tilde{f}(\lambda)}{N_c}\, n_{12}(\sigma)\, m}$$

- Dispersions are gapless; (ε , $\boldsymbol{k}$) explains the factor (1/$\boldsymbol{Nc}$ , $\boldsymbol{m}$)
- Cannot excite the maximal giant graviton ($\boldsymbol{r}$ = $[1^{\boldsymbol{Nc}}]$ )

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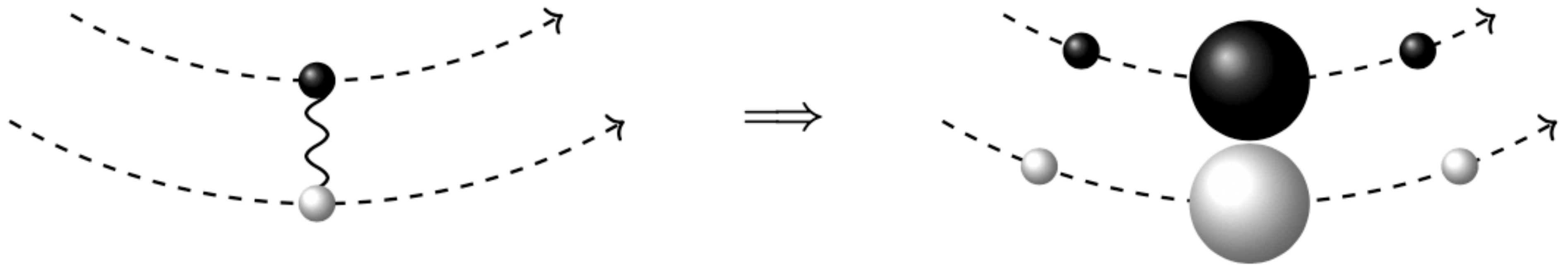
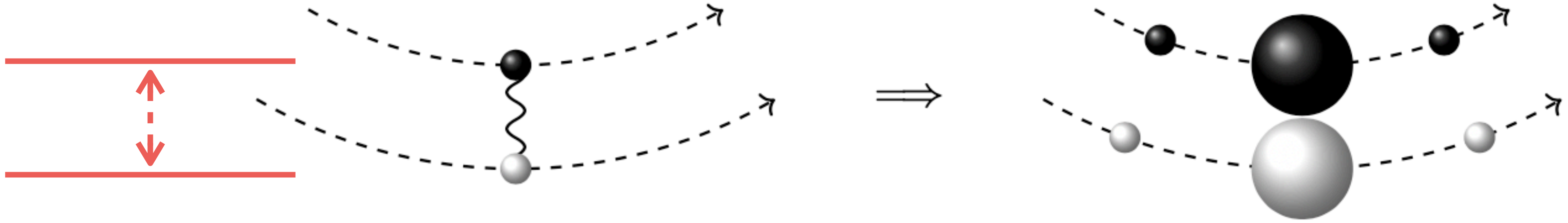
- Dispersions are gapless; (ε , $k$) explains the factor (1/$Nc$ , $m$)
- Cannot excite the maximal giant graviton ($r$ = [$1^{Nc}$] )
- Non-abelian DBI should explain $n_{ij}$(σ) ? Probably no



Open strings of *a finite length* costs a lot of energies;
After oscillating D-branes, open strings decouple at large $N_c$

# Length scale of open strings between multiple giants



**Large $N_c$ continuum limit $\rightarrow$ Distance between two non-maximal giants $\sim O(\sqrt{N_c})$**

$$\mathcal{Q}_{km,ij} \simeq (2N_c)^m : \left( \frac{\alpha}{4} y_{ij}^2 - \frac{1}{\alpha} \frac{\partial^2}{\partial y_{ij}^2} \right)^k :$$

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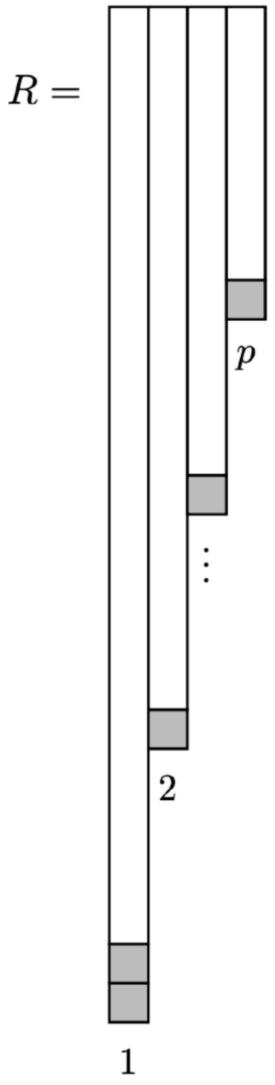

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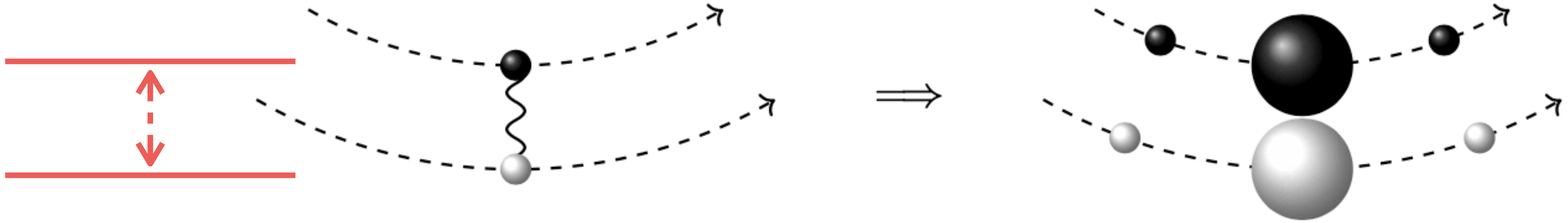
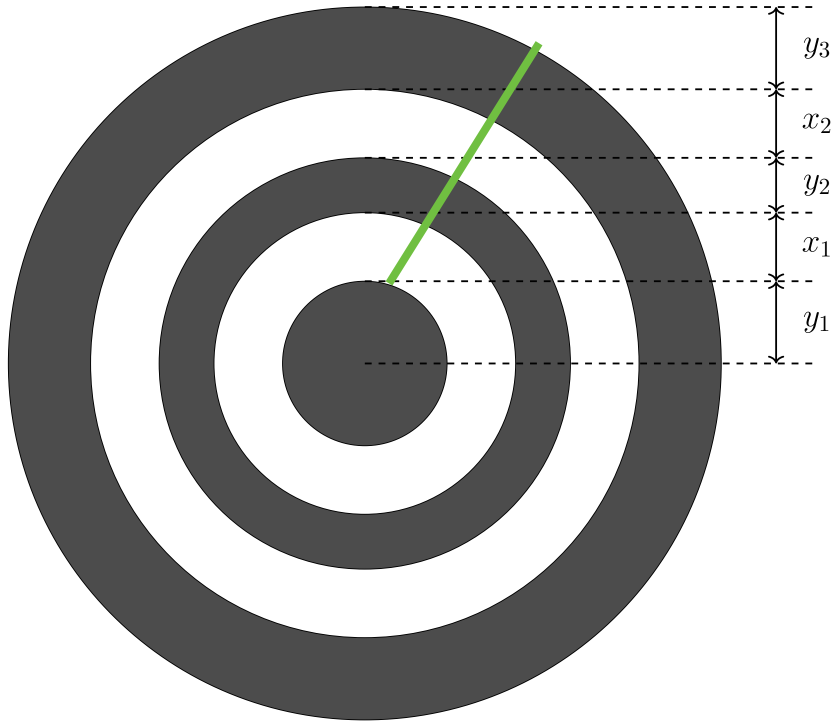


**Large $N_c$ continuum limit → Distance between two non-maximal giants $\sim O(\sqrt{N_c})$**

**Typical scale in the LLM plane $\sim O(N_c)$**

Multiple non-maximal giants are almost coincident in LLM,
except for the outermost giant (= 1st column of $\boldsymbol{R}$ )



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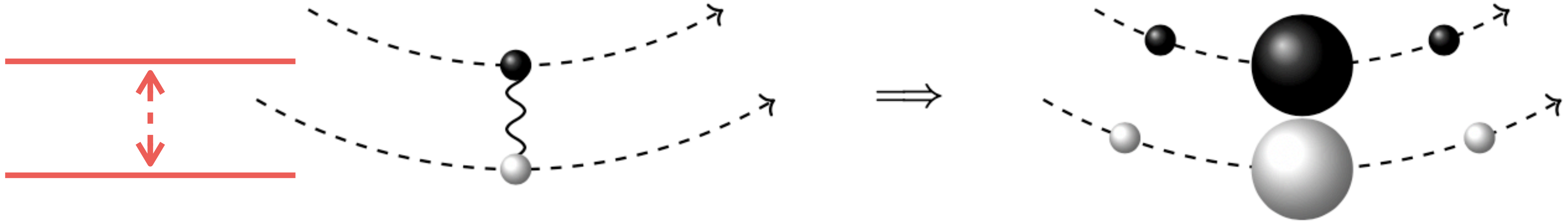

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**Typical scale in the LLM plane $\sim O(N_c)$**

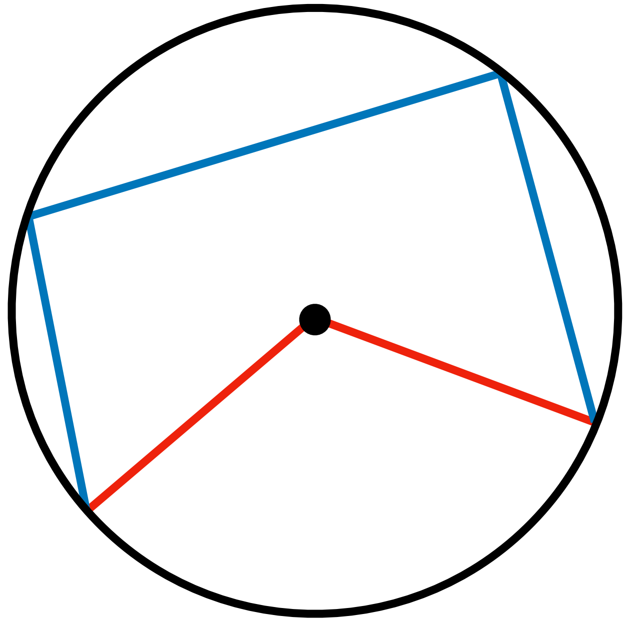
Multiple non-maximal giants are almost coincident in LLM,
except for the outermost giant (= 1st column of $\boldsymbol{R}$ )

- The open string energy (= tension × length) between different giants should be large (or very large) at strong coupling $\Rightarrow$ should decouple
- The open string tension is negligible at weak coupling $\Rightarrow$ no need to decouple

Multiple branes are easy to see only at weak coupling

# Integrable open string spectrum

The $Z$=0 maximal giant gravitons
with open strings attached



Almost single-trace operator ending
on the determinant of $Z$'s

$$\mathcal{O} = \epsilon^{i_1 i_2 \ldots i_{N_c}}_{j_1 j_2 \ldots j_{N_c}} Z^{j_1}_{i_1} Z^{j_2}_{i_2} \ldots Z^{j_{N_c-1}}_{i_{N_c-1}} \times$$

$$(\chi \ldots ZZ \ldots \psi_1 \ldots \psi_2 \ldots ZZ \ldots \chi)^{j_{N_c}}_{i_{N_c}}$$

boundary mode    bulk mode

[Hofman, Maldacena (2007)]

Both states can be described by an $\mathbf{su(2|2)}$ integrable spin chain with boundaries

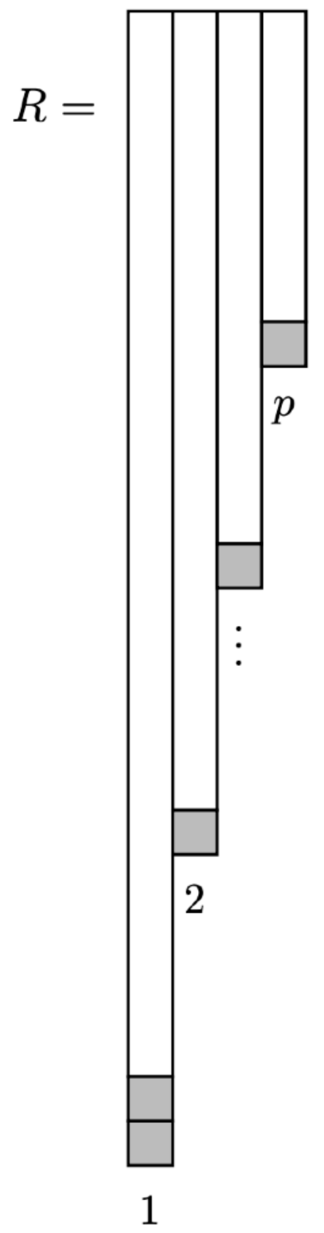
$$E - J = \sum_i \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_i}{2}} \quad \rightarrow \quad (\text{tension}) \times (\text{string length})$$

This dispersion relation is *gapped* (unless $\lambda \ll 1$ )

# Possible AdS/CFT scenarios

## Weak coupling

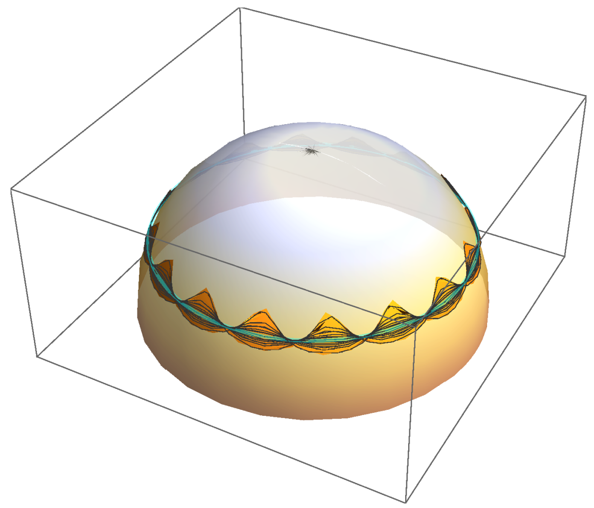
Finite oscillator from effective $\mathbf{U(p)}$ theory



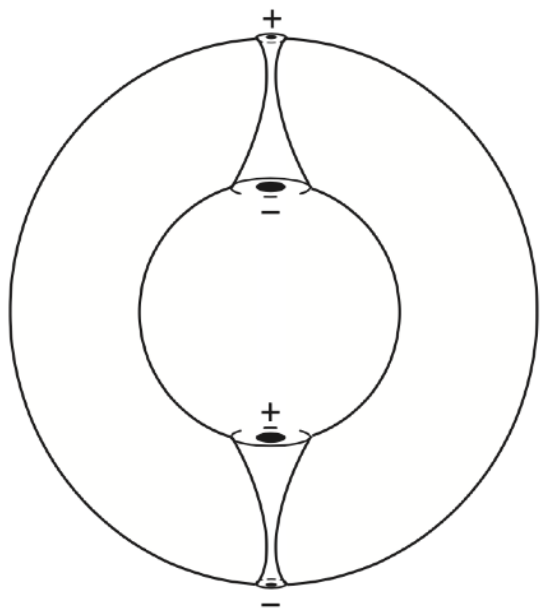

(1) all-loop commutation:
gapless dispersion,
open string decouples

(2) no commutation:
gapped dispersion,
anomalous dimensions grow large

## Strong coupling



oscillating giants

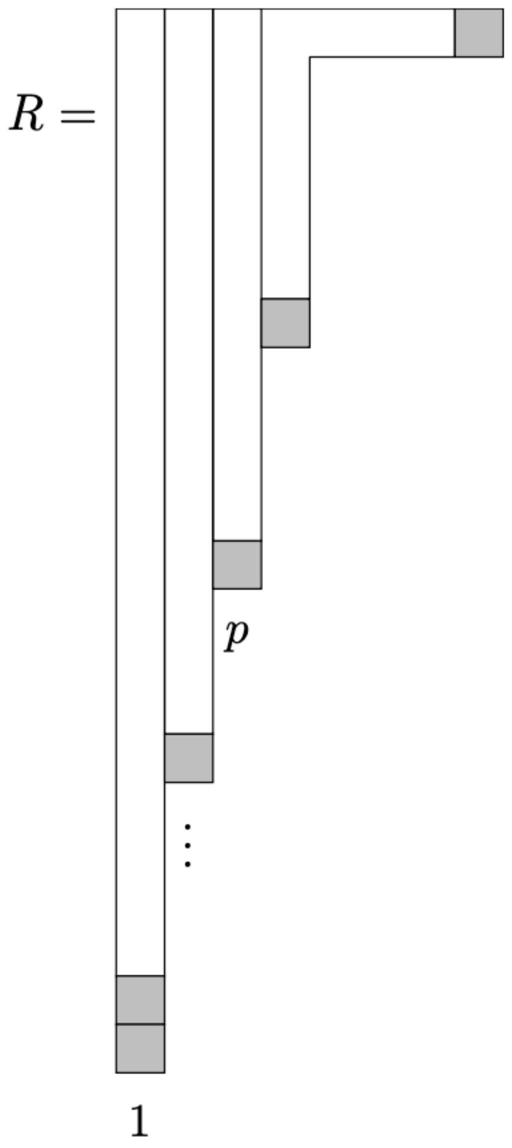



spiky strings
(∞ energy?)

[Sadri, Sheikh-Jabbari (2003)]

# Central extension of su(2|2)?

$$\mathcal{O}_{\det} = \sum_{\substack{i_1,i_2,\ldots,i_{N_c},\\ j_1,j_2,\ldots,j_{N_c}=1}}^{N_c} \epsilon^{i_1 i_2 \ldots i_{N_c}}_{j_1 j_2 \ldots j_{N_c}} \underline{Z^{j_1}_{i_1} Z^{j_2}_{i_2} \ldots Z^{j_{N_c-1}}_{i_{N_c-1}}} \underline{(\chi_L \ldots ZZ \ldots \psi_1 \ldots \psi_2 \ldots ZZ \ldots \chi_R)^{j_{N_c}}_{i_{N_c}}}$$




The centrally-extended $\mathbf{su}(2|2)$ justifies all-loop ansatz?

A non-trivial central extension of $\mathbf{su}(2|2)$ is known for coherent states of non-maximal giant gravitons

[Berenstein (2013,2014)], [Berenstein,Dzienkowski (2013)]

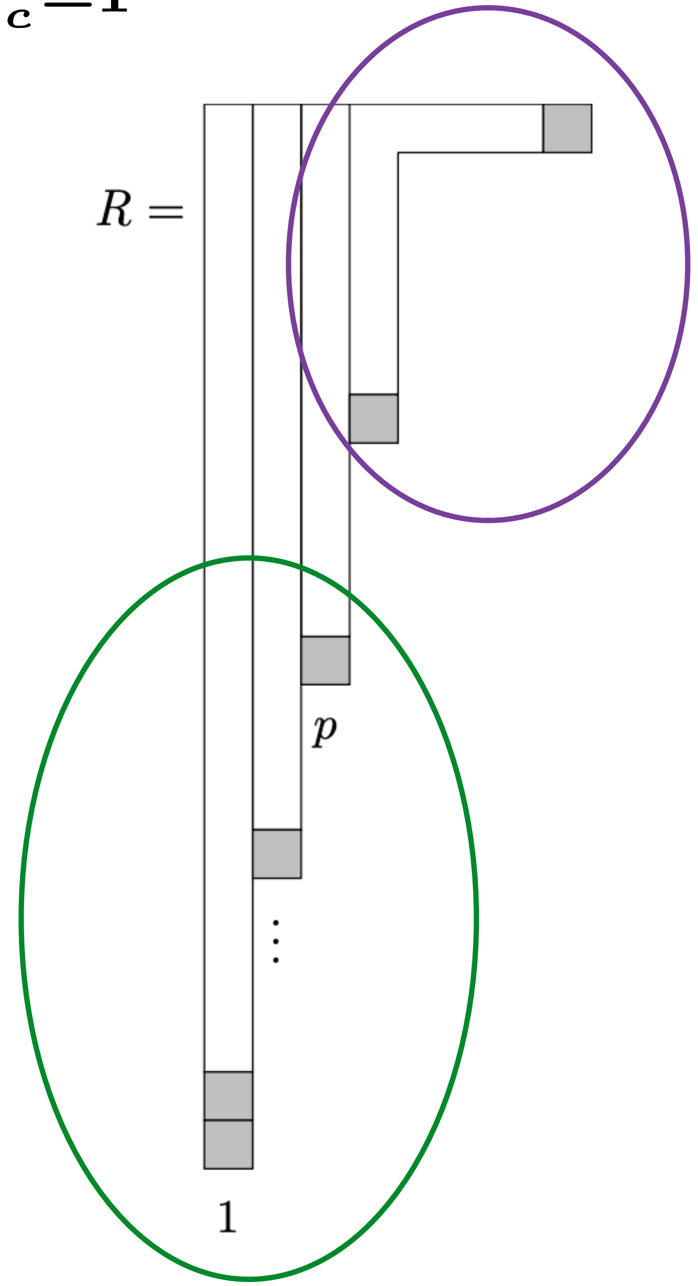
Open string on multiple giants = add a "single-trace" by attaching a single-hook next to $p$ columns

[Kristjansen, Plefka, Semenoff, Staudacher (2002)]
[April, Drummond, Heslop, Paul, Sanflippo, Santagata, Stewart (2020)]

Do the centers act non-trivially on the long columns?

# Central extension of su(2|2)?

$$\mathcal{O}_{\text{det}} = \sum_{\substack{i_1,i_2,\ldots,i_{N_c},\\ j_1,j_2,\ldots,j_{N_c}=1}}^{N_c} \epsilon^{i_1 i_2 \ldots i_{N_c}}_{j_1 j_2 \ldots j_{N_c}} \underline{Z^{j_1}_{i_1} Z^{j_2}_{i_2} \ldots Z^{j_{N_c-1}}_{i_{N_c-1}}} \; \underline{(\chi_L \ldots ZZ \ldots \psi_1 \ldots \psi_2 \ldots ZZ \ldots \chi_R)^{j_{N_c}}_{i_{N_c}}}$$




The centrally-extended $\mathbf{su}(2|2)$ justifies all-loop ansatz?

A non-trivial central extension of $\mathbf{su}(2|2)$ is known
for coherent states of non-maximal giant gravitons

[Berenstein (2013,2014)], [Berenstein,Dzienkowski (2013)]

Open string on multiple giants = add a “single-trace”
by attaching a single-hook next to $p$ columns

[Kristjansen, Plefka, Semenoff, Staudacher (2002)]
[April, Drummond, Heslop, Paul, Sanflippo, Santagata, Stewart (2020)]

Do the centers act non-trivially on the long columns?

# Summary and Outlook

# Summary

- Studied AdS/CFT in the non-planar large $N_c$ limit
- Mixing of huge operators;

  Gauss Graph basis $\Rightarrow$ Effective $\mathbf{U(p)}$ theory

- Proposed all-loop ansatz and a new AdS/CFT example;

  Finite-oscillators at weak coupling $=$ Oscillating giants at strong coupling

# Outlook

- Justify the commutation relations (or “one-loop exactness”)
- More data to check the proposal; <(giant)(giant)(single-trace)>

  [Bak, Chen, Wu (2013)] [Bissi, Kristjansen, Young, Zoubos (2013)] , …, [Jiang, Komatsu, Wu, Yang (2021)]

- (Non-planar) integrability?
- Can we classify the classical motion of D-branes?
- How to find the Gauss graph constraints in the string-brane system?

# Representation matrices

Examples:

$$(2,3)\ \begin{ytableau} 1 & 2 & 3 \\ 4 \end{ytableau} = \begin{ytableau} 1 & 2 & 3 \\ 4 \end{ytableau}\,, \qquad (2,3)\ \begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau} = -\frac{1}{2}\ \begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau} + \frac{\sqrt{3}}{2}\ \begin{ytableau} 1 & 3 & 4 \\ 2 \end{ytableau}$$

- We can proceed without choosing explicit matrix representations (use central elements)
- Explicit matrices are useful for computing the one-loop mixing
- e.g. if two boxes are separated by a long distance, the matrix elements are trivial

$$D^R\Big((a,a+1)\Big)\ |R,i\rangle = \frac{1}{d_{a,a+1}}\ |R,i\rangle + \sqrt{1-\frac{1}{d_{a,a+1}^2}}\ |R,(a,a+1)i\rangle$$

$$\to\ |R,(a,a+1)i\rangle\,, \qquad (|d_{a,a+1}| \gg 1)$$

# Restricted Schur characters

**Restrict $S_L$ to $S_{n_Y} \otimes S_{n_Z}$ with $L = n_Y + n_Z$**

**In the split basis, the representation matrices almost block diagonal**

**(block diagonal if $\alpha \in S_{n_Y} \otimes S_{n_Z}$)**

$$D^R_{IJ}(\alpha) = B^T \begin{pmatrix} D^{r_1\otimes s_1}_{i_1 j_1}(\alpha) & * & * & * \\ * & D^{r_2\otimes s_2,11}_{i_2 j_2}(\alpha) & D^{r_2\otimes s_2,12}_{i_2 j_2}(\alpha) & * \\ * & \boxed{D^{r_2\otimes s_2,21}_{i_2 j_2}(\alpha)} & D^{r_2\otimes s_2,22}_{i_2 j_2}(\alpha) & * \\ * & * & * & \ddots \end{pmatrix} B$$

↓ trace

$$\chi^{r_2\otimes s_2,21}(\alpha) = \sum_{i_2} D^{r_2\otimes s_2,21}_{i_2 i_2}(\alpha)$$

# U(N) structure in Schur-Weyl duality

Introduce another notation, $\phi_{i_1}(x_1)\,\phi_{i_2}(x_2)\,\ldots\,\phi_{i_L}(x_L) \equiv \phi_{i_1 i_2 \ldots i_L}\ (= |12\ldots L\rangle)$

**Lowering operators:** $E_{i-1,i} = \phi_{i-1}\dfrac{\partial}{\partial \phi_i} \quad (i = 2, 3, \ldots, N), \quad E_{i-1,i}\cdot \phi_{111\ldots} = 0$

**Highest weight state: Substitute $\phi_a$ to the $a$-th row**

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \;\to\; \begin{array}{|c|c|}\hline a & a \\ \hline b & \\ \cline{1-1}\end{array} \quad (a = 1, b = 2)$$

In this procedure, the permutation-group structure is (apparently) not manifest

**Two-column states are spanned by** $(\phi_{112}\,, \phi_{121}\,, \phi_{211})$

**The states should be orthogonal to the single-column state,** $\phi_{111}$

$\Rightarrow$ **Possible space of HWS are two-dimensional, not four**

# Example at $L = 3$ :

$$\phi_{i_1}(x_1)\,\phi_{i_2}(x_2)\,\ldots\,\phi_{i_L}(x_L) \equiv |1,2,\ldots,L\rangle\,, \quad (i_k = 1,2,\ldots,N,\, N \geq L)$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} = \frac{1}{\sqrt{6}}\left(|123\rangle + |231\rangle + |312\rangle + |132\rangle + |321\rangle + |213\rangle\right)$$

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} = \frac{1}{\sqrt{6}}\left(|123\rangle + |231\rangle + |312\rangle - |132\rangle - |321\rangle - |213\rangle\right)$$

$$\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}\right)_1 = \frac{1}{\sqrt{12}}\left(2\,|123\rangle + 2\,|213\rangle - |321\rangle - |312\rangle - |132\rangle - |231\rangle\right)$$

$$\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}\right)_1 = \frac{1}{2}\left(|132\rangle + |231\rangle - |321\rangle - |312\rangle\right)$$

irrep

multiplicity two

$$\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}\right)_2 = \frac{1}{2}\left(|132\rangle + |312\rangle - |321\rangle - |231\rangle\right)$$

$$\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}\right)_2 = \frac{1}{\sqrt{12}}\left(2\,|123\rangle - 2\,|213\rangle + |321\rangle - |312\rangle + |132\rangle - |231\rangle\right)$$

irrep

## Example at $L = 3$ : in QCD, $(\phi_1, \phi_2, \phi_3) = (u, d, s)$

$$\phi_{i_1}(x_1)\,\phi_{i_2}(x_2)\,\ldots\,\phi_{i_L}(x_L) \equiv \phi_{i_1 i_2 \ldots i_L} \quad (= |12\ldots L\rangle)$$

$$\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}\right)_{\mathrm{HWS}} = \phi_{111}$$

$$\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}\right)_{\mathrm{HWS}} = \frac{1}{\sqrt{6}} \sum_{\sigma \in S_3} \phi_{\sigma(1)\sigma(2)\sigma(3)}$$

$$\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\right)_{\mathrm{HWS},1} = \frac{1}{\sqrt{6}}\left(2\phi_{112} - \phi_{211} - \phi_{121}\right)$$

$$\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right)_{\mathrm{HWS},1} = \frac{1}{\sqrt{2}}\left(\phi_{211} - \phi_{121}\right)$$

irrep

$$\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\right)_{\mathrm{HWS},2} = 0$$

$$\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right)_{\mathrm{HWS},2} = \frac{1}{\sqrt{2}}\left(\phi_{121} - \phi_{211}\right)$$

irrep

Interchanged if symmetrized inside the same columns

# Is $J = J_Z + J_Y$ or $J=J_Y$

Question: The dispersion relation is gapped if $J=J_Z$

Then can we excite an open string on the strong coupling side?

Answer: The D-brane wrapping on $S^3$ cannot have non-zero $J_Y$

So indeed, we must introduce an open string carrying $J_Y$

But this open string decouples from D-brane after oscillating it,

because the length between D-branes is too large