Lectures on superstring theories

Abstract

This article is based on the lectures of *superstring theory* given by Nathan Berkovits at ICTP-SAIFR in 2016, São Paulo; note taken by Ryo Suzuki.[1]

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[1]For any questions, comments, corrections of typo, please contact to `rsuzuki.mp_at_gmai.com`

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0 Plan of the lectures

We are going to survey several formulations of perturbative superstring. The lectures are mostly about tree-level. We also work on general curved backgrounds.

As prerequisites, the readers are supposed to have the knowledge of bosonic string and some superstring.

We first explain Ramond-Neveu-Schwarz (RNS) and Green-Schwarz (GS) superstring theories. RNS has the $\mathcal{N} = 1$ worldsheet supersymmetry. GS has the κ symmetry, and has the manifest spacetime supersymmetry. Then we discuss Pure-Spinor (PS) and Twistor superstring theories. PS can be used to study the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background, though this part is less established.

All of the RNS, GS and PS theories have the same spectrum as the light-cone (LC) superstring:

$$\begin{tikzcd} \boxed{\text{RNS} : \mathcal{N} = 1 \text{ worldsheet susy}} \arrow[dr] & \\ \boxed{\text{GS} : \kappa \text{ symmetry}} \arrow[r] & \boxed{\text{LC superstring}} \\ \boxed{\text{PS} : \text{Super-Poincaré}} \arrow[ur] & \end{tikzcd}$$

Twistor theories do not fit into this framework, but they are useful for studying $\mathcal{N} = 4$ Super-Yang-Mills (SYM) in 4 dimensions, or $\mathcal{N} = 1$ SYM in 10 dimensions. The following two twistor string theories may be related to the $\alpha' = 0$ limit of PS.

- Ambitwistor string [1, 2, 3].
- Witten's twistor string, which describes SYM plus conformal supergravity [4].

There is also a "doubly supersymmetric" superstring theory based on twistors, meaning that it has both worldsheet and spacetime supersymmetry [5].

The first half of this lecture course is a review; please consult textbooks.

1 RNS superstring

1.1 Lecture 1

1.1.1 Heterotic action

Heterotic superstring has $\mathcal{N}=(1,0)$ worldsheet supersymmetry. Let us define worldsheet superfield

$$\mathbb{X}^m = X^m + \kappa\psi^m, \qquad \kappa^2 = 0, \tag{1.1}$$

and superderivative,

$$D = \frac{\partial}{\partial\kappa} + \kappa\frac{\partial}{\partial z}\,, \qquad D^2 = \frac{\partial}{\partial z}\,. \tag{1.2}$$

We introduce the action[2]

$$\begin{aligned} S_0 &= \int dz d\bar{z} \int d\kappa\, D\mathbb{X}^m \bar{\partial}\mathbb{X}_m = \iint (\psi + \kappa\partial X)\left(\bar{\partial}X + \kappa\bar{\partial}\psi\right) \\ &= \int dz d\bar{z}\left(\partial X^m \bar{\partial} X_m - \psi^m \bar{\partial}\psi_m\right). \end{aligned} \tag{1.3}$$

This system has the central charges $(c_L, c_R) = (15, 10)$. Introduce the ghosts

$$S_{\text{gh}} = \int dz d\bar{z}\Big(\underbrace{\beta\bar{\partial}\gamma}_{(11,0)} + \underbrace{b\bar{\partial}c}_{(-26,0)} + \underbrace{\bar{b}\partial\bar{c}}_{(0,-26)}\Big) \tag{1.4}$$

Thus, $S_0 + S_{\text{gh}}$ has $(c_L, c_R) = (0, -16)$. Usually we add right movers, like

$$S_1 = \int \xi^I \partial\xi^I \qquad (I = 1, 2, \ldots, 32), \tag{1.5}$$

where ξ^I are $SO(32)$ vectors, and have the conformal weight $(0, 1/2)$.

Let us rewrite ghosts in superspace, so that $\mathcal{N}=(1,0)$ worldsheet susy is manifest:

$$\mathbb{B} = \beta + \kappa b, \qquad \mathbb{C} = c + \kappa\gamma, \qquad \bar{\mathbb{B}} = \bar{b} + \kappa k, \qquad \bar{\mathbb{C}} = \bar{c} + \kappa h, \tag{1.6}$$

Then

$$\begin{aligned} \int dz d\bar{z} \int d\kappa\, \mathbb{B}\bar{\partial}\mathbb{C} &= \int dz d\bar{z}\left(\beta\bar{\partial}\gamma + b\bar{\partial}c\right), \\ \int dz d\bar{z} \int d\kappa\, \bar{\mathbb{B}} D\bar{\mathbb{C}} &= \int dz d\bar{z}\left(\bar{b}\partial\bar{c} + kh\right). \end{aligned} \tag{1.7}$$

One finds that the right-moving components (k, h) do not propagate.

[2]The factors like $1/(4\pi\alpha')$ in actions will be omitted. See Appendix A.1 for the correct numbers.

We introduce supersymmetric stress-energy tensors [6, 7],

$$\begin{aligned}
\mathbb{T} &= :-\frac{1}{2} D\mathbb{X}^m \partial \mathbb{X}_m: \\
&= -\frac{1}{2}\partial X^m \psi_m + \kappa \left(-\frac{1}{2}\partial X^m \partial X_m + \frac{1}{2}\psi^m \partial \psi_m \right), \qquad (1.8) \\
\mathbb{T}_{\rm gh} &= :-\mathbb{C}(D^2\mathbb{B}) + \frac{1}{2}(D\mathbb{C})(D\mathbb{B}) - \frac{3}{2}(D^2\mathbb{C})\mathbb{B}:, \\
&= :-c(\partial\beta) + \frac{1}{2}\gamma b - \frac{3}{2}(\partial c)\beta + \kappa \left\{ c(\partial b) + 2(\partial c) b - \frac{1}{2}\gamma(\partial\beta) - \frac{3}{2}(\partial\gamma)\beta \right\}:. \qquad (1.9)
\end{aligned}$$

1.1.2 Type II action

We introduce the $\mathcal{N}=(1,1)$ worldsheet superfield

$$\begin{aligned}
\mathbb{X}^m &= X^m + \kappa\psi^m + \bar\kappa\bar\psi^m + \kappa\bar\kappa F^m, \\
D = \frac{\partial}{\partial\kappa} + \kappa\frac{\partial}{\partial z}&, \qquad \bar D = \frac{\partial}{\partial\bar\kappa} + \bar\kappa\frac{\partial}{\partial\bar z}.
\end{aligned} \qquad (1.10)$$

The type II action is,

$$\begin{aligned}
S_0 &= \int d^2z \int d^2\kappa D\mathbb{X}^m \bar D \mathbb{X}_m = \iint \left(\psi + \kappa\partial X + \bar\kappa F + \kappa\bar\kappa\partial\bar\psi\right)(\text{c.c.}) \\
&= \int d^2z \left(\partial X^m \bar\partial X_m - \psi^m\bar\partial\psi_m - \bar\psi^m\partial\bar\psi_m - F^m F_m\right).
\end{aligned} \qquad (1.11)$$

F is an auxiliary field. This system has $(c_L, c_R) = (15, 15)$. The supersymmetric ghosts are

$$\mathbb{B} = \beta + \kappa b + \bar\kappa f + \kappa\bar\kappa g, \qquad \mathbb{C} = c + \kappa\gamma + \bar\kappa h + \kappa\bar\kappa\ell, \qquad (1.12)$$

leading to

$$\begin{aligned}
-\int d^2z \int d^2\kappa\, \mathbb{B}\bar D\,\mathbb{C} &= -\iint (\beta + \kappa b + \ldots)\left(h + \bar\kappa\bar\partial c + \kappa\ell - \kappa\bar\kappa\bar\partial\gamma\right) \\
&= \int d^2z \left(b\bar\partial c + \beta\bar\partial\gamma - gh - f\ell\right),
\end{aligned} \qquad (1.13)$$

and similarly for $\bar{\mathbb{B}} D\,\bar{\mathbb{C}}$. The first two terms of (1.13) give $(c_L, c_R) = (-26+11, 0)$, and the last two terms are auxiliary.

1.1.3 Vertex operators

Vertex operators can be found by coupling external fields and evaluating the action.

The vertex operator in open superstring is

$$\begin{aligned}
A_m(\mathbb{X}) &= A_m(X) + \kappa\psi^n\partial_n A_m(X), \\
\iint :A_m D\mathbb{X}^m: &= \iint :(A + \kappa\psi\partial A)(\psi + \kappa\partial X + \bar\kappa F + \ldots): \\
&= \int d^2z :\left(A_m\partial X^m + \psi^m\psi^n\partial_{[m}A_{n]} + \partial^m A_m\right): \ .
\end{aligned} \qquad (1.14)$$

We will neglect the last term $\partial^m A_m$ which is the gauge variation of $X^m A_m$. No ghosts enter here. On the boundary we impose the conditions

$$D\mathbb{X}^m = \bar{D}\mathbb{X}^m \qquad \begin{cases} z=\bar{z}, \quad \kappa=\bar{\kappa} & (\mathrm{Re}\, z>0) \\ z=\bar{z}, \quad \kappa=\pm\bar{\kappa} & (\mathrm{Re}\, z<0). \end{cases} \tag{1.15}$$

It implies

$$\partial X^m = \bar{\partial} X^m \quad (z=\bar{z}), \qquad \psi^m = \begin{cases} +\bar{\psi}^m & (z=\bar{z}, \quad \mathrm{Re}\, z>0) \\ \pm\bar{\psi}^m & (z=\bar{z}, \quad \mathrm{Re}\, z<0) \end{cases} \tag{1.16}$$

where $+$ corresponds to NS fermions and $-$ corresponds to R fermions. To see this, recall that

$$z = e^{\tau+i\sigma} \equiv e^{\rho}, \qquad \psi(\rho) = \left(\frac{\partial\rho}{\partial z}\right)^{1/2} \psi(z) = \frac{\psi(z)}{\sqrt{z}}\,. \tag{1.17}$$

The condition $\psi(\sigma=\pi)=\bar{\psi}(\sigma=\pi)$ is mapped to $\psi(z<0)=-\bar{\psi}(\bar{z}<0)$, which kills NS (integer) modes of $\psi(z)$.

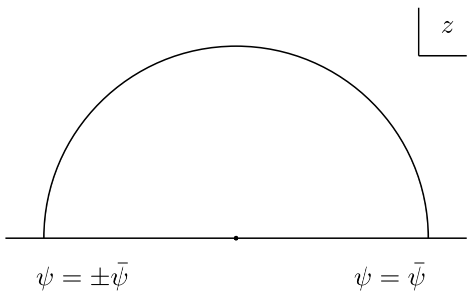

Figure 1: Boundary conditions on fermions on z-plane.

The NS vertex operator in heterotic superstring follows from

$$S = \int d^2 z \int d\kappa \Big\{ D\mathbb{X}^m \bar{\partial}\mathbb{X}^n \left(g_{mn}(\mathbb{X}) + b_{mn}(\mathbb{X})\right) + D\mathbb{X}^m A_m^A\, \xi_I \xi_J\, T_{IJ}^A(\mathbb{X}) + T(\mathbb{X}) \Big\}, \tag{1.18}$$

where T is tachyon field which is classically massless. We will omit T. By expanding the first term of (1.18) in κ,[3]

$$\begin{aligned} S &= \int \Big\{ (g_{mn}+b_{mn}) \left(\partial X^m \bar{\partial} X^n + \psi^m \bar{\partial}\psi^n\right) + \psi^m \psi^p \partial_p (g_{mn}+b_{mn}) \bar{\partial} X^n \Big\}, \\ &= \int \left(\partial X^m \bar{\partial} X_m + \psi^m \bar{\nabla} \psi_m\right), \end{aligned} \tag{1.19}$$

[3] Here we used $g_{mn}(\mathbb{X}) = g_{mn}(X) + \kappa\, \psi^p \partial_p g_{mn}(X)$ and $g_{mn} = g_{mn}(X)$.

where[4]

$$\bar\nabla\psi_m = (g_{mn}+b_{mn})\,\bar\partial\psi^n + \psi^p(\omega_{pmn}+H_{pmn})\bar\partial X^n, \qquad \partial_{[p}g_{m]n} \equiv \omega_{pmn}\,. \tag{1.20}$$

The fermion bilinear term, $\psi\psi(\omega+H)\bar\partial X$, resembles $\psi\psi\partial A$ in (1.14). On the curved spacetime, we should use $\psi^a\bar\nabla\psi_a$ in place of $\psi^m\bar\nabla\psi_m$, where $\psi_a = e^a_m\psi^m$ and e^a_m is the vierbein. In textbooks, the vertex operators are given in Fourier space, using

$$g_{mn}(X) = \eta_{mn} + h_{mn}(X) = \eta_{mn} + \int d^Dk\,\tilde h_{mn}(k)e^{ikX}\,. \tag{1.21}$$

The NS vertex operator in type II superstring follows from

$$S = \int d^2z\int d\kappa\Big\{D\mathbb{X}^m\bar D\mathbb{X}^n\,(g_{mn}(\mathbb{X})+b_{mn}(\mathbb{X}))\Big\}, \tag{1.22}$$

$$\begin{aligned}&= \int d^2z\Big\{\left(\partial X^m\bar\partial X^n + \psi^m\bar\partial\psi^n + \bar\psi^m\partial\bar\psi^n + F^mF^n\right)(g_{mn}+b_{mn})\\ &\qquad + \partial X^m\bar\psi^n\psi^p\partial_p(g_{mn}+b_{mn}) + (\text{c.c.}) + \psi^m\bar\psi^n\psi^p\bar\psi^q\,\partial_p\bar\partial_q(g_{mn}+b_{mn})\Big\},\end{aligned}$$

$$\simeq \int\Big\{\left(\partial X^m\bar\partial X_m + \psi^m\bar\nabla\psi_m + \bar\psi^m\nabla\bar\psi_m\right) + \psi^m\bar\psi^n\psi^p\bar\psi^q\,(R_{mnpq}+\partial_mH_{npq})\Big\}, \tag{1.23}$$

where we neglected F^2 in the last line, and

$$\nabla\psi_m = (g_{mn}+b_{mn})\,\partial\psi^n + \psi^p\bar\partial X^n(\omega_{pmn}-H_{pmn}). \tag{1.24}$$

Switching $z\leftrightarrow\bar z$ flips the sign of b_{mn}, and thus $H\to -H$ in (1.24).

All terms in (1.22) are necessary to preserve worldsheet supersymmetry, though historically they were discovered by imposing the conformal invariance. PS vertex operators have similar structure.

1.2 Lecture 2

1.2.1 Tree amplitudes

The tree superstring amplitudes can be computed in several "pictures". We first introduce a usual way which can be readily extended to loop amplitudes. Then we discuss unusual ways, which are valid only for trees, but related to PS amplitudes.

We consider open superstrings, using the worldsheet superfield

$$\mathbb{X}^m = X^m + \kappa\psi^m + \bar\kappa\bar\psi^m + \kappa\bar\kappa F^m, \tag{1.25}$$

and imposing the boundary conditions $D\mathbb{X}^m = \bar D\mathbb{X}^m|_{\kappa=\pm\bar\kappa}$. We take the vertex operators

$$V = c\,\partial X^mA_m\,, \qquad U = \{b,V\} = \partial X^mA_m\,, \tag{1.26}$$

[4] Here the indices are raised or lowered by $g_{mn}+b_{mn}$.

where $A_M(X)$ can be written as

$$A_m = e^{ikX}\epsilon_m\,, \qquad k^m\epsilon_m = 0, \qquad k^2 = 0. \tag{1.27}$$

An N-point bosonic tree amplitude is given by

$$\begin{aligned}\mathcal{A}_N &= \langle V_1(z_1)V_2(z_2)V_3(z_3)\int dz_4 U(z_4)\dots\int dz_N U(z_N)\rangle \\ &= \underbrace{\langle c(z_1)c(z_2)c(z_3)\rangle}_{=(z_1-z_2)(z_2-z_3)(z_3-z_1)}\langle U_1(z_1)U_2(z_2)U_3(z_3)\int dz_4 U(z_4)\dots\int dz_N U(z_N)\rangle_{\mathrm{mat}}\end{aligned} \tag{1.28}$$

For supersymmetric amplitudes, we take

$$V = \mathcal{C}\,D\mathbb{X}^m A_m(\mathbb{X}) = \mathcal{C}\left\{\psi^m A_m + \kappa\left(\partial X^m A_m + \psi^m\psi^n\partial_{[m}A_{n]}\right)\right\}. \tag{1.29}$$

The question is how to choose $\mathcal{C}$. One guess is

$$\begin{aligned} V \overset{?}{=} W &\equiv \mathbb{C}D\mathbb{X}^m A_m(\mathbb{X}) \\ &\sim c\,\psi A + \kappa\left(\gamma\psi A + c\,\partial X A + c\,\psi\psi\partial A\right).\end{aligned} \tag{1.30}$$

This quantity has manifest worldsheet supersymmetry, but its conformal weight is not same as (1.26) due to κ. Thus we use

$$\tilde{V} = \int d\kappa W = \gamma\,\psi^m A_m + c\left(\partial X^m A_m + \psi^m\psi^n\partial_{[m}A_{n]}\right), \tag{1.31}$$

$$\tilde{U} = \{b, \tilde{V}\} \quad = \partial X^m A_m + \psi^m\psi^n\partial_{[m}A_{n]}\,. \tag{1.32}$$

Now $\tilde{U}$ has the conformal weight 0.

Recall that we introduced an unintegrated vertex operator V in (1.26) to kill the zero mode integration of the tree amplitude (1.28). In superstring, we also need to kill the zero modes of $\beta\gamma$ ghosts by introducing he delta function,

$$\int dc_0 c_0 = 1 \quad \leftrightarrow \quad \int d\gamma_0\,\delta(\gamma_0) = 1. \tag{1.33}$$

A common way to define $\delta(\gamma_0)$ is to bosonize $\beta\gamma$ ghosts,

$$\underbrace{\gamma}_{\text{weight } -\frac{1}{2}} = \underbrace{\eta}_{1}\underbrace{e^{\varphi}}_{-\frac{3}{2}}, \qquad \underbrace{\beta}_{\text{weight } \frac{3}{2}} = \underbrace{e^{-\varphi}}_{\frac{1}{2}}\underbrace{\partial\xi}_{1}, \tag{1.34}$$

where η, ξ are anti-commuting fields satisfying[5]

$$\eta(y)\eta(z) \sim 0, \qquad \eta(y)\xi(z) \sim (y-z)^{-1}, \qquad \xi(y)\xi(z) \sim 0, \tag{1.35}$$

[5] The OPE (1.36) says that $e^{\pm\varphi}$ are anti-commuting, $:e^{\varphi(z)}::e^{\varphi(w)}: \sim -:e^{\varphi(w)}::e^{\varphi(z)}:$.

and $\varphi(z)$ is a chiral boson satisfying

$$\varphi(y)\varphi(z) \sim -\log(y-z), \qquad e^{m\varphi(y)}\, e^{n\varphi(z)} \sim (y-z)^{-mn}. \tag{1.36}$$

Thus we identify

$$\delta(\gamma_0) \sim e^{-\varphi}. \tag{1.37}$$

It follows that

$$\langle \delta(\gamma(y))\delta(\gamma(z))\rangle \sim (y-z)^{-1}. \tag{1.38}$$

Since the bosonization changes the $\mathbb{BC}$ OPE, the ghost supersymmetric stress-energy tensor takes a different form.[6] The new $\mathbb{T}_{\rm gh}$ is given by

$$\begin{aligned}\mathbb{T}'_{\rm gh} &= :-\mathbb{C}'(D^2\mathbb{B}') + \frac{1}{2}(D\mathbb{C}')(D\mathbb{B}') - \frac{3}{2}(D^2\mathbb{C}')\mathbb{B}': \qquad (1.39)\\ &= :-2c\,\partial(e^{-\varphi}\,\partial\xi) + \eta e^{\varphi}\, b - 3(\partial c)\, e^{-\varphi}\,\partial\xi + \kappa\left\{c(\partial b) + 2(\partial c)b - \eta(\partial\xi) - \frac{1}{2}(\partial\varphi)^2 - \partial^2\varphi\right\}: .\end{aligned}$$

Let us count the number of supermoduli on sphere. A primary field ϕ with conformal weight (h,h) behaves as

$$\phi \to |z|^{-2h} \qquad (|z| \to \infty), \tag{1.40}$$

which implies that ϕ with $h<0$ blows up at infinity. For $h=-1$ we have $\phi \to z^2$, leaving us three zero modes $O(1), O(z), O(z^2)$. In general,

$$\text{The number of zero modes on sphere} = -2h+1, \tag{1.41}$$

where h is the conformal weight of ghost. Concretely, c has conformal weight -1, and it kills 3 zero modes. γ has conformal weight $-1/2$, and kills 2 zero modes. Thus we need two $\delta(\gamma)$'s in the tree amplitude.

An N-point superstring tree amplitude is given by

$$\mathcal{A}_N = \langle V^{(-1)}(z_1)V^{(-1)}(z_2)V^{(0)}(z_3)\int dz_4\tilde{U}(z_4)\ldots\int dz_N U(z_N)\rangle, \tag{1.42}$$

where

$$\begin{aligned}V^{(-1)} &= c\, e^{-\varphi}\psi^m A_m\,,\\ V^{(0)} &= \int d\kappa\,\mathbb{C}\, D\mathbb{X}^m\mathbb{A}_m(X) \quad = \eta\, e^{\varphi}\,\psi^m A_m + c\left(\partial X^m A_m + \psi^m\psi^n\partial_{[m}A_{n]}\right),\end{aligned} \tag{1.43}$$

using (1.14). Note that $V^{(0)} = \tilde{V}$ is given in (1.31). Thus

$$\mathcal{A}_N = (z_2-z_3)(z_3-z_1)\,\langle (D\mathbb{X}A)_1\,(D\mathbb{X}A)_2\,\tilde{V}_3\int dz_4\tilde{U}(z_4)\ldots\int dz_N\tilde{U}(z_N)\rangle_{\rm mat}. \tag{1.44}$$

The matter part of the super-amplitude (1.44) is not same as the matter part of the bosonic amplitude (1.28), because the former does not contain tachyons while the latter does. This method can be generalized to loops.

[6] For example, $\gamma(z)\beta(0) = O(z^{-1}) + O(1) + :z\left\{\eta\partial\xi + \frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}\partial^2\varphi\right\}: + \ldots$, and the $O(z)$ terms contribute to $\mathbb{T}'_{\rm gh}$.

1.2.2 Picture changing

The coefficient in front of φ is called picture ($V^{(n)} \sim e^{n\varphi}$), which is measured by

$$\mathcal{P} = \oint dz\,(\partial\varphi + \xi\eta)\,. \tag{1.45}$$

It follows that γ, β have picture 0.

For each picture, we have a copy of the same Hilbert space

$$\mathcal{H}_{\text{phys}} \subset \mathcal{H}_{\text{ex}} = \left\{ e^{n\varphi}\,|v\rangle \,\middle|\, v \in \mathcal{H}_{\text{phys}}\,,\ n \in \mathbb{Z} \right\}, \tag{1.46}$$

which introduces too many states.[7] The extended Hilbert space $\mathcal{H}_{\text{ex}}$ is needed because the super-Poincare algebra does not close on a fixed picture. The amplitude (1.44) is called F2 picture. An alternative but equivalent expression is

$$\mathcal{A}_N = (z_1 - z_2)(z_2 - z_3)(z_3 - z_1)\,\langle \tilde{V}_1 \tilde{V}_2 \tilde{V}_3 \int dz_4 \tilde{U}(z_4) \ldots \int dz_N \tilde{U}(z_N)\rangle_{\text{mat}}, \tag{1.47}$$

which is called F1 picture.

Let us introduce the picture-changing operator

$$Z = \{Q, \xi\}, \qquad V^{(p+1)} =: Z V^{(p)} :\,. \tag{1.48}$$

Here Q is the BRST charge,

$$\begin{aligned} Q &= -\oint \frac{dz}{2\pi i} \int d\kappa \ :\mathbb{C}\left(\mathbb{T} + \frac{1}{2}\,\mathbb{T}_{\text{gh}}\right): \\ &= \oint :-\frac{1}{4}\,\gamma^2 b + \frac{1}{2}\,\gamma\,\psi^m \partial X_m + c\,T_{\text{total}} + \partial(\ldots): \\ c\,T_{\text{total}} &= -\frac{c}{2}\,\partial X^m \partial X_m + \frac{c}{2}\,\psi^m \partial\psi_m + bc\partial c - \frac{c}{2}\,\gamma\partial\beta - \frac{3c}{2}\,\beta\partial\gamma. \end{aligned} \tag{1.49}$$

The BRST charge is nilpotent for the critical superstring. If we redefine (β, γ) to $(x^{-1}\beta, x\gamma)$, it changes Q but not the OPE. Thus Q remains nilpotent for any x. By choosing $x = 2$ and rewriting Q in terms of the bosonized variables (1.39), we find

$$\begin{aligned} Q &= \oint :-\eta\partial\eta\, e^{2\varphi}\, b + \eta\, e^{\varphi}\, \psi^m \partial X_m + c\,T'_{\text{total}} + \partial(\ldots): \\ c\,T'_{\text{total}} &= -\frac{c}{2}\,\partial X^m \partial X_m - \frac{c}{2}\,\psi^m \partial\psi_m + bc\partial c - c\eta\partial\xi - \frac{c}{2}(\partial\varphi)^2 - c\,\partial^2\varphi. \end{aligned} \tag{1.50}$$

One can check $\{Q, \xi V^{(-1)}\} = V^{(0)}$ from (1.43), using the fact that $\langle e^{-\varphi} e^{2\varphi}\rangle$ gives a double zero.

[7] The zero mode of ξ was ambiguous in $\beta = \partial\xi\, e^{-\varphi}$, which is why we needed the extended Hilbert space; compare (1.51) and (1.52).

The picture-changing operator is not BRST-trivial on the physical Hilbert space $\mathcal{H}_{\rm phys}$, because the physical BRST-exact operator f should be written as

$$f = \{Q, F\}, \qquad F = F(X, b, c, \beta, \gamma). \tag{1.51}$$

However, $\partial Z = \{Q, \partial\xi\}$ is BRST-trivial, because

$$\{Q, \partial\xi\}\, e^{-\varphi} \simeq \{Q, \partial\xi e^{-\varphi}\} = \{Q, \beta\}. \tag{1.52}$$

Let us introduce the normalization

$$\langle ZZ\, c(\partial c)(\partial^2 c)\, e^{-2\varphi}\rangle = 1, \tag{1.53}$$

with

$$Z = c\partial\xi - \frac{1}{2}\, e^{\varphi}\, \psi^m \partial X_m - \frac{1}{4}\, (\partial\eta) e^{2\varphi}\, b - \frac{1}{4}\partial(\eta e^{2\varphi} b). \tag{1.54}$$

Recall that we do not have Z's in the usual $\beta\gamma$ system, and we do not have Z nor $e^{-2\varphi}$ in the bc system. We rewrite (1.44) as

$$\begin{aligned} \mathcal{A}_N &= \langle (ZV^{(0)})_1\, (ZV^{(0)})_2\, V_3^{(0)} \int dz_4 \tilde{U}(z_4) \ldots \int dz_N \tilde{U}(z_N) \rangle_{\rm mat}\,, \\ &= \langle ZZ\, c(\partial c)(\partial^2 c)\, e^{-2\varphi}\rangle \langle \tilde{V}_1 \tilde{V}_2 \tilde{V}_3 \int dz_4 \tilde{U}(z_4) \ldots \int dz_N \tilde{U}(z_N) \rangle_{\rm mat}\,. \end{aligned} \tag{1.55}$$

W may assign Z to other $V^{(0)}$ because ∂Z is BRST trivial. Up to normalization, this quantity (1.55) is same as the tree amplitude in the F1 picture (1.47).

We do not know pictures of picture-changing in PS or twistor formalism. We know the PS analog of the gluon vertex operator $\tilde{V} \sim \partial X^m A_m + \psi^m \psi^n \partial_{[m} A_{n]}$.

2 Green-Schwarz superstring

2.1 Lecture 3

2.1.1 Light-cone gauge

RNS and GS formalisms have the following dynamical degrees of freedom in light-cone (LC) gauge,

$$\text{RNS:} \quad X^m, \psi^m, bc, \beta\gamma \quad \xrightarrow{\text{LC}} \quad X^j, \psi^j \quad (j = 1, 2, \dots, 8), \tag{2.1}$$

$$\text{GS:} \quad X^m, \theta^\alpha, \quad \xrightarrow{\text{LC}} \quad X^j, \theta^A = (\gamma^+\theta)^A \quad (A = 1, 2, \dots, 8), \tag{2.2}$$

where $\gamma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ kills the first 8 components of θ^α.

From the $SO(8)$ triality, vectors ψ^j, spinors θ^A, anti-spinors $\Sigma^{\dot{A}}$ are equivalent. One can bosonize ψ as

$$\psi^{2k-1} \pm i\psi^{2k} = e^{\pm i\sigma_k} \qquad (k = 1, 2, 3, 4). \tag{2.3}$$

The σ's are chiral spinors satisfying

$$\sigma_J(z)\sigma_K(0) \sim \delta_{JK} \log(z). \tag{2.4}$$

They are related to θ^A and $\Sigma^{\dot{A}}$ as

$$\theta^A = \left\{\theta^{++++}, \theta^{++--}, \dots, \theta^{----}\right\} = \left\{e^{\frac{i}{2}(\sigma_1+\sigma_2+\sigma_3+\sigma_4)}, \dots, e^{-\frac{i}{2}(\sigma_1+\sigma_2+\sigma_3+\sigma_4)}\right\}, \tag{2.5}$$

$$\Sigma^{\dot{A}} = \left\{\Sigma^{+---}, \dots\right\}, \qquad = \left\{e^{\frac{i}{2}(\sigma_1-\sigma_2-\sigma_3-\sigma_4)}, \dots\right\}. \tag{2.6}$$

It implies that $e^{\frac{i}{2}\sigma_K}$ has the weight 1/8 since $\theta^A, \Sigma^{\dot{A}}$ have the weight 1/2. The actions in LC gauge are given by

$$S = \int d^2x \left(\partial X^j \bar{\partial} X^j + \psi^j \bar{\partial} \psi^j\right) \qquad \text{(RNS)}, \tag{2.7}$$

$$= \int d^2x \left(\partial X^j \bar{\partial} X^j + \theta^A \bar{\partial} \theta^A + \Sigma^{\dot{A}} \partial \Sigma^{\dot{A}}\right) \qquad \text{(GS, IIA)}, \tag{2.8}$$

$$= \int d^2x \left(\partial X^j \bar{\partial} X^j + \theta^A \bar{\partial} \theta^A + \bar{\theta}^A \partial \bar{\theta}^B\right) \qquad \text{(GS, IIB)}, \tag{2.9}$$

2.1.2 Covariant particle action

We seek for the covariant description. The Nambu-Goto action for supersymmetric massive particles is given by[8]

$$S = -m \int \sqrt{\left(\dot{X}^m - \frac{1}{2}\,\dot{\theta}\gamma^m\theta\right)^2}, \tag{2.10}$$

[8] The conjugation $\bar{\theta} = \theta\gamma^0$ is not necessary because our γ_0 is proportional to the identity matrix $\mathbf{1}_{16}$.

having the global spacetime supersymmetry

$$\delta X^m = \frac{1}{2}\,\theta\gamma^m\epsilon, \quad \delta\theta = \epsilon. \tag{2.11}$$

Note that the worldsheet susy is no longer manifest. In the first order form, it becomes

$$S = \int P_m\left(\dot{X}^m - \frac{1}{2}\,\dot{\theta}\gamma^m\theta\right) + e\left(P_m P^m - m^2\right). \tag{2.12}$$

We can set $m = 0$ in this expression, giving the constraint $P_m^2 = 0$. The momentum conjugate to θ^α is

$$p_\alpha \equiv \frac{\partial\mathcal{L}}{\partial\dot{\theta}^\alpha} = -\frac{1}{2}\,P^m(\gamma_m\theta)_\alpha\,, \tag{2.13}$$

namely

$$d_\alpha \equiv p_a + \frac{1}{2}(\not{P}\theta)_\alpha \approx 0, \tag{2.14}$$

where $\approx$ means that it vanishes on-shell. It turns out that $d_\alpha \approx 0$ is a second-class constraint,

$$\{d_\alpha\,, d_\beta\} = (\gamma^m)_{\alpha\beta}P_m\,, \tag{2.15}$$

where $\{,\}$ is the Poisson bracket. One also finds that $(\not{P}d)^\alpha$ is a first-class constraint,

$$\{(\not{P}d)^\alpha\,, (\not{P}d)^\beta\} = (\not{P}\not{P}\not{P})^{\alpha\beta} \propto P_m^2 = 0. \tag{2.16}$$

The first-class constraints imply gauge symmetry.[9] So the equation (2.16) suggests that $(\not{P}d)^\alpha$ are the generators of the so-called κ-symmetry. We use κ_α for 16 Grassmann-odd gauge variation parameters. The κ-symmetry generators act as[10]

$$\{\kappa_\alpha(\not{P}d)^\alpha, \theta^\beta\} = (\kappa\not{P})^\beta \qquad \Rightarrow \qquad \delta\theta^\alpha = (\not{P}\kappa)^\alpha\,, \tag{2.17}$$

$$\{\kappa_\alpha(\not{P}d)^\alpha, X^m\} = \frac{1}{2}\,\kappa\not{P}\gamma^m\theta \quad \Rightarrow \quad \delta X^m = -\frac{1}{2}\,\theta\gamma^m\not{P}\kappa\,, \tag{2.18}$$

as well as $\delta e = \frac{1}{2}\,\dot{\theta}^\alpha\kappa_\alpha$, where we used $d_\alpha \approx 0$.

Let us fix the κ-symmetry gauge by $\gamma^-\theta = 0$. Then

$$\dot{\theta}\gamma^m\theta = \frac{1}{2}\,\dot{\theta}\{\gamma^-, \gamma^+\}\gamma^m\theta = \frac{1}{2}\,\dot{\theta}\gamma^+\gamma^-\gamma^m\theta = \frac{1}{2}\,\dot{\theta}\gamma^+\gamma^-\gamma^+\theta = \dot{\theta}\gamma^+\theta\,, \tag{2.19}$$

where we used $\{\gamma^m, \gamma^n\} = 2\,\eta^{mn}$. The action (2.12) with $m = 0$ becomes

$$S = \int\left(P_m\dot{X}^m + P_+\,\dot{\theta}\gamma^+\theta + eP_mP^m\right). \tag{2.20}$$

We can rescale θ to absorb P_+ and diagonalize γ^+. Then θ's are 8 free fermions satisfying[11]

$$\{\theta^A, \theta^B\} = 2\delta^{AB}\,. \tag{2.21}$$

[9] Such gauge symmetry is local. See [8] for the quantization of constrained systems.

[10] Here P_m in $(\kappa\not{P})^\alpha$ is a parameter for gauge variation, and should not act on X^m in (2.18).

[11] The momentum conjugate to θ is proportional to θ.

Since $\{,\}$ is the Poisson bracket, we can think of θ's as four coordinates and four momenta. Or θ may be regarded as γ-matrices. The wave-function of one-particle state, which contains 8 bosonic and 8 fermionic degrees of freedom can be written by using θ^A as

$$f(X,\theta)=\begin{cases} f^j(X)+\theta^A(\sigma^j)_{A\dot{A}}\, f^{\dot{A}}(X) \\ f_{\dot{A}}(X)+\theta^A(\sigma^j)_{A\dot{A}}\, f_j(X). \end{cases} \tag{2.22}$$

2.1.3 Superstring action

Consider the action of heterotic superstring,

$$\begin{gathered} S_{\text{het}}=\int d\tau d\sigma\left(\Pi^m\overline{\Pi}_m+\ldots\right), \\ \Pi^m=\partial X^m-\frac{1}{2}\,\partial\theta\gamma^m\theta, \qquad \overline{\Pi}^m=\bar{\partial}X^m-\frac{1}{2}\,\bar{\partial}\theta\gamma^m\theta. \end{gathered} \tag{2.23}$$

We chose the conformal gauge, and the Virasoro constraints $\Pi_m^2=\overline{\Pi}_m^2=0$ are imposed. Note that $\Pi^m,\overline{\Pi}^m$ are invariant under the spacetime supersymmetry (2.11). Additional terms are needed in the action in order to preserve κ-symmetry and to reproduce the LC spectrum.

Again, the spacetime supersymmetry shows up with the first-class constraints: the κ-symmetry. The κ-symmetry transformation on the coordinates (θ^α, X^m) can be obtained by replacing P^m with Π^m in (2.17) and (2.18),

$$\delta\theta^\alpha=(\not{\Pi}\kappa)^\alpha, \qquad \delta X^m=-\frac{1}{2}\,\theta\gamma^m\not{\Pi}\kappa. \tag{2.24}$$

It follows that

$$\begin{aligned} \delta\Pi^m &= -\frac{1}{2}\partial\left(\theta\gamma^m\not{\Pi}\kappa\right)+\cdots=-(\partial\theta)\gamma^m\not{\Pi}\kappa, \\ \delta\overline{\Pi}^m &= -(\bar{\partial}\theta)\gamma^m\not{\Pi}\kappa \qquad (\text{not } \ \not{\overline{\Pi}}). \end{aligned} \tag{2.25}$$

The action (2.23) transforms as

$$\delta S_{\text{het}}=-\int\left(\partial\theta\not{\overline{\Pi}}\not{\Pi}\kappa+\bar{\partial}\theta\not{\Pi}\not{\Pi}\kappa+\ldots\right). \tag{2.26}$$

The second term is proportional to Π_m^2, which vanishes by the Virasoro constraints. To cancel the first term, more terms have to be added to the action.

Consider the new action

$$\begin{gathered} S_{\text{het}}=\int\Big(\Pi^m\overline{\Pi}_m+W^m\overline{\Pi}_m-\Pi^m\overline{W}_m\Big), \\ W_m=\frac{1}{2}\,\partial\theta\gamma_m\theta, \qquad \overline{W}_m=\frac{1}{2}\,\bar{\partial}\theta\gamma_m\theta. \end{gathered} \tag{2.27}$$

This action is invariant under the global spacetime supersymmetry (2.11). We can find it from

$$\begin{aligned}\delta W_m\overline{\Pi}^m - \delta\overline{W}_m\Pi^m = &-\frac{1}{2}\Big[\partial\left(\theta\gamma^m\epsilon\,\bar{\partial}X_m\right) - \bar{\partial}\left(\theta\gamma^m\epsilon\,\partial X_m\right)\Big] \\ &+\frac{1}{12}\Big[\partial\left(\theta\gamma^m\epsilon\,\bar{\partial}\theta\gamma_m\theta\right) - \bar{\partial}\left(\theta\gamma^m\epsilon\,\partial\theta\gamma_m\theta\right)\Big] \\ &-\frac{1}{6}\Big[(\partial\theta\gamma_m\theta)(\bar{\partial}\theta\gamma^m\epsilon) + (\theta\gamma_m\bar{\partial}\theta)(\partial\theta\gamma^m\epsilon) + (\bar{\partial}\theta\gamma_m\partial\theta)(\theta\gamma^m\epsilon)\Big],\end{aligned} \tag{2.28}$$

where the last term vanishes from the Fierz-like identity (A.44). The action is also invariant under the κ-symmetry. To see it, consider

$$\delta S_{\text{het}} = \int \Pi^m\delta(\overline{\Pi}^m - \overline{W}^m) + \delta(\Pi^m + W^m)\,\overline{\Pi}_m + (W^m\delta\overline{\Pi}_m - \delta\Pi^m\overline{W}_m). \tag{2.29}$$

The first term is proportional to $\not{\Pi}^2 = 0$. The second term vanishes by the equations of motion

$$\partial\bar{\partial}X^m - \partial\overline{W}^m = \partial\overline{\Pi}^m = 0. \tag{2.30}$$

The last term takes the form similar to (2.28), and vanishes from the Fierz-like identity.

The added term in (2.27) can be interpreted as a B-field. Let us combine bosonic and fermionic coordinates as $Z^M = (X^m, \theta^\alpha) \in \mathbb{R}^{1,9|16}$ and write

$$B = B_{MN}\,\partial Z^M\bar{\partial}Z^N, \quad B_{\alpha m} = -B_{m\alpha} = (\gamma_m)_{\alpha\beta}\theta^\beta, \quad B_{\alpha\beta} = B_{mn} = 0. \tag{2.31}$$

We write the three-form $H = dB$ by using the "worldvolume" coordinates (τ, σ, ξ) as

$$\begin{gathered}H = \partial_{[P}B_{MN]}\partial Z^M\bar{\partial}Z^N\partial_\xi Z^P, \qquad S_b \equiv \int d\tau d\sigma B = \int d\tau d\sigma \int_0^1 d\xi H, \\ Z^M(\tau,\sigma,\xi=1) = Z^M(\tau,\sigma), \qquad Z^M(\tau,\sigma,\xi=0) = 0.\end{gathered} \tag{2.32}$$

The two-form B has gauge symmetry[12]

$$\delta B_{MN} = \partial_{[M}\Lambda_{N]}\,, \quad \delta S_b = \int \partial(\Lambda\bar{\partial}Z) - \bar{\partial}(\Lambda\partial Z). \tag{2.33}$$

By using the gauge degrees of freedom one choose H such that $H_{\alpha\beta m} = (\gamma_m)_{\alpha\beta}$ are the only non-vanishing components. H becomes manifestly spacetime supersymmetric if we replace ∂Z^m to Π^m. The κ-symmetry variation is

$$\begin{aligned}\delta H &= \gamma_{m\alpha\beta}\,\delta\left(\Pi^m\,\bar{\partial}\theta^\alpha\,\partial_\xi\theta^\beta\right) + (\text{cyclic}), \\ &= -(\partial\theta\gamma^m)\not{\Pi}\kappa(\bar{\partial}\theta\gamma_m\partial_\xi\theta) + \bar{\partial}(\kappa\not{\Pi})\,\not{\Pi}\,\partial_\xi\theta + \bar{\partial}\theta\,\not{\Pi}\,\partial_\xi(\not{\Pi}\kappa) + (\text{cyclic}).\end{aligned} \tag{2.34}$$

The first term vanishes from the Fierz-like identity if the cyclic permutations are added, and the remaining terms vanish from $\not{\Pi}^2 = 0$.

[12]The chain rule $\partial\Lambda = \partial Z^M\partial_M\Lambda$ is used.

For type II superstring, we double the fermionic coordinates,

$$\Pi^m = \partial X^m - \frac{1}{2}\,\partial\theta\gamma^m\theta - \frac{1}{2}\,\partial\bar{\theta}\gamma^m\bar{\theta}, \qquad \overline{\Pi}^m = \bar{\partial} X^m - \frac{1}{2}\,\bar{\partial}\theta\gamma^m\theta - \frac{1}{2}\,\bar{\partial}\bar{\theta}\gamma^m\bar{\theta}, \tag{2.35}$$

$$W^m = \frac{1}{2}\left(\partial\theta\gamma^m\theta - \partial\bar{\theta}\gamma^m\bar{\theta}\right), \qquad \overline{W}^m = \frac{1}{2}\left(\bar{\partial}\theta\gamma^m\theta - \bar{\partial}\bar{\theta}\gamma^m\bar{\theta}\right). \tag{2.36}$$

The action in (2.27) now takes the form:

$$S_{\rm II} = \int \Big(\Pi^m\overline{\Pi}_m + W^m\,\overline{\Pi}_m - \overline{W}^m\,\Pi_m + \frac{1}{4}\left(\partial\theta\gamma^m\theta\,\bar{\partial}\bar{\theta}\gamma^m\bar{\theta} - \bar{\partial}\theta\gamma^m\theta\,\partial\bar{\theta}\gamma^m\bar{\theta}\right)\Big). \tag{2.37}$$

This action is invariant under a pair of κ-symmetry transformations,

$$\delta\theta^\alpha = (\not{\Pi}\kappa)^\alpha\,, \qquad \delta\bar{\theta}^{\hat{\alpha}} = 0, \qquad \delta X^m = -\frac{1}{2}\,\theta\gamma^m\not{\Pi}\kappa, \tag{2.38}$$

$$\bar{\delta}\theta^\alpha = 0\,, \qquad \bar{\delta}\bar{\theta}^{\hat{\alpha}} = (\not{\overline{\Pi}}\bar{\kappa})^{\hat{\alpha}}, \qquad \bar{\delta} X^m = -\frac{1}{2}\,\bar{\theta}\gamma^m\not{\overline{\Pi}}\bar{\kappa}. \tag{2.39}$$

The type IIB action has the symmetry $\theta \leftrightarrow \bar{\theta}$ and $z \leftrightarrow \bar{z}$. In type IIA $\theta, \bar{\theta}$ have the opposite chirality. By using $Z^M = (X^m, \theta^\alpha, \bar{\theta}^{\hat{\alpha}}) \in \mathbb{R}^{1,9|32}$ in type IIB, we get

$$B_{\alpha\hat{\beta}} = (\gamma^m)_{\alpha\gamma}\theta^\gamma(\gamma_m)_{\hat{\beta}\hat{\gamma}}\bar{\theta}^{\hat{\gamma}}, \qquad H_{\alpha\beta m} = (\gamma_m)_{\alpha\beta}, \qquad H_{\hat{\alpha}\hat{\beta}m} = -(\gamma_m)_{\hat{\alpha}\hat{\beta}}\,. \tag{2.40}$$

In the LC gauge $\gamma^-\theta = \gamma^-\bar{\theta} = 0$, the IIB action becomes

$$S_{\rm II} = \int \Big\{\partial X^m\bar{\partial}X_m - (\theta\gamma^+\bar{\partial}\theta)\partial X_+ - (\bar{\theta}\gamma^+\partial\bar{\theta})\bar{\partial}X_+ + \ldots\Big\}, \tag{2.41}$$

where $\ldots$ are the terms higher order in $\theta, \bar{\theta}$.

2.2 Lecture 4

We continue the discussion on GS. The action is

$$S = \int d^2z\left(\Pi^m\overline{\Pi}_m + B\right). \tag{2.42}$$

For type II, the equation (2.37) reads

$$\Pi^m = \partial X^m - \frac{1}{2}\,\partial\theta\gamma^m\theta - \frac{1}{2}\,\partial\bar{\theta}\gamma^m\bar{\theta}, \qquad \overline{\Pi}_m = \bar{\partial} X^m - \frac{1}{2}\,\bar{\partial}\theta\gamma^m\theta - \frac{1}{2}\,\bar{\partial}\bar{\theta}\gamma^m\bar{\theta}, \tag{2.43}$$

$$B = -\frac{1}{2}\,\partial X^m\left(\bar{\partial}\theta\gamma_m\theta - \bar{\partial}\bar{\theta}\gamma_m\bar{\theta}\right) + \frac{1}{4}\left(\partial\bar{\theta}\gamma^m\bar{\theta}\right)\left(\bar{\partial}\theta\gamma_m\theta\right) - \left(\partial \leftrightarrow \bar{\partial}\right). \tag{2.44}$$

The second term can be written as $\int d^2zB = \int d^3zH$, where

$$\begin{aligned} H = dB &= \gamma_{\alpha\beta m}\,\partial_{[1}\theta^\alpha\partial_2\theta^\beta\partial_{3]}X^m - \gamma_{\hat{\alpha}\hat{\beta}m}\,\partial_{[1}\bar{\theta}^{\hat{\alpha}}\partial_2\bar{\theta}^{\hat{\beta}}\partial_{3]}X^m + \ldots \\ &\equiv H_{ABC}\,\partial Z^A\partial Z^B\partial Z^C. \end{aligned} \tag{2.45}$$

For open superstring, we impose the boundary conditions such as

$$\text{D9-brane} \quad \begin{cases} \partial X^m = \bar{\partial} X^m \;\big|_{z=\bar{z}} \\ \theta^\alpha = \bar{\theta}^\alpha \;\big|_{z=\bar{z}} \end{cases} \tag{2.46}$$

$$\text{D7-brane} \quad \begin{cases} \partial X^m = \bar{\partial} X^m \;\big|_{z=\bar{z}} & m = 0, 1, \dots 7 \\ \partial X^m = -\bar{\partial} X^m \;\big|_{z=\bar{z}} & m = 8, 9 \\ \theta^\alpha = \left(i\gamma^{89}\bar{\theta}\right)^\alpha \;\big|_{z=\bar{z}} \end{cases} \tag{2.47}$$

Note that $i\gamma^{89}$ has the eigenvalues ± 1. The D7-brane boundary conditions can also be written as

$$\gamma^{89}\,\bar{\not{\Pi}}\,\theta = \not{\Pi}\,\bar{\theta} \quad \leftarrow \quad \begin{cases} \Pi^m = \bar{\Pi}^m \;\big|_{z=\bar{z}} & m = 0, 1, \dots 7 \\ \Pi^m = -\bar{\Pi}^m \;\big|_{z=\bar{z}} & m = 8, 9 \\ \theta^\alpha = \left(i\gamma^{89}\bar{\theta}\right)^\alpha \;\big|_{z=\bar{z}} \end{cases} \tag{2.48}$$

Massless vertex operators can be constructed by the marginal deformation of the action. For gluons and gluinos, we deform the action by the boundary term as

$$\begin{aligned} S &\to S + e \int dz\, A_M(x,\theta)\partial Z^M, \\ &= S + e \int dz \Big(A_m(x)\partial X^m + F_{mn}(x)\,(\theta\gamma^{mnp}\theta)\,\partial X_p + \dots \Big). \end{aligned} \tag{2.49}$$

In RNS formalism, we had $V = A_m \partial X^m$ for bosonic string and $V = A_m \partial X^m + F_{mn}\psi^m\psi^n$ for superstring. Thus, $p_\alpha \theta^\alpha$ roughly corresponds to $\psi^m\psi^n$, which can be seen from

$$p_\alpha \equiv (\gamma^m\theta)_\alpha \partial X^m, \qquad \{\theta^\alpha, p_\beta\} = \delta^\alpha_\beta \quad \longleftrightarrow \quad \{\psi^m, \psi^n\} = \delta^{mn}\,. \tag{2.50}$$

2.2.1 Curved backgrounds

In curved backgrounds, we replace (2.42) by

$$\begin{gathered} S = \int d^2 z \left(\eta_{ab}\, \Pi^a \overline{\Pi}^b + B \right), \\ \Pi^a = E^a_M\, \partial Z^M, \quad \overline{\Pi}^a = E^a_N\, \bar{\partial} Z^N, \quad B = B_{MN}\, \partial Z^M \bar{\partial} Z^N\,. \end{gathered} \tag{2.51}$$

where E^a_M is a super-vielbein. In the flat space it reduces to $E^a_m = \delta^a_m$ and $E^a_\alpha = (\gamma^a)_{\alpha\beta}\theta^\beta$. The action (2.51) is not κ-invariant unless the background spacetime solves the equations of motion. Nor is it supersymmetric in spacetime unless the Killing spinor exists.

We expand the first term of (2.51) in $\theta, \bar\theta$ and obtain component fields. The on-shell degrees of freedom for type IIB are[13]

$$
\begin{aligned}
G_{mn} &= g_{mn} + (\theta\gamma_m)_\alpha\, \chi_n^\alpha + (\bar\theta\gamma_n)_{\hat\beta}\, \hat\chi_n^{\hat\alpha} + (\theta\gamma_m)_\alpha(\bar\theta\gamma_n)_{\hat\beta}\, f^{\alpha\hat\beta} + \ldots\,, \\
G_{\alpha\beta} &= (\theta\gamma^m)_\alpha(\theta\gamma^n)_\beta\, g_{mn} + \ldots\,, \\
G_{\alpha\hat\beta} &= (\theta\gamma^m)_\alpha(\bar\theta\gamma^n)_{\hat\beta}\, g_{mn} + \ldots\,,
\end{aligned}
\tag{2.52}
$$

with[14]

$$
g_{mn} = \eta_{ab} E_m^a E_n^b\,, \qquad f^{\alpha\hat\beta} = F_a\,(\gamma^a)^{\alpha\hat\beta} + F_{abc}\,(\gamma^{abc})^{\alpha\hat\beta} + F_{abcde}\,(\gamma^{abcde})^{\alpha\hat\beta}. \tag{2.53}
$$

Here $F_a\,, F_{abc}\,, F_{abcde}$ are RR fields, which come from the odd number of θ's and the odd number of $\bar\theta$'s. From this result, one can read off the massless vertex operators corresponding to each field. We can do the same computation for the second term of (2.51),

$$
B_{mn} = b_{mn} + (\theta\gamma_m)_\alpha(\bar\theta\gamma_n)_{\hat\beta}\, \tilde f^{\alpha\hat\beta}, \qquad \ldots. \tag{2.54}
$$

Compare the GS action (2.51) with the bosonic part of RNS,

$$
S = \int \Big[g_{mn}\partial X^m \bar\partial X^n + b_{mn}\partial X^m \bar\partial X^n + \alpha'\varphi R \Big]. \tag{2.55}
$$

One finds that the dilaton term is missing in GS action. In general, the classical κ symmetry imposes the (generalized) supergravity equations of motion on G and B, but does not fix the dilaton coupling. Weyl symmetry may be broken as in non-critical strings.

2.2.2 $AdS_5 \times S^5$

In the flat spacetime, superstring possesses super-Poincaré symmetry containing $\mathfrak{so}(1,9)$ satisfying

$$
\{q_\alpha\,, q_\beta\} = (\gamma^m)_{\alpha\beta}\, p_m\,, \qquad \{\bar q_{\hat\alpha}\,, \bar q_{\hat\beta}\} = (\gamma^m)_{\hat\alpha\hat\beta}\, p_m\,, \tag{2.56}
$$

which leads to the symmetry

$$
\delta\theta^\alpha = \epsilon^\alpha, \qquad \delta\bar\theta^{\hat\alpha} = \bar\epsilon^{\hat\alpha}, \qquad \delta x^m = \epsilon\gamma^m\theta + \bar\epsilon\gamma^m\bar\theta. \tag{2.57}
$$

In $AdS_5 \times S_5$, superstring has the $\mathfrak{psu}(2,2|4)$ symmetry containing $\mathfrak{su}(2,2) \times \mathfrak{su}(4) \simeq \mathfrak{so}(2,4) \times \mathfrak{so}(6)$. The generators can be represented as a 32×32 matrix

$$
\text{Generators} \quad \sim \quad \begin{pmatrix} 15_B & 16_F \\ 16'_F & 15_B \end{pmatrix}, \qquad \begin{pmatrix} \mathfrak{su}(2,2) & \\ & \mathfrak{su}(4) \end{pmatrix}. \tag{2.58}
$$

[13]It is straightforward but tedious to relate each component field to the dynamical degrees of freedom, because one needs to solve the equations of motion. At the leading order in θ, one can square the degrees of freedom of super Yang-Mills in 10 dimensions. We discuss more details in Section 3.

[14]The γ matrix with mixed indices $(\gamma^a)^{\alpha\hat\beta}$ does not exist in the flat space.

A	a	$\hat{a}$	$[ab]$	$[\hat{a}\hat{b}]$	α	$\hat{\alpha}$
Range	$0 \sim 4$	$5 \sim 9$	$\dim = 10$	$\dim = 10$	$\dim = 16$	$\dim = 16$
Direction	AdS_5	S^5	$SO(1,4)$	$SO(5)$	Fermionic	Fermionic

Table 1: The values of A. The generators $M_{[AB]}$ of $SO(6)$ decompose into $\{M_{[ab]}\,, M_{[a6]}\}$ of $SO(5)$, where $A, B = 1, \ldots, 6$ and $a, b = 1, \ldots, 5$.

Let us denote the 16_F generators by $\bar{q}^J{}_K$, and $16'_F$ by $q^{\bar{K}}{}_{\bar{J}}$, $K = 1, \ldots 4$ and $\bar{K} = \bar{1}, \ldots, \bar{4}$. They satisfy

$$\{q^{\bar{K}}{}_{\bar{J}}\,, \bar{q}^J{}_K\} = R^{\bar{K}}{}_K\, \delta^J_{\bar{J}} + R^J{}_{\bar{J}}\, \delta^{\bar{K}}_K \tag{2.59}$$

where $R^{\bar{K}}{}_K\,, R^J{}_{\bar{J}}$ are some bosonic generators.

The $AdS_5 \times S^5$ action can be written by super-vielbein describing a supercoset,

$$AdS_5 = \frac{SO(2,4)}{SO(1,4)}\,, \qquad S^5 = \frac{SO(6)}{SO(5)}\,, \qquad g(x, \theta, \bar{\theta}) \in \frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}\,. \tag{2.60}$$

The supercoset has 10 bosonic and 32 fermionic variables. The group element g is defined modulo $g \sim gh$ with $h \in SO(1,4) \times SO(5)$, and $\Sigma \in PSU(2,2|4)$ acts by $g \to \Sigma g$. The 1-form currents are defined by

$$J^A = \left(g^{-1}\partial g\right)^A\,, \qquad \bar{J}^A = \left(g^{-1}\bar{\partial} g\right)^A\,, \tag{2.61}$$

The label A takes the values of (2.58), which is summarized in Table 1. Under the gauge transformation $g \to gh$, the 1-form behaves as $J \to h^{-1}Jh + h^{-1}\partial h$.

$$J^A \ \to \ \begin{cases} (h^{-1}Jh + h^{-1}\partial h)^A & A = [ab],\ [\hat{a}\hat{b}] \\ (h^{-1}Jh)^A & \text{otherwise} \end{cases}\,. \tag{2.62}$$

The group and coset elements can be parametrized as

$$g = \exp\left(x^a P^a + \bar{x}^{\hat{a}} \bar{P}^{\hat{a}} + \theta^\alpha Q_\alpha + \bar{\theta}^{\hat{\alpha}} \bar{Q}_{\hat{\alpha}}\right)\,, \qquad h = h^{[ab]} M_{[ab]} + \hat{h}^{[\hat{a}\hat{b}]} \hat{M}_{[\hat{a}\hat{b}]}\,. \tag{2.63}$$

We define the super-vielbein as

$$\Pi^a = J^a = (g^{-1}\partial_M g)^a \partial Z^M \qquad \Leftrightarrow \qquad E^a_M = (g^{-1}\partial_M g)^a, \tag{2.64}$$

and similarly for $\Pi^{\hat{a}}$. The three-form $H = dB$ is

$$H_{ABC} = \nabla_{[A} B_{BC]} + T_{[AB}{}^D B_{C]D}\,, \tag{2.65}$$

where $T_{AB}{}^C$ is the super-torsion,

$$\left[\nabla_A, \nabla_B\right] = R_{ABC}{}^D M_C{}^D + T_{AB}{}^C \nabla_C\,. \tag{2.66}$$

In the flat background, we have

$$T_{\alpha\beta}{}^m = (\gamma^m)_{\alpha\beta}\,, \qquad T_{\hat\alpha\hat\beta m} = (\gamma_m)_{\hat\alpha\hat\beta}\,, \tag{2.67}$$

which is required by supersymmetry $\{Q_\alpha, Q_\beta\} = (\gamma^m)_{\alpha\beta}\,\partial_m$, and[15]

$$H_{ABC} = T_{ABC}\,. \tag{2.68}$$

The other components of the torsion tensor vanish.

In $AdS_5 \times S^5$, the torsion tensor has other non-vanishing components,

$$\left[\nabla_\alpha\,, \partial_m\right] = T_{\alpha m}{}^{\hat\beta}\nabla_{\hat\beta}\,, \qquad T_{\alpha m}{}^{\hat\beta} = (\gamma_m)_{\alpha\gamma}\,\frac{\delta^{\gamma\hat\beta}}{R_{AdS}}\,, \qquad \delta^{\gamma\hat\beta} \equiv (\gamma^{01234})^{\gamma\hat\beta}\,, \tag{2.69}$$

where R_{AdS} is the radius of AdS_5 and S^5. The powers of R_{AdS} are introduced to express that $T_{\alpha m}{}^{\hat\beta}$ has the mass dimension one.[16] The new torsion appears for the following reason. Let us organize the supersymmetry charges

$$q_\alpha + i\bar q_{\hat\alpha} \;\to\; q^K{}_J\,, \qquad q_\alpha - i\bar q_{\hat\alpha} \;\to\; \bar q^J{}_K\,. \tag{2.70}$$

Then $q, \bar q$ commute in the flat background, but $\{q, \bar q\} \neq 0$ from (2.59) in $AdS_5 \times S^5$. In other words, R-R or NS-NS fluxes are related to the torsion. For R-R

$$f^{\alpha\hat\beta} = (\gamma^{abcde})^{\alpha\hat\beta} \neq 0, \tag{2.71}$$

and for NS-NS

$$H_{\alpha\beta b} = \nabla_{[\alpha}B_{\beta b]} + T_{\alpha\beta}{}^a B_{ab} + T_{\alpha b}{}^{\hat\alpha} B_{\hat\alpha\beta} \neq 0, \tag{2.72}$$

as well as $H_{\hat\alpha\hat\beta b} \neq 0$. Other components of H are zero, so (2.68) is not satisfied in $AdS_5 \times S^5$.[17] In the flat background, a constant and Lorentz-covariant B-field does not solve the equation (2.68). In $AdS_5 \times S^5$, a constant and $\mathfrak{psu}(2,2|4)$-covariant B-field can solve (2.72) as,

$$B \propto \delta_{\alpha\hat\beta}\left(J^\alpha \bar J^{\hat\beta} - \bar J^\alpha J^{\hat\beta}\right). \tag{2.73}$$

From (2.51), the GS action on $AdS_5 \times S^5$ becomes

$$S = \int d^2z\left(\eta_{ab}\,J^a \bar J^b + \kappa\,\delta_{\alpha\hat\beta}\left(J^\alpha \bar J^{\hat\beta} - \bar J^\alpha J^{\hat\beta}\right)\right), \tag{2.74}$$

The coefficient κ can be fixed by sugra equations of motion, or by the κ-symmetry.

We do not know how to covariantly quantize the GS action. Neither we know how to construct massive vertex operators in a $\mathfrak{psu}(2,2|4)$-covariant way. Massless vertex operators can be obtained by the deformation of the action, which correspond to supergravity states, dilaton mode and β-deformations.

[15] See (A.34) for an explanation of (2.68).

[16] The mass dimension of torsion components is given by (the dimensions of lower indices) − (the dimensions of upper indices).

[17] Here is an intuitive argument for $H_{AB}{}^C \neq T_{AB}{}^C$. The indices of H can be raised by $\delta^{\gamma\hat\beta}/R_{AdS}$, which is the unique dimensionful quantity. However, R_{AdS} should measure the strength of R-R flux, not NS-NS.

3 Pure spinor superstring

3.1 Lecture 5

3.1.1 Superparticle action

Let X^m, θ^α be ten bosonic and sixteen fermionic coordinates, $m = 0, \ldots, 9$ and $\alpha = 1, \ldots, 16$. The Brink-Schwarz action for a massless superparticle ($m = 0$ in (2.12)) is

$$S = \int d\tau \left(P_m \Big(\dot{X}^m - \frac{1}{2} \theta \gamma^m \dot{\theta} \Big) + e P^m P_m \right). \tag{3.1}$$

Its covariant quantization using Dirac brackets ends up with solving eight first-class and eight second class constraints, $10 - 2 = 16 - 8$.

The PS action for a massless superparticle is

$$S = \int d\tau \left(P_m \dot{X}^m + p_\alpha \dot{\theta}^\alpha + \omega_\alpha \dot{\lambda}^\alpha + e P^m P_m + \tilde{e}\, \lambda \gamma^m \lambda \right). \tag{3.2}$$

The pairs $(p_\alpha, \theta^\alpha)$ are 16 worldsheet fermions in the Majorana-Weyl representations of $SO(1,9)$. The pairs $(\omega_\alpha, \lambda^\alpha)$ are 16 worldsheet bosons in the complex Weyl representations of $SO(1,9)$.

The last term of (3.2) is the PS condition, which removes 5 complex bosons from λ^α. In addition, another 5 complex bosons of ω_α are redundant, because the action is invariant under the gauge transformation

$$\delta \omega_\alpha = \Lambda^m (\gamma_m \lambda)_\alpha \,. \tag{3.3}$$

Let us count the degrees of freedom in detail. We perform Wick rotation to $SO(10)$ and take a $U(5)$-covariant basis, $\lambda^\alpha \to \lambda^{s_1, s_2, s_3, s_4, s_5}$ with $s_k = \pm 1$. Here $\lambda^{s_1, s_2, s_3, s_4, s_5}$ are created or annihilated by $\gamma_{k,-} \equiv (\gamma_{2k-1} \pm i \gamma_{2k})$ as

$$\gamma_{k,-}\, \lambda^{s_1, s_2, s_3, s_4, s_5} \sim (1 \mp s_k)\, \lambda^{s_1, s_2, s_3, s_4, s_5}. \tag{3.4}$$

See Appendix A.3 for details. The 32 components of $\lambda^{s_1, s_2, s_3, s_4, s_5}$ split into those with an even or odd number of $-$'s. Each group has a definite chirality according to (3.4).

Suppose a chiral fermion has an even number of $-$'s. We start from $\lambda^{+++++} \neq 0$ and generate the other 15 components by applying $\gamma_{k,-}$ repeatedly. There are 10 components like λ^{+++--}, and 5 components like λ^{+----}, which corresponds to $\mathbf{16} = \mathbf{1} \oplus \overline{\mathbf{10}} \oplus \mathbf{5}$ under $SO(10) \to U(5)$. We can write

$$\lambda^+ = \lambda^{+++++}, \quad \lambda_{ab} = u_{ab}\, \lambda^+, \qquad \lambda^a = -\frac{\epsilon^{abcde}}{8} \frac{\lambda_{bc} \lambda_{de}}{\lambda^+} \qquad (a = 1, \ldots, 5), \tag{3.5}$$

which will be explained in Appendix A.3. Here $u_{ab} = -u_{ba}$ are the harmonic variables of the coset $SO(10)/U(5)$, and the λ^+ fixes the overall scale.[18] Thus

$$\dim_{\mathbb{C}} \left(\frac{SO(10)}{U(5)} \times \mathbb{C} \right) = 11. \tag{3.6}$$

[18] The generators of $so(10)$ can be written as $\{t_a^b, u_{[ab]}, v^{[ab]}\}$, where $\{t_a^b\}$ generate $u(5)$.

Similarly, an eight-dimensional PS has seven components from $\dim_{\mathbb{C}}\left(\frac{SO(8)}{U(4)}\times\mathbb{C}\right)$.

By analogy with the Brink-Schwarz action, let us introduce

$$d_\alpha \equiv p_\alpha + \frac{1}{2}(\not{P}\theta)_\alpha\,, \qquad Q \equiv \lambda^\alpha d_\alpha\,, \tag{3.7}$$

and interpret Q as the BRST charge. Q is nilpotent owing to the PS condition,

$$\{Q,Q\} = \lambda^\alpha\lambda^\beta \not{P}_{\alpha\beta} = 0. \tag{3.8}$$

In GS, we removed extra spacetime fermions by the LC gauge. In PS, we gauge them away by putting the gauge variation parameters λ^α on the curved background.

Roughly speaking, the BRST quantization of the Brink-Schwarz particle action gives the PS particle action. Both actions lead to the spectrum of a supersymmetric gauge theory in ten-dimensions. This is a surprising fact, because the gauge theory shows up from the particle action without gauge fields, thanks to supersymmetry. This gauge symmetry is abelian.[19] It is not known if one can obtain non-abelian gauge theories or $\mathcal{N}=1, d=10$ supergravity from a particle action.

The equivalence between (3.1) and (3.2) can be shown in the LC gauge [9]. One can also compute the BRST cohomology to see that it agrees with the spectrum of supersymmetric gauge theory in ten-dimensions [10]. However, the computation of cohomology is tedious. Below we introduce gauge fields in superspace from the beginning and discuss the BRST cohomology.

3.1.2 Gauge theory in superspace

Define

$$\nabla_m = \partial_m + A_m(X,\theta), \quad \nabla_\alpha = D_\alpha + A_\alpha(X,\theta), \quad D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}(\gamma^m\theta)_\alpha \frac{\partial}{\partial X^m}\,, \tag{3.9}$$

where A_m, A_α are ten-dimensional on-shell superfields,[20] and D_α is the covariant superderivative satisfying

$$\{D_\alpha\,, D_\beta\} = (\gamma^m)_{\alpha\beta}\frac{\partial}{\partial X^m}\,, \tag{3.10}$$

and ∇_α satisfies

$$\{\nabla_\alpha\,, \nabla_\beta\} = (\gamma^m)_{\alpha\beta}\nabla_m + F_{\alpha\beta}\,, \qquad F_{\alpha\beta} = D_\alpha A_\beta + D_\beta A_\alpha - (\gamma^m)_{\alpha\beta}\, A_m\,. \tag{3.11}$$

Notice the similarity between D_α and d_α in (3.7).

A symmetric bispinor in $d=10$ can be decomposed into the direct sum of 1-form and 5-form,

$$F_{\alpha\beta} = F_m(\gamma^m)_{\alpha\beta} + F_{mnpqr}(\gamma^{mnpqr})_{\alpha\beta}\,. \tag{3.12}$$

[19] In non-abelian gauge theories, the nilpotency condition (3.8) should become $\{Q+V, Q+V\}=0$.
[20] For partially off-shell formulation, see [11].

We should impose

$$F_{\alpha\beta} = D_\alpha A_\beta + D_\beta A_\alpha - (\gamma^m)_{\alpha\beta}\, A_m = 0, \tag{3.13}$$

to describe gauge theory, which has 2-form field strength only. The residual gauge transformation is

$$\delta A_m = \partial_m \Omega, \qquad \delta A_\alpha = D_\alpha \Omega. \tag{3.14}$$

Let us expand ten-dimensional superfields into components as

$$\begin{aligned} \Omega &= f + g_\alpha\, \theta^\alpha + h_{\alpha\beta}\, \theta^\alpha\theta^\beta + \dots, \\ A_\alpha &= a_\alpha + a_{\alpha\beta}\, \theta^\beta + a_{\alpha\beta\gamma}\, \theta^\beta\theta^\gamma + \dots, \\ A_m &= a_m + a_{m\alpha}\, \theta^\alpha + \dots. \end{aligned} \tag{3.15}$$

The gauge transformation $\delta A_\alpha = g_a + g_{\alpha\beta}\, \theta^\beta + \dots$ removes a_α and anti-symmetric part of $a_{\alpha\beta}$. The symmetric part of $a_{\alpha\beta}$ is determined by $F_{\alpha\beta} = 0$ as

$$a_{\alpha\beta} + a_{\beta\alpha} = (\gamma^m)_{\alpha\beta}\, a_m \quad \Rightarrow \quad a_{\alpha\beta}(\gamma^{mnpqr})^{\alpha\beta} = 0. \tag{3.16}$$

In this way, we obtain all on-shell degrees of freedom in $d = 10$ super Maxwell or super Yang-Mills.

We also define

$$[\nabla_a\,, \nabla_m] = F_{\alpha m}\,, \qquad [\nabla_m\,, \nabla_n] = F_{mn}\,, \tag{3.17}$$

and impose the Bianchi identity

$$[\{\nabla_{(\alpha}\,, \nabla_\beta\}, \nabla_{\gamma)}] = -(\gamma^m)_{(\alpha\beta} F_{\gamma)m} = 0, \tag{3.18}$$

where we used $F_{\alpha\beta} = 0$. From (A.44), we write a solution of this identity as

$$F_{\alpha m} = (\gamma_m)_{\alpha\beta}\, W^\beta \quad \Rightarrow \quad W^\alpha = \frac{1}{10}\,(\gamma^m)^{\alpha\beta} F_{\beta m} = \frac{1}{10}\,(\gamma^m)^{\alpha\beta}\,(\partial_m A_\beta - D_\beta A_m)\,. \tag{3.19}$$

The Bianchi identity with (m, n, α) gives[21]

$$\nabla_\alpha F_{mn} + \nabla_m(\gamma_n W)_\alpha - \nabla_n(\gamma_m W)_\alpha = 0, \tag{3.20}$$

and the Bianchi identity with (α, β, m) is

$$\begin{aligned} &\{\nabla_\alpha, [\nabla_\beta, \nabla_m]\} - \{\nabla_\beta, [\nabla_m, \nabla_\alpha]\} + [\nabla_m, \{\nabla_\alpha, \nabla_\beta\}] = 0 \\ &\Rightarrow \quad \nabla_\alpha\, (\gamma_m\, W)_\beta + \nabla_\beta(\gamma_m\, W)_\alpha - (\gamma^n)_{\alpha\beta} F_{mn} = 0. \end{aligned} \tag{3.21}$$

Multiplying the last equation by $(\gamma^m)^{\epsilon\alpha}$ we find

$$(\gamma^m)^{\epsilon\alpha}\, \nabla_\alpha W^\delta (\gamma_m)_{\delta\beta} + 10\, \nabla_\beta W^\epsilon - (\gamma^{mn})^\epsilon{}_\beta F_{mn} = 0. \tag{3.22}$$

We take the ansatz $\nabla_\alpha W^\beta = x\, \delta_\alpha{}^\beta + y\, (\gamma^{pq})^\beta{}_\alpha F_{pq}$. Using (A.38), we get

$$\nabla_\alpha W^\beta = \frac{1}{4}\,(\gamma^{mn})^\beta{}_\alpha F_{mn}\,. \tag{3.23}$$

[21] We use the notation $\nabla_\alpha F_{mn} = D_\alpha F_{mn} + [A_\alpha, F_{mn}]$, and similarly for $\nabla_m W^\alpha$. The commutator term vanishes in super-Maxwell theories.

3.2 Lecture 6

3.2.1 BRST cohomology of superparticles

We rederive the spectrum of super-Maxwell in PS formalism as the cohomology of the BRST operator,

$$Q = \lambda^\alpha d_\alpha, \qquad d_\alpha = p_\alpha + \frac{1}{2}(\gamma^m\theta)_\alpha P_m\,. \tag{3.24}$$

The nilpotency $Q^2 = 0$ follows from the PS condition on λ^α. Here d_α is the worldline version of the supercovariant derivative (3.9), and the gauge parameters λ^α parametrize the curved space $\frac{SO(10)}{U(5)} \times \mathbb{C}$.[22]

We assign the ghost number $+1$ to the pure spinor λ^α and BRST operator Q, so that $J = \omega_\alpha\lambda^\alpha$ is the ghost number current. For super-Maxwell theory, we assign the ghost number $+1$ to the photon vertex operator $V = \lambda^\alpha A_\alpha(X,\theta)$ and require that V belongs to the cohomology of Q,

$$QV = 0. \tag{3.25}$$

In addition, we impose gauge symmetry. The gauge transformation should be written as $\delta V = Q\Omega$ for some $\Omega(X,\theta)$ having zero ghost number. Since $d_\alpha\Omega = D_\alpha\Omega$ for superparticles, we find

$$\delta V = Q\Omega = \lambda^\alpha D_\alpha\Omega \quad \Rightarrow \quad \delta A_\alpha = D_\alpha\Omega. \tag{3.26}$$

The condition $QV = 0$ gives

$$\begin{aligned} 0 &= \lambda^\alpha D_\alpha\left(\lambda^\beta A_\beta\right) = \lambda^\alpha\lambda^\beta D_{(\alpha}A_{\beta)} \\ &= \frac{1}{16}\Big\{(\lambda\gamma^m\lambda)(\gamma_m)^{\alpha\beta} + (\lambda\gamma^{m_1\ldots m_5}\lambda)(\gamma_{m_1\ldots m_5})^{\alpha\beta}\Big\}D_\alpha A_\beta\,, \end{aligned} \tag{3.27}$$

where we used an identity for a symmetric bispinor $\lambda^{(\alpha}\lambda^{\beta)}$; see (A.39).

The first term of (3.27) vanishes due to the PS condition, so

$$(\gamma_{m_1\ldots m_5})^{\alpha\beta} D_\alpha A_\beta = 0. \tag{3.28}$$

Since $(\gamma^m)_{\alpha\beta}(\gamma_{npqrs})^{\alpha\beta} = 0$, from (3.13) we can write

$$A_m = \frac{1}{8}(\gamma_m)^{\alpha\beta} D_\alpha A_\beta\,. \tag{3.29}$$

For non-Abelian gauge symmetry (SYM), we modify (3.25) to $QV + [V,V] = 0$. The gluon vertex operator now contains Chern-Paton factor, $V^I = \lambda^\alpha A^I_\alpha$. See [12] for the non-Abelian case.

The (massless) spectrum depends on the ghost number of V. If V has the ghost number zero, then $QV = 0$ gives $D_\alpha V = 0$. Therefore V is constant. If V has the ghost number two, we get

$$V = \lambda^\alpha\lambda^\beta B_{\alpha\beta}(X,\theta), \qquad \delta B_{\alpha\beta} = D_\alpha\Omega_\beta\,. \tag{3.30}$$

[22] The BRST cohomology would be trivial if λ^α lived in the flat space, since Q reduces to a linear combination of $\frac{\partial}{\partial\theta^\alpha}$ at zero momentum $P_m = 0$.

It means that one can get rid of 1-form by gauge transformation. Let us choose the gauge $(\gamma^m)_{\alpha\beta}B^{\alpha\beta} = 0$. Actually $B_{\alpha\beta}$ is an anti-field of SYM in BV formalism, giving the same spectrum as SYM. The superfield and anti-superfield are related by

$$
\begin{aligned}
A_\alpha &= \underbrace{a_m}_{\text{gluon}}(\gamma^m\theta)_\alpha + (\underbrace{\chi\gamma_m\theta}_{\text{gluino}})(\gamma^m\theta)_\alpha + \dots, \\
B_{\alpha\beta} &= (\underbrace{\chi^*\gamma_{mn}\theta}_{\text{anti-gluino}})(\gamma^m\theta)_\alpha(\gamma^n\theta)_\beta + \underbrace{a^*_m}_{\text{anti-gluon}}(\gamma^m\theta)_\alpha(\gamma^n\theta)_\beta(\theta\gamma_n\theta) + \dots .
\end{aligned}
\tag{3.31}
$$

The role of the equations of motion and gauge transformations is interchanged between fields and anti-fields. The gluon and gluino obey

$$
\partial^m\partial_{[m}a_{n]} = 0, \qquad \partial^m(\gamma_m\chi) = 0, \qquad \delta a_m = \partial_m\Lambda, \tag{3.32}
$$

while the anti-fields obey

$$
\delta a^*_m = \partial^n\partial_{[n}\Lambda^*_{m]}, \qquad \delta\chi^*_\alpha = \partial^m(\gamma_m\kappa)_\alpha, \qquad \partial^m a^*_m = 0, \tag{3.33}
$$

where κ is another gauge transformation parameter. In BV formalism, the BRST cohomology has the duality of flipping the ghost number.

If V has the ghost number three, we write

$$
V = \lambda^\alpha\lambda^\beta\lambda^\gamma\, C_{\alpha\beta\gamma}(X,\theta), \qquad QV = 0, \qquad \delta C_{\alpha\beta\gamma} = D_\alpha\Omega_{\beta\gamma}\,. \tag{3.34}
$$

Since $(\lambda\gamma^m\theta)$ is BRST exact at zero-momentum, we rewrite

$$
V = (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)\,\mathcal{C}_{mnp}(X,\theta). \tag{3.35}
$$

The 3-form $\mathcal{C}_{mnp}$ is related to an anti-symmetric bispinor by $\mathcal{C}_{mnp}(\gamma^{mnp})_{\alpha\beta} \equiv \mathcal{C}_{\alpha\beta}$, and the gauge transformation of (3.34) precisely removes this degree of freedom. Thus V is a constant,

$$
V = (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)\,(\theta\gamma_{mnp}\theta) \sim O(\lambda^3\theta^5). \tag{3.36}
$$

This term itself cannot be BRST exact, because the Lorentz-invariant combination of $O(\lambda^2\theta^6)$ is annihilated by Q,[23]

$$
Q_\alpha \cdot \Big\{(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\theta\gamma_{mpq}\theta)(\theta\gamma_{npq}\theta)\Big\} = 0. \tag{3.37}
$$

There is no vertex operator with $\#_{gh}(V) < 0$. We need ω_α to create such states, but ω_α is not gauge invariant. The vertex operator with $\#_{gh}(V) \geq 4$ contains $\lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta$, and such an operator does not contribute to the (perturbative) string amplitudes.[24]

The BRST cohomology of an open bosonic string (in RNS) can be studied similarly. The BRST operator is

$$
Q = \int \big(c\,\partial X\bar{\partial}X + bc\,\partial c\big)\,. \tag{3.38}
$$

[23] Recall $(\gamma_{mnp})_{\alpha\beta} = -(\gamma_{mnp})_{\beta\alpha}$ and that λ's are bosons and θ's are fermions on the worldsheet.

[24] See Footnote 29 for further discussion.

In the sector of the ghost number $\#_{gh} = 0, 1, 2, 3$, one finds the states

$$1, \qquad c\,\partial X^m A_m(X), \qquad c\,\partial c\,\partial X^m A_m^*(X), \qquad c\,\partial c\,\partial^2 c. \tag{3.39}$$

Here A_m^* is the anti-field of A_m. The spectrum is invariant under $\#_{gh} \to 3 - \#_{gh}$. It is not possible to find a non-trivial state with the ghost number three, because $W = c\,\partial c\,\partial^2 c f(X)$ is BRST exact:

$$\{Q, c\,\partial^2 c f(X)\} = c\,\partial c\,\partial^2 c f(X). \tag{3.40}$$

Let us return to the superparticle case. Massive vertex operators can be found in the sector with the same ghost number as the massless vertex operators. In super-Maxwell or SYM, the first massive vertex operator has one derivative as

$$V_{\text{massive}} = \lambda^\alpha \left\{ \partial X^m B_{m\alpha}(X, \theta) + \partial\theta^\beta B_{\alpha\beta}(X, \theta) \right\}. \tag{3.41}$$

In open superstring, the first massive vertex operators have the conformal weight one [13],[25]

$$V_{\text{massive}} = \begin{cases} c\,\partial X^m \partial X^n\, B_{mn}(X) & (\#_{gh} = 1), \\ c\,\partial c\,\partial X^m \partial X^n\, B_{mn}^*(X) & (\#_{gh} = 2). \end{cases} \tag{3.42}$$

3.2.2 Heterotic superstring in PS

The PS formalism can be generalized from superparticles to superstrings. The action for the heterotic (or open) superstring is

$$S = \int d^2 z \Big(\frac{1}{2}\, \partial X^m \bar\partial X_m + p_\alpha \bar\partial \theta^\alpha - \omega_\alpha \bar\partial \lambda^\alpha \Big). \tag{3.43}$$

The pairs $(p_\alpha, \theta^\alpha)$ are the free fermionic bc system on the worldsheet with conformal dimensions $(1, 0)$, describing 16 spacetime fermions. The pairs $(\omega_\alpha, \lambda^\alpha)$ are the curved $\beta\gamma$ system on the worldsheet with conformal dimensions $(1, 0)$, describing 11 spacetime bosons.[26] The (classical) energy-momentum tensor T is

$$T = -\frac{1}{2} \underbrace{\partial X^m \partial X_m}_{10} - \underbrace{p_\alpha \partial\theta^\alpha}_{-32} - \underbrace{\omega_\alpha \partial\lambda^\alpha}_{22} = -\frac{1}{2}\, \Pi^m \Pi_m - d_\alpha \partial\theta^\alpha - \omega_\alpha \partial\lambda^\alpha\,, \tag{3.44}$$

where the numbers indicate the central charges, explained in Appendix A.1.

The BRST charge is[27]

$$Q = \int dz\, \lambda^\alpha d_\alpha\,, \qquad d_\alpha = p_\alpha - \frac{1}{2}\, \partial X^m (\gamma_m \theta)_\alpha + \frac{1}{8}\, (\gamma^m \theta)_\alpha (\partial\theta \gamma_m \theta). \tag{3.45}$$

[25] Recall that the conformal weight of c is -1, and that of ∂ is $+1$.

[26] We flipped the sign in front of $\omega_\alpha \bar\partial \lambda^\alpha$ for later purposes.

[27] The sign of the second term of d_α is flipped owing to the OPE $X^m(z, \bar z) X^n(0) \sim -\eta^{mn} \log |z|^2$. Also, the symbol for normal-ordering will be omitted.

The last term did not appear in the super-particle case, because the equations of motion read $\dot{\theta} = 0$. This d_α satisfies[28]

$$\{d_\alpha\,, d_\beta\} = -(\gamma_m)_{\alpha\beta}\,\Pi^m\,, \quad [d_\alpha\,, \Pi^m] = -(\gamma\partial\theta)_\alpha\,, \quad \Pi^m = \partial X^m - \frac{1}{2}\,\partial\theta\gamma^m\theta. \tag{3.46}$$

The integrated massless vertex operator (without the plane-wave factor e^{ikX}) is

$$\begin{aligned} V &= \lambda^\alpha A_\alpha(x,\theta), \\ &= a_m\,(\lambda\gamma^m\theta)_\alpha + (\chi\gamma_m\theta)(\lambda\gamma^m\theta) + \partial^m a^n(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta) + \dots . \end{aligned} \tag{3.47}$$

with $QV = 0$. The components at higher orders in θ are determined by the lower-order terms through the equations of motion. Below we compute the tree-level amplitude of superstrings. We will not discuss loop amplitudes in this lecture.

3.2.3 Tree-level amplitudes

Consider the 3-point amplitude on a disk $\langle V_1V_2V_3\rangle$. In the bosonic case, we normalize the vev as $\langle c\,\partial c\,\partial^2 c\rangle = 1$. In supersymmetric case, we normalize

$$\langle V_\theta\rangle = 1, \qquad V_\theta = (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta) \sim \lambda^3\theta^5, \tag{3.48}$$

because V_θ is BRST trivial as shown in (3.34). Therefore, the amplitude $\langle V_1V_2V_3\rangle$ can be computed by picking up the term $O(\lambda^3\theta^5)$ after the substitution of (3.47).[29]

We find

$$\langle V_1V_2V_3\rangle = a_1^m a_2^n\,\partial_m(a_3)_n + \chi_1^\alpha(\not{a}_2)_{\alpha\beta}\,\chi_3^\beta + (\text{cyclic}), \tag{3.49}$$

which is the standard 3-point interaction of ten-dimensional SYM.

The 4-point amplitude is given by

$$\langle V_1(z_1)V_2(z_2)V_3(z_3)\int_{\partial\Sigma} dz_4 U(z_4)\rangle, \qquad V = \lambda^\alpha A_\alpha\,, \qquad QU = \partial V, \tag{3.50}$$

where we integrate over the boundary of the disk (Figure 2). The integrations over the interval $[z_1, z_2], [z_2, z_3], [z_3, z_1]$ correspond to different channels in gauge theory, which guarantees the crossing symmetry of the 4pt amplitude.

The unintegrated massless vertex operator U is given by

$$U = \partial\theta^\alpha A_\alpha + \Pi^m A_m + d_\alpha W^\alpha + \frac{1}{2}\,N_{mn}F^{mn}\,, \qquad N_{mn} = \frac{1}{2}\,\omega\gamma_{mn}\lambda, \tag{3.51}$$

[28]The notation $\{A, B\} = C$ or $[A, B] = C$ means $\operatorname*{Res}_{z=0} A(z)B(0) = C(0)$.

[29]Another explanation goes as follows. The 11 zero modes of λ cancel out part of 16 zero modes of θ, leaving θ^5 behind. To explain λ^3, notice that the ghost current $J = \omega\lambda$ has the ghost number anomaly of (-8). This can be cancelled by the 11-dimensional integral of PS zero mode, giving $11 - 8 = 3$ [14].

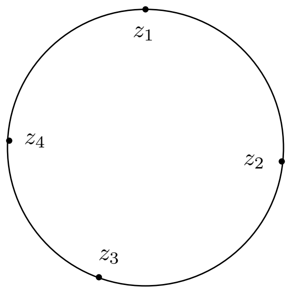

Figure 2: 4-point functions on a disk.

up to BRST-exact terms. Here $A_m(X,\theta)$ is an on-shell vector superfield, N_{mn} are the $SO(1,9)$ generators, F_{mn} and W^α are the field strengths defined in (3.17) and (3.19). The equations of motion (3.29) and the Bianchi identity (3.23) imply that

$$A_m = (\gamma_m)^{\alpha\beta} D_\alpha A_\beta \,, \qquad \nabla_\alpha W^\beta = \frac{1}{4}(\gamma^{mn})^\beta{}_\alpha F_{mn} \,. \tag{3.52}$$

The component at $O(\theta^0)$ is[30]

$$U\Big|_{\theta=0} = a_m \partial X^m + \partial_{[m} a_{n]} \left[\frac{1}{2}\omega\gamma^{mn}\lambda + \frac{1}{2}p\gamma^{mn}\theta\right]. \tag{3.53}$$

In bosonic string theory, we have $V = c\,A_m \partial X^m$ and $U = A_m \partial X^m$, which agrees with the bosonic part of (3.53).

One can check $QU = \partial V$ by using the OPE which follow from (3.43). More explicitly, we have

$$\partial(\lambda^\alpha A_\alpha) = \underbrace{\lambda^\alpha \partial_m A_\alpha \partial X^m}_{(1)} + \underbrace{\lambda^\alpha \partial_\beta A_\alpha \partial\theta^\beta}_{(2)} + \underbrace{\partial\lambda^\alpha A_\alpha}_{(3)} \,, \tag{3.54}$$

which should match

$$\begin{aligned} QU = &- \underbrace{(\lambda^\beta D_\beta A_\alpha)\partial\theta^\alpha}_{(2)} + \underbrace{\partial\lambda^\alpha A_\alpha}_{(3)} + \underbrace{(\lambda^\beta D_\beta A_m)\Pi^m}_{(1)} + \underbrace{A_m(\lambda\gamma^m\partial\theta)}_{(2)} \\ &- \underbrace{(\lambda^\beta D_\beta W^\alpha) d_\alpha}_{(4)} - \underbrace{(\not{\Pi}\lambda)_\alpha W^\alpha}_{(1)} + \frac{1}{2}(\lambda^\beta D_\beta F^{mn}) N_{mn} + \underbrace{\frac{1}{4}(\gamma_{mn}\lambda)^\alpha d_\alpha F^{mn}}_{(4)} \,. \end{aligned} \tag{3.55}$$

term by term. From the equations of motion in Section 3.1.2 and PS constraint, we find

$$\begin{aligned} QU - \partial V = &\ \lambda^\beta \left\{D_\beta A_m - W^\alpha(\gamma_m)_{\alpha\beta}\right\}\Pi^m - \lambda^\alpha \partial_m A_\alpha \partial X^m \\ &- \lambda^\beta \left\{D_\beta A_\alpha - A_m(\gamma^m)_{\alpha\beta} + \partial_\alpha A_\beta\right\}\partial\theta^\alpha \\ &- \lambda^\beta \left\{D_\beta W^\alpha - \frac{1}{4}(\gamma_{mn})^\alpha{}_\beta F^{mn}\right\} d_\alpha + \frac{1}{2}\omega_\alpha (\gamma^{mn})^\alpha{}_\gamma \lambda^\gamma \lambda^\beta D_\beta F_{mn} \\ = &\ 0. \end{aligned} \tag{3.56}$$

[30]The last term comes from $d_\alpha W^\alpha$, where $W^\alpha\Big|_{\theta=0}$ is gauge degrees of freedom.

The identity $(\gamma^{mn})^\alpha{}_\gamma \lambda^\gamma \lambda^\beta D_\beta F_{mn} = 0$ can be seen by multiplying $\lambda^\alpha \lambda^\gamma D_\gamma$ to the abelian version of (3.23).

The unintegrated vertex operator in RNS formalism is given by (1.32)

$$U\Big|_{\theta=0} = \partial X^m a_m + \psi^m \psi^n \partial_{[m} a_{n]}\,. \tag{3.57}$$

Here $M^{mn} = \psi^m \psi^n$ is the Lorentz current. This is also the generator of level $k = 1$ Kac-Moody algebra,

$$J^A(y) J^B(z) \sim \frac{\alpha' k\, \delta^{AB}}{(y-z)^2} + \frac{f^{ABC} J^C}{y-z}\,, \tag{3.58}$$

because

$$M^{mn}\, M^{pq} \;\sim \frac{\eta^{(mq}\eta^{np)}}{(y-z)^2} + \frac{M^{mq}\,\eta^{np}}{y-z} + (\text{cyclic}). \tag{3.59}$$

BY comparing (3.57) and (3.53), one identifies

$$M^{mn} \;\leftrightarrow\; \hat{M}^{mn} \equiv \frac{1}{2}\,\omega\gamma_{mn}\lambda + \frac{1}{2}\,p\gamma_{mn}\theta. \tag{3.60}$$

The first term on RHS has level -3, and the second term has level 4. Thus $\hat{M}^{mn}$ has level 1, as expected.

It is straightforward to compute n-point tree-level super-amplitude in PS formalism. In RNS, the computation becomes harder as the number of gluinos increases:

# (gluinos)	0	2	4	6	8
Complexity	simple	doable	hard	PhD thesis	No results

In GS formalism, conformal gauge is troublesome due to the kinetic term of fermions. In the LC gauge, we need to introduce branch points to compute general n-point amplitudes. The computation gets quickly involved since the roots of polynomials of order $(n-2)$ appear.

3.3 Lecture 7

3.3.1 Closed superstring in PS

We discuss massless vertex operators for type II closed superstring, which describe ten-dimensional type II supergravity.

We review PS superstring for type II in the flat background, generalizing the case of open string in Section 3.2.2. The action is

$$S = \int d^2 z \left(\frac{1}{2} \partial X^m \bar{\partial} X_m + p_\alpha \bar{\partial}\theta^\alpha - \omega_\alpha \bar{\partial}\lambda^\alpha + \bar{p}_{\hat{\alpha}} \partial \hat{\theta}^{\hat{\alpha}} - \hat{\omega}_{\hat{\alpha}} \partial \hat{\lambda}^{\hat{\alpha}} \right). \tag{3.61}$$

The holomorphic part of the BRST charge is given by (3.45). The unintegrated vertex operator is

$$V = \lambda^\alpha \hat{\lambda}^{\hat{\alpha}} A_{\alpha\hat{\alpha}}(x,\theta), \qquad QV = \bar{Q}V = 0, \tag{3.62}$$

where

$$\begin{aligned} A_{\alpha\hat{\alpha}} &= (g_{mn} + b_{mn} + \varphi\,\eta_{mn})\,(\gamma^m\theta)_\alpha(\gamma^n\hat{\theta})_{\hat{\alpha}} \\ &\quad + \xi_m^{\hat{\beta}}\,(\gamma^m\theta)_\alpha\,\hat{\theta}_{\hat{\alpha}}\hat{\theta}_{\hat{\beta}} + \xi_m^{\beta}\,\theta_\alpha\theta_\beta\,(\gamma^m\hat{\theta})_{\hat{\alpha}} + F^{\beta\hat{\beta}}\,\theta_\alpha\theta_\beta\,\hat{\theta}_{\hat{\alpha}}\hat{\theta}_{\hat{\beta}} + \ldots . \end{aligned} \tag{3.63}$$

This generalizes the vertex operator of bosonic string,

$$V = c\bar{c}\,\partial X^m \bar{\partial} X^n\,(g_{mn} + b_{mn} + \varphi\,\eta_{mn})\,. \tag{3.64}$$

The integrated vertex operator satisfies $QU = \partial V$ and $\bar{Q}U = \bar{\partial}V$, which gives

$$\begin{aligned} U &= A_\alpha \hat{A}_{\hat{\beta}}\,\partial\theta^\alpha\bar{\partial}\hat{\theta}^{\hat{\beta}} + A_\alpha \bar{A}_m\,\partial\theta^\alpha\Pi^m + A_m\hat{A}_{\hat{\alpha}}\,\Pi^m\bar{\partial}\hat{\theta}^{\hat{\alpha}} + A_m\bar{A}_n\,\Pi^m\bar{\Pi}^n \\ &\quad + W^\alpha\hat{W}^{\hat{\alpha}} d_\alpha \hat{d}_\alpha + \frac{1}{4}\,F^{mn}\bar{F}^{pq}N_{mn}\bar{N}_{pq} + \ldots \end{aligned} \tag{3.65}$$

To compute the tree-amplitude, we normalize the states by $\langle\lambda^3\theta^5\hat{\lambda}^3\hat{\theta}^5\rangle = 1$.

In general, the closed string vertex operator is the "square" of the open string vertex operators, because the BRST charges Q and $\hat{Q}$ do not interact each other. If $\mathcal{A}_L\,,\mathcal{A}_R$ are in the cohomology of $Q_L\,,Q_R$, respectively, then $\mathcal{A}_{LR} \equiv \mathcal{A}_L\,\mathcal{A}_R$ is in the cohomology of $Q_{LR} \equiv Q_L + Q_R$.

The equation (3.65) is roughly equivalent to

$$U \sim (G_{MN} + B_{MN})\,\partial Z^M\bar{\partial}Z^N + (F_{\rm RR})^{\alpha\hat{\beta}} d_\alpha \hat{d}_{\hat{\beta}} + (R + \partial H)^{mnpq}\,N_{mn}\bar{N}_{pq}\,, \tag{3.66}$$

where $Z^M = (X^m, \theta^\alpha, \hat{\theta}^{\hat{\alpha}}) \in \mathbb{R}^{1,9|32}$, $F_{\rm RR}$ is the RR flux, and R^{mnpq} is the Riemann tensor. The GSO projection is manifest in (3.66) if we assign $F = +1$ to the worldsheet fermions with upper index (θ^α), and $F = -1$ to those with lower index (d_α). There are gauge degrees of freedom coming from the super-Poincaré rotation and gauge symmetry of B, which can be fixed by

$$0 = G_{\alpha\beta} = G_{\hat{\alpha}\hat{\beta}} = B_{\alpha\beta} = B_{\hat{\alpha}\hat{\beta}}\,. \tag{3.67}$$

This gauge choice is also consistent from the spectrum of ten-dimensional supergravity in the flat spacetime. We have $-G_{\alpha\hat{\beta}} = G_{\hat{\beta}\alpha} = B_{\alpha\hat{\beta}} = B_{\hat{\beta}\alpha}$. The combination $\Big(G_{\alpha\hat{\beta}} + B_{\alpha\hat{\beta}}\Big)$ never appears because the equations of motion tell $\partial\hat{\theta} = \bar{\partial}\theta = 0$.

The field A_α of open string satisfies the equations of motion (3.28) and (3.29). In closed string, we have[31]

$$0 = D_\alpha A_{\beta\hat{\alpha}}\,(\gamma^{mnpqr})^{\alpha\beta} = \bar{D}_{\hat{\alpha}}\hat{A}_{\alpha\hat{\beta}}\,(\gamma^{mnpqr})^{\hat{\alpha}\hat{\beta}}\,, \qquad D_\alpha A_{\beta\hat{\alpha}} + D_\beta A_{\alpha\hat{\alpha}} = (\gamma^m)_{\alpha\beta}G_{m\hat{\alpha}}\,, \tag{3.68}$$

[31] Here $A_{\beta\hat{\alpha}} = A_\beta\,\hat{A}_{\hat{\alpha}}$ and $G_{m\hat{\alpha}} = A_m\hat{A}_\alpha$.

in the gauge choice (3.67).

Recall that the constant dilaton does not appear in GS action, since it is not known how to incorporate the α' corrections without breaking κ symmetry. In PS, we can add the dilaton to (3.66) by modifying $(R + \partial H)$ to $(R + \partial H + \alpha' \Phi\, r)$.[32] At the non-linear level, g_{mn} and φ in (3.63) mix as $G_{mn} = g_{mn}\, e^{\varphi_0} + \theta^2 \partial\varphi$.

The NS vertex operator in RNS was given by (1.23),

$$\begin{aligned} U &= \int d^2\kappa \Big(g_{mn}(\mathbb{X}) + b_{mn}(\mathbb{X}) \Big) D\mathbb{X}^m \bar{D}\mathbb{X}^n, \\ &= (g_{mn} + b_{mn})\, \partial X^m \bar\partial X^n + \Omega_{pmn} \partial X^p \bar\psi^m \psi^n + (R_{mnpq} + \partial_m H_{npq})\, \psi^m \psi^n \bar\psi^p \bar\psi^q, \end{aligned} \tag{3.69}$$

which looks very similar to U in (3.66).

3.3.2 The b ghost

We introduce b ghost as the solution of

$$\{Q, b\} = T, \tag{3.70}$$

where T is the chiral energy-momentum tensor given by (3.44),

$$T = -\frac{1}{2}\, \Pi^m \Pi_m - d_\alpha \partial\theta^\alpha - \omega_\alpha \partial\lambda^\alpha\,. \tag{3.71}$$

We may add a total derivative $\sim \partial^2 \log(\bar\lambda\lambda)$ as α' corrections. To obtain b, consider

$$\begin{aligned} \{Q, d_\alpha\} &= -(\not{\Pi}\lambda)_\alpha \\ \{Q, \bar\lambda \not{\Pi} d\} &= -\bar\lambda \{Q, \not{\Pi}\} d - \bar\lambda \not{\Pi}\not{\Pi}\lambda = -(\bar\lambda\gamma_m d)\,(\lambda\gamma^m\partial\theta) - \Pi_m\Pi^m(\bar\lambda\lambda) \\ \{Q, \omega\partial\theta\} &= \omega\partial\lambda + d\partial\theta, \end{aligned} \tag{3.72}$$

where we used the OPE (A.17). Here $\bar\lambda_\alpha$ is a fixed constant pure spinor, not the complex conjugate of λ. Thus, a solution of (3.70) is

$$b = \frac{\Pi^m(\bar\lambda\gamma_m d)}{2\lambda^\alpha\bar\lambda_\alpha} - \omega_\alpha\partial\theta^\alpha + \frac{(\omega\gamma^m\bar\lambda)(\lambda\gamma_m\partial\theta)}{2\lambda^\alpha\bar\lambda_\alpha}\,. \tag{3.73}$$

The last term also guarantees the invariance under the gauge transformation $\delta\omega_\alpha = \Lambda^n(\gamma_n\lambda)_\alpha$ appeared in (3.3), which can be checked by the formula (A.46).[33]

The fixed pure spinor breaks the Lorentz covariance, unless we introduce the extended PS formalism with non-minimal variables. In the extended PS formalism, we get the $\mathcal{N} = 2$ worldsheet superconformal symmetry, as will be explained further in Section 4. Assuming $\{b, b\} = 0$ and $\{b, Q\} = \{Q, b\}$, we also obtain the $\mathcal{N} = 1$ superconformal generator

$$G \equiv Q + b \quad \Rightarrow \quad \{G, G\} = 2T. \tag{3.74}$$

[32] See [15] for the discussion on the coupling to the worldsheet curvature.

[33] The last two terms can be written as $-\mathcal{P}^\alpha_\beta\, \omega_\alpha \partial\theta^\beta$ by using $\mathcal{P}^\alpha_\beta$ in (A.18).

The worldsheet superconformal symmetry is realized in different ways in RNS and PS. There is no c ghost in PS; just λ and Q are ghost-like. In addition, the superconformal generators look different. From (1.8), one finds

$$G_{\rm PS} \simeq Q + b, \qquad G_{\rm RNS} \simeq \psi\partial X + c\partial\beta + b\gamma. \tag{3.75}$$

4 Untwisting pure spinor

4.1 Lecture 8

We will explain untwisting formalism, which provides pure spinor superstring with manifest $\mathcal{N} = 1$ worldsheet superconformal symmetry. The motivations for the untwisting are

- To relate PS and RNS
- To describe RR-flux background using $\mathcal{N} = 1$ worldsheet SCFT
- To study loop amplitudes, by somehow avoiding $1/(\lambda\bar{\lambda})$ problem.
- To study AdS/CFT using the bi-twistor $\Lambda\gamma^m\Lambda \sim X^m$.

In the original PS action, all fields have integer conformal dimensions. To define a worldsheet superconformal generator, we need some fields with half-integer conformal dimensions. This can be done by twisting the energy-momentum tensor. The twisting also changes the central charge, so we should extend the PS action by adding non-minimal PS variables. In the extended PS formalism, the BRST charge and the b-ghost generate the $\mathcal{N} = 2$ worldsheet superconformal symmetry.

Below we follow the notation of [16].

4.1.1 Worldsheet superconformal symmetry

We look for the $\mathcal{N} = 2$ worldsheet superconformal symmetry from the open superstring action (3.43). The generators are a twisted version of the energy-momentum tensor T in (3.71), $G^+ = Q, G^- = b$, and the ghost current J.[34] They are given by

$$\begin{aligned} T &= -\frac{1}{2}\Pi^m\Pi_m - d_\alpha\partial\theta^\alpha - \frac{1}{2}\omega_\alpha\partial\lambda^\alpha + \frac{1}{2}(\partial\omega_\alpha)\lambda^\alpha + \dots \\ G^+ &= \lambda^\alpha d_\alpha + \dots \\ G^- &= \frac{\bar{\lambda}\not{\Pi}d}{2\lambda\bar{\lambda}} - \omega_\alpha\partial\theta^\alpha + \frac{(\omega\gamma^m\bar{\lambda})(\lambda\gamma_m\partial\theta)}{2\lambda\bar{\lambda}} + \dots \\ J &= -\omega_\alpha\lambda^\alpha + \dots, \end{aligned} \tag{4.1}$$

The $\dots$ represent α' corrections (total derivatives and the terms with non-minimal variables), which we will neglect below. These generators satisfy the $\mathcal{N} = 2$ OPE (A.10) by choosing $\dots$ properly. The ghost current J satisfies[35]

$$\lim_{y\to z} J(y)\, G^\pm(z) \sim \frac{\pm G^\pm}{y-z}\,. \tag{4.3}$$

[34] J is conserved on the flat spacetime, but not on general backgrounds.

[35] To compute the OPE with $1/\lambda\bar{\lambda}$, use

$$\omega_\alpha f(\lambda\bar{\lambda}) = \omega_\alpha\,(\lambda\bar{\lambda})\,\frac{\partial f}{\partial(\lambda\bar{\lambda})}\,. \tag{4.2}$$

The complex structure of $\mathcal{N}=2$ algebra is specified by $\bar{\lambda}$. The worldsheet fields transform as

$$\delta_+ X^m = \frac{1}{2}\lambda\gamma^m\theta, \quad \delta_+\theta^\alpha = \lambda^\alpha, \quad \delta_+ N^{mn} = \frac{1}{2}\left(d\gamma^{mn}\lambda\right), \quad \delta_+ d_\alpha = -\Pi^m\left(\gamma_m\lambda\right), \tag{4.4}$$

and

$$\delta_- X^m = \frac{(\bar{\lambda}\gamma^m d)}{\lambda\bar{\lambda}} + \frac{(\bar{\lambda}\not{\Pi}\,\gamma^m\theta)}{2\lambda\bar{\lambda}}, \quad \delta_-\theta^\alpha = \frac{(\bar{\lambda}\not{\Pi})^\alpha}{2\lambda\bar{\lambda}}, \quad \delta_- N^{mn} = \ldots \tag{4.5}$$

The chiral or anti-chiral operators are defined by $G^+ O = 0$ or $G^- O = 0$. We may introduce $\mathcal{N}=(1,1)$ superfields annihilated by G^+ and G^-. However, the formulas become messy.

Let use the language of $\mathcal{N}=1$ worldsheet superconformal symmetry. We set[36]

$$G = G^+ + G^-, \qquad \{G, G\} = T, \qquad \delta X^m = \delta_+ X^m + \delta_- X^m, \quad \cdots. \tag{4.6}$$

In the original theory, G_+ had the conformal weight $+1$ and G_- had $+2$. Now T is given in (4.1), with an extra total derivative to guarantee that $G^\pm$ have the conformal weight $\frac{3}{2}$. These non-minimal variables, $(\bar{\omega}^\alpha, \bar{\lambda}_\alpha)$ and their worldsheet superpartner (s^α, r_α), should be added to cancel the extra central charges.[37]

The superconformal primaries have a single pole for the OPE with G. Since X^m and θ^α are conformal primaries, we promote them to superfields:

$$\mathbb{X}^m = X^m + \kappa\psi^m, \qquad \delta X^m = \psi^m \equiv \frac{1}{2}\lambda\gamma^m\theta - \frac{(\bar{\lambda}\gamma^m d)}{2\lambda\bar{\lambda}} + \frac{(\bar{\lambda}\not{\Pi}\gamma^m\theta)}{4\lambda\bar{\lambda}}, \tag{4.7}$$

$$\Theta^\alpha = \theta^\alpha + \kappa\Lambda^\alpha, \qquad \delta\theta^\alpha = \Lambda^\alpha \equiv \lambda^\alpha + \frac{(\bar{\lambda}\not{\Pi})^\alpha}{2\lambda\bar{\lambda}}. \tag{4.8}$$

We regard X^m, θ^α as dynamical degrees of freedom. Then Λ^α is a new unconstrained spinor which depends on X^m through $\not{\Pi}$. We remove the X^m dependence by imposing[38]

$$0 = \mathbb{\Pi}^m\gamma_m\bar{\lambda} \equiv \left(D\mathbb{X}^m - \frac{1}{2}D\Theta\gamma^m\Theta\right)\gamma_m\bar{\lambda} = \left(\psi^m - \frac{1}{2}\Lambda\gamma^m\theta\right)\gamma_m\bar{\lambda} + O(\kappa). \tag{4.9}$$

Introduce a new primary superfield

$$\Phi_\alpha \equiv \Omega_\alpha + \kappa h_\alpha\,, \tag{4.10}$$

$$\Omega_\alpha = \omega_\alpha - \frac{1}{2(\lambda\bar{\lambda})}(\omega\gamma_m\bar{\lambda})(\gamma^m\lambda)_\alpha\,, \tag{4.11}$$

$$h_\alpha = d_\alpha - \frac{1}{2(\lambda\bar{\lambda})}(d\gamma_m\bar{\lambda})(\gamma^m\lambda)_\alpha - \frac{1}{2(\lambda\bar{\lambda})^2}\bar{\lambda}_\alpha(\bar{\lambda}\gamma^m d)\Pi_m\,. \tag{4.12}$$

This superfield satisfies

$$\bar{\lambda}\gamma^n\Phi = 0, \tag{4.13}$$

[36] In terms of OPE, the second equation is $\lim_{y\to 0} G(y)G(0) \sim T(0)/y$.

[37] The non-minimal variables are not related to the fixed constant PS $\bar{\lambda}_\alpha$ used in $(\omega_a, \lambda^\alpha)$ OPE in (A.18). We will use $\bar{\lambda}_\alpha$ as a fixed constant PS.

[38] Recall $D = \frac{\partial}{\partial\kappa} + \kappa\frac{\partial}{\partial z}$ and $\gamma^m_{\alpha\beta} = \gamma^m_{\beta\alpha}$.

so that $\mathbb{\Phi}$ has 11 components. This equation can be shown by the formula (A.46). It will be useful to remember

$$
\begin{aligned}
\psi^m - \frac{1}{2}\Lambda\gamma^m\theta &= -\frac{(\bar\lambda\gamma^m d)}{2\lambda\bar\lambda}\,, \\
h_\alpha - d_\alpha = -\frac{(d\gamma_m\bar\lambda)}{2(\lambda\bar\lambda)}\left\{(\gamma^m\lambda)_\alpha + \frac{(\bar\lambda\not{\Pi}\gamma^m)_\alpha}{2\lambda\bar\lambda}\right\} &= \left(\psi^m - \frac{1}{2}\Lambda\gamma^m\theta\right)(\Lambda\gamma_m)_\alpha\,,
\end{aligned}
\tag{4.14}
$$

where (A.46) is used in the last line.

4.1.2 Heterotic PS superstring

We write the heterotic PS superstring action using the $\mathcal{N}=1$ superfield:[39]

$$
\begin{aligned}
S &= \int d^2z d\kappa\left[\frac{1}{2}\,\Pi^m_k\bar\Pi^m + B + \mathbb{\Phi}_\alpha\bar\partial\mathbb{\Theta}^\alpha + (\bar\lambda\gamma_m L)\,(\Pi_{km}) + M_m(\bar\lambda\gamma^m\tilde{\mathbb{\Phi}})\right], \\
&= \int d^2z d\kappa\left[\frac{1}{2}\,D\mathbb{X}^m\bar\partial\mathbb{X}_m + \mathbb{\Phi}_\alpha\bar\partial\mathbb{\Theta}^\alpha + L_\alpha(\not{\mathbb{\Pi}}\bar\lambda)^\alpha + M_m(\bar\lambda\gamma^m\mathbb{\Phi}) + O(\mathbb{\Theta})\right],
\end{aligned}
\tag{4.15}
$$

where $L_\alpha\,, M_m$ are Lagrangian multipliers, and

$$
\begin{aligned}
\Pi^m_k = D\mathbb{X}^m - \frac{1}{2}D\mathbb{\Theta}\gamma^m\mathbb{\Theta}, \quad \Pi^m = \partial\mathbb{X}^m - \frac{1}{2}\partial\mathbb{\Theta}\gamma^m\mathbb{\Theta}, \quad \bar\Pi^m = \bar\partial X^m - \frac{1}{2}\bar\partial\theta\gamma^m\theta, \\
B = \frac{1}{4}\Big[(D\mathbb{\Theta}\gamma^m\mathbb{\Theta})\,\bar\partial X_m - D\mathbb{X}^m\left(\bar\partial\mathbb{\Theta}\gamma_m\mathbb{\Theta}\right)\Big].
\end{aligned}
\tag{4.16}
$$

Let us solve the constraints and reproduce the original action. $\not{\mathbb{\Pi}}\bar\lambda = 0$ gives[40]

$$
0 = \left(\psi^m - \frac{1}{2}\Lambda\gamma^m\theta\right)(\gamma_m\bar\lambda)^\alpha\,, \tag{4.17}
$$

$$
0 = \left(\partial X^m - \frac{1}{2}\partial\theta\gamma^m\theta - \frac{1}{2}\Lambda\gamma^m\Lambda\right)(\gamma_m\bar\lambda)^\alpha\,. \tag{4.18}
$$

The first line is consistent with (4.14).

Let us expand the action (4.15) by component fields,

$$
S = \int d^2z\left(\frac{1}{2}\partial X^m\partial X_m + \frac{1}{2}\,\psi^m\bar\partial\psi_m + \Omega_\alpha\bar\partial\Lambda^\alpha + h_\alpha\bar\partial\theta^\alpha + \dots\right), \tag{4.19}
$$

where $\dots$ represents the terms higher order in Θ. From (4.14) and the PS constraints, we obtain[41]

$$
\begin{aligned}
h_\alpha\bar\partial\theta^\alpha &= d_\alpha\bar\partial\theta^\alpha - \frac{(\bar\lambda\gamma^m d)}{2\lambda\bar\lambda}(\Lambda\gamma_m\bar\partial\theta) \\
\frac{1}{2}\,\psi^m\bar\partial\psi_m &= \frac{(\bar\lambda\gamma^m d)}{2\lambda\bar\lambda}\bar\partial(\Lambda\gamma_m\theta) + (\text{total derivative}).
\end{aligned}
\tag{4.20}
$$

[39] See (1.3) for RNS and (2.27) for GS heterotic superstring actions.

[40] One finds from (4.18) that the $O(\kappa)$ component of $\mathbb{\Pi}^m$ is equal to $\Pi^m - \frac{1}{2}\Lambda\gamma^m\Lambda$.

[41] To see the second equation, take the derivative of $(\Lambda\gamma_m\theta)(\Lambda\gamma^m\theta) = 0$ and use $\theta\gamma_m\theta = 0$.

The sum of the two gives $p_\alpha \bar\partial\theta^\alpha \sim d_\alpha \bar\partial\theta^\alpha$. Similarly, to obtain $\omega\bar\partial\lambda$, recall

$$\Omega_\alpha\, \bar\partial\Lambda^\alpha = \Omega_\alpha\, \bar\partial\left(\frac{(\bar\lambda \not{\Pi})^\alpha}{\lambda\bar\lambda} + \lambda^\alpha\right). \tag{4.21}$$

The first term vanishes because $\bar\lambda\gamma^m\Phi \sim \bar\lambda\gamma^m\Omega = 0$. We then identify $\Omega_\alpha \sim \omega_\alpha$. By working out the details, one can show that the new action (4.15) is equivalent to the old one (3.43).

In the untwisted formalism, we consider only the $\mathcal{N}=1$ superconformal symmetry generated by the sum $G = Q_{\mathrm{BRST}} + b$. Roughly speaking, we replace the BRST-closed condition $Q_{\mathrm{BRST}}V = 0$ by the superconformal primary condition $GV = 0$. The two conditions are more or less equivalent in Siegel gauge $b_0 V = 0$. The $\mathcal{N}=2$ structure is not completely lost, because G^+ and G^- have different ghost numbers.

In ordinary PS, the b ghost does not propagate in Siegel gauge, and the BRST-closed condition is sufficient for computing tree-level amplitudes. However, the vertex operator like $V_0 = \lambda^\alpha A_\alpha$ is not a conformal primary. The new vertex operator V in (4.24) is a conformal primary, but more complicated than V_0.

4.2 Lecture 9

4.2.1 Vertex operators

We look for $\mathcal{N}=1$ superconformal primary

$$G(y)V(z) \sim \frac{DV(z)}{y-z}\,, \qquad T(y)V(z) \sim \frac{\partial V(z)}{y-z} = \frac{D^2V(z)}{y-z}\,. \tag{4.22}$$

The vertex operator $V = \lambda_\alpha A^\alpha$ in (3.47) is not superconformal primaries since its OPE with G^- has a double pole. A supersymmetric version

$$V = D\Theta^\alpha A_\alpha(\mathbb{X},\Theta) = \Lambda^\alpha A_\alpha + O(\kappa) \tag{4.23}$$

is not Lorentz covariant because of the dependence on $\bar\lambda$ in Λ.[42] We tentatively give up the Lorentz covariance, and try an ansatz for superconformal primaries:

$$V = \sum_n V_n \equiv D\Theta^\alpha A_\alpha + \Pi^m A_m + \Phi_\alpha W^\alpha\,, \tag{4.24}$$

where n is the ghost number;

$$J(y)V_n(z) \sim \frac{nV_n(z)}{y-z}\,, \qquad J\cdot V_n = nV_n\,. \tag{4.25}$$

[42]This is because $N_{mn}(z)\,\bar\lambda(0) \sim 0$. The Lorentz covariance is restored only if the constraints $\not{\Pi}\bar\lambda = \bar\lambda\gamma^m\Phi = 0$ are imposed.

We assign the ghost number $+1$ to λ and -1 to ω from (4.1). Let us write down each term of (4.24),[43]

$$
\begin{aligned}
D\Theta^\alpha A_\alpha \simeq \Lambda^\alpha A_\alpha \qquad\qquad & = \left(\lambda^\alpha + \frac{(\bar\lambda \not{\Pi})^\alpha}{\lambda\bar\lambda}\right) A_\alpha && (4.26)\\
\Pi^m A_m \simeq \left(\psi^m - \frac{1}{2}\Lambda\gamma^m\theta\right) A_m & = -\frac{(\bar\lambda\gamma^m d)}{2\lambda\bar\lambda} A_m && (4.27)\\
\Phi_\alpha W^\alpha \simeq \Omega_\alpha W^\alpha \qquad\qquad & = -\frac{1}{4\lambda\bar\lambda}\Big((\bar\lambda\gamma_{mn})_\alpha N^{mn} + \bar\lambda_\alpha J\Big) W^\alpha\,, && (4.28)
\end{aligned}
$$

where we used (A.45), (3.51). The term $\lambda^\alpha A_\alpha$ belongs to V_1, and the others belong to V_{-1}. Thus $V = V_1 + V_{-1}$ in (4.24). Since V is a superconformal primary,

$$
G^+V_1 = 0, \qquad G^+V_{-1} + G^-V_{-1} = 0, \qquad G^-V_{-1} = 0. \tag{4.29}
$$

Consider the κ-integrated vertex operator $U = \int d\kappa V$. Recall that[44]

$$
\begin{aligned}
&U = DV, \quad DU = \partial V, \quad D = G^+ + G^-,\\
&G^+U_0 = \partial V_1, \quad G^-U_0 + G^+U_{-2} = \partial V_{-1}.
\end{aligned} \tag{4.30}
$$

We require "BRST invariance" by imposing

$$
(DV)_{n>0} = 0 \qquad \Leftrightarrow \qquad V = \sum_{n=-\infty}^{1} V_n\,, \tag{4.31}
$$

and require Lorentz covariance by imposing

$$
\int dz \left(\int d\kappa V\right)_0 = \int dz\,(DV)_0 \quad \text{is independent of } \bar\lambda. \tag{4.32}
$$

This condition is non-trivial, and not gauge-fixing for the superconformal symmetry. Note that $\int (DV)_0$ agrees with the usual pure spinor vertex operator. If the background spacetime has a Killing spinor, we can make the vertex operator independent of $\bar\lambda$ by choosing $\bar\lambda$ in the Killing direction.

By writing

$$
U = \sum_m U_m = \underbrace{U_0}_{\text{Lorentz covariant}} + \underbrace{U_{-2}}_{\text{not covariant}}\,, \tag{4.33}
$$

we want to show that only U_0 contributes to the tree-level amplitude, which is equivalent to

$$
U_0 = \partial\theta^\alpha A_\alpha + \Pi^m A_m + d_\alpha W^\alpha + \frac{1}{2} N_{mn} F^{mn}\,, \tag{4.34}
$$

[43] The last line can be derived by applying the identity (A.45) to (4.11).

[44] From (1.2), $U = GV$ up to a total derivative in z.

as in (3.51). We apply the chain rule to $U = \int d\kappa \left(D\Theta^\alpha \mathbb{A}_\alpha + \mathbb{\Pi}^m \mathbb{A}_m + \mathbb{\Phi}_\alpha \mathbb{W}^\alpha\right)$ as

$$
\begin{aligned}
\int d\kappa f(\mathbb{X}, \Theta) &= \left(D\mathbb{X}^m \partial_m + D\Theta^\alpha \partial_\alpha\right) f \Big|_{\kappa=0} \\
&= \left(\psi^m - \frac{1}{2}\Lambda\gamma^m\theta\right) \partial_m f + \Lambda^\alpha D_\alpha f \,,
\end{aligned}
\tag{4.35}
$$

where D_α is the spacetime super-covariant derivative of (3.9). We obtain

$$
\begin{aligned}
U = \partial\theta^\alpha A_\alpha + \Lambda^\alpha &\Big[\left(\psi^m - \frac{1}{2}\Lambda\gamma^m\theta\right) \partial_m A_\alpha + \Lambda^\beta D_\beta A_\alpha\Big] \\
&+ \left(\partial X^m - \frac{1}{2}\partial\theta\gamma^m\theta - \frac{1}{2}\Lambda\gamma^m\Lambda\right) A_m \\
&- \left(\psi^m - \frac{1}{2}\Lambda\gamma^m\theta\right) \Big[\left(\psi^n - \frac{1}{2}\Lambda\gamma^n\theta\right) \partial_n A_m + \Lambda^\alpha D_\alpha A_m\Big] \\
&+ h_\alpha W^\alpha + \Omega_\alpha \Big[\left(\psi^m - \frac{1}{2}\Lambda\gamma^m\theta\right) \partial_m W^\alpha + \Lambda^\beta D_\beta W^\alpha\Big].
\end{aligned}
\tag{4.36}
$$

Then we extract the $n = 0$ terms, by recalling that $(\psi^m - \frac{1}{2}\Lambda\gamma^m\theta)$ has $n = -1$ from (4.14). We find

$$
\begin{aligned}
U_0 = \partial\theta^\alpha A_\alpha + \Pi^m A_m &+ \left\{d_\alpha + \left(\psi^m - \frac{1}{2}\Lambda\gamma^m\theta\right) (\lambda\gamma_m)_\alpha\right\} W^\alpha + \Omega_\alpha \, \lambda^\beta D_\beta W^\alpha \\
&+ \lambda^\alpha(\psi^m - \frac{1}{2}\Lambda\gamma^m\theta)\,(\partial_m A_\alpha - D_\alpha A_m) + \frac{1}{2}\left[\Lambda^\alpha\Lambda^\beta\left(2D_{(\alpha}A_{\beta)} - \gamma^m_{\alpha\beta}A_m\right)\right]_{n=0} .
\end{aligned}
\tag{4.37}
$$

By using the identities in Section 3.1.2, one finds that this agrees with (4.34). We use the identity (3.23) to obtain $\frac{1}{2}\, N^{mn} F_{mn}$, and the extra term $(\Omega_\alpha - \omega_\alpha)$ vanishes from (A.38). The first term in the second line cancels $(h_\alpha - d_\alpha)W^\alpha$ in the first line.

4.2.2 Scattering amplitude

After untwisting, the vertex operator V_i has conformal weight $\frac{1}{2}$. The string amplitude at tree-level is given by

$$
\begin{aligned}
A_{\text{tree}} &= \mathfrak{X}\,\langle (GV_1)(z_1)(GV_2)(z_2)(GV_3)(z_3) \int GV_4 \int GV_5 \ldots\rangle, \\
\mathfrak{X} &= (z_1 - z_2)(z_2 - z_3)(z_3 - z_1).
\end{aligned}
\tag{4.38}
$$

This does not depend on $z_{1,2,3}$, since GV_i has conformal weight 1. Each factor in A_{tree} can have the component with zero ghost number.[45] The component of A_{tree} with zero ghost number is special, because j_{BRST} is conserved.

The tree-level amplitude should satisfy superconformal Ward identities when V_i are superconformal primaries. We do not know any prescription to compute loop amplitudes.

[45] In usual PS, we assigned the ghost number $(0, 1, 2, 3)$ to $(1, V, V^*, \lambda^3\theta^5)$.

The vertex operator GV is written as

$$GV = A_m \partial X^m + \frac{F_{mn}}{4} (d\gamma^{mn}\theta + \omega\gamma^{mn}\lambda) + \dots . \tag{4.39}$$

One can show that the terms in the parenthesis have the same OPE as $\psi\psi$ which appears in the gluon tree amplitude in RNS formalism,

$$A^{\mathrm{RNS}}_{\mathrm{tree}} = \langle V^{\mathrm{RNS}}(z_1) V^{\mathrm{RNS}}(z_2) V^{\mathrm{RNS}}(z_3) \int V^{\mathrm{RNS}} \dots \rangle, \qquad V^{\mathrm{RNS}} = A\partial X + F\psi\psi. \tag{4.40}$$

4.2.3 Unsolved problems

We will discuss the following problems will be discussed in the forthcoming sections.

The first problem is the relation between PS and RNS. The degrees of freedom in NS sector look similar, but those in R sector look different.

The second problem is to apply the untwisting to $\mathrm{AdS}_5 \times \mathrm{S}^5$.

The third problem is possible relation to twistors. By expanding the constraint (4.42) we obtain

$$0 = \left(\partial X^m - \frac{1}{2} \partial\theta\gamma^m\theta - \frac{1}{2} \Lambda\gamma^m\Lambda \right) (\gamma_m \bar{\lambda})^\alpha \,, \tag{4.41}$$

which implies that Λ is a bosonic spinor coordinate related to X^m as in twistor theory.

4.3 Lecture 10

4.3.1 Relation to RNS

Recall that we imposed the constraint

$$0 = \left(D\mathbb{X}^m - \frac{1}{2} D\Theta\gamma^m\Theta \right) \gamma_m \bar{\lambda}, \tag{4.42}$$

which relate two worldsheet superfields $\mathbb{X}^m$ and Θ. This constraint was solved by

$$\mathbb{X}^m = X^m + \kappa \left(\frac{1}{2} \lambda\gamma^m\theta - \frac{(\bar{\lambda}\gamma^m d)}{2\lambda\bar{\lambda}} + \frac{(\bar{\lambda}\not{\Pi}\gamma^m\theta)}{4\lambda\bar{\lambda}} \right), \tag{4.43}$$

$$\Theta^\alpha = \theta^\alpha + \kappa \left(\lambda^\alpha + \frac{(\bar{\lambda}\not{\Pi})^\alpha}{2\lambda\bar{\lambda}} \right), \tag{4.44}$$

$$\Pi^m = \partial X^m - \frac{1}{2} \partial\theta\gamma^m\theta. \tag{4.45}$$

We define a new variable

$$\Theta'^\alpha = \Theta^\alpha + \mathbb{K}_m (\gamma^m \bar{\lambda})^\alpha, \tag{4.46}$$

such that the above condition is rewritten as

$$\left(D\Theta'\gamma^m\Theta' \right) \gamma_m \bar{\lambda} = 0. \tag{4.47}$$

Now Θ' is independent of $\mathbb{X}^m$, and more closely related to the worldsheet fermions in RNS formalism.

The fermionic superfield $\mathbb{K}_m$ in (4.46) satisfies the equation[46]

$$
\begin{aligned}
&\Big[D\mathbb{X}_m + D\mathbb{K}_m(\bar\lambda\Theta') + \mathbb{K}_m(\bar\lambda D\Theta')\Big]\gamma^m\bar\lambda = 0 \\
\Rightarrow \qquad &\mathbb{K}_m = -\frac{1}{\bar\lambda D\Theta'}D\mathbb{X}_m + \frac{1}{2}\frac{\bar\lambda\Theta'}{\bar\lambda D\Theta'}\left(\frac{D\mathbb{X}_m}{\bar\lambda D\Theta'}\right),
\end{aligned} \tag{4.48}
$$

assuming $D\bar\lambda = 0$ and $\bar\lambda D\Theta \neq 0$, which is the supersymmetrization of $\bar\lambda\lambda \neq 0$.

The heterotic action (4.15) can be written as

$$
S = \int d^2z d\kappa \left[\frac{1}{2}D\mathbb{X}^m\bar\partial\mathbb{X}_m + \tilde\Phi_\alpha\bar\partial\Theta'^\alpha + (\bar\lambda\gamma_m\tilde L)\left(D\Theta'\gamma^m\Theta'\right) + \tilde M_m(\bar\lambda\gamma^m\tilde\Phi) + O(\Theta')\right], \tag{4.49}
$$

where $\tilde\Phi = \tilde\Omega_\alpha + \kappa\tilde h_\alpha$ differs from Φ by a shift which absorbs the difference $(\Theta' - \Theta)$.

We can ignore the $O(\Theta')$ terms because they do not contribute to the tree amplitude. Let us explain this point in NS sector. Recall that λ had the $U(1)$ charge $+1$ and ω had -1 under $J \sim -\lambda^\alpha\Omega_\alpha$ in (4.1). The vertex operators had non-positive $U(1)$ charge as in (4.33). Since the worldsheet supersymmetry relates $\Omega \sim \omega$ with h, gauge-invariant quantities with zero $U(1)$ charge with respect to $J = -\lambda^\alpha\tilde\Omega_\alpha$ satisfy

$$
\text{Number of } (\theta) - \text{Number of } (h) \geq 0. \tag{4.50}
$$

At the same time, we need

$$
\text{Number of } (\theta) - \text{Number of } (h) = 0 \tag{4.51}
$$

to have the non-vanishing tree amplitude (4.38). Thus, the $O(\Theta')$ terms in worldsheet do not contribute to the tree amplitude.

We want to argue that the tree amplitude of gluons based on (4.49) is equivalent to that of RNS formalism (4.40). For this purpose it is sufficient to prove that the components of Θ, Φ do not contribute to the tree amplitude. Vertex operators in NS sector are given in (4.34) or (4.39). If we rewrite the vertex operator using Θ', we obtain

$$
GV = A_m\partial X^m + \frac{1}{2}F_{mn}M'^{mn} + \ldots, \qquad M'^{mn} = \frac{1}{2}\left(\tilde\Omega\gamma^{mn}\lambda + \tilde h\gamma^{mn}\theta'\right). \tag{4.52}
$$

where

$$
\Theta'^\alpha = \theta'^\alpha + \kappa\lambda^\alpha + O(\theta^2), \qquad \theta'\gamma^{mn}\lambda = 0. \tag{4.53}
$$

The new variable M'^{mn} have the OPE

$$
M'_{mn}(z)M'_{pq}(0) \sim \frac{M'_{m(p}\eta_{n)q}}{z}, \tag{4.54}
$$

[46] To show the second line, use $(\bar\lambda\Theta')(\bar\lambda\Theta') = 0$.

instead of (3.59). The tree amplitude consists of gauge-invariant quantities with zero $U(1)$ charges, like M'^{mn}. However, M' does not contribute to the tree amplitude. If A_{tree} contains the term $\langle (M')^n \rangle$, one apply the OPE (4.54) repeatedly to reduce it to $\langle M' \rangle$, which is zero. We conjecture that only the superconformal primaries with the superfield $\mathbb{X}^m$ contribute to the tree amplitude in NS sector.

What happens in R sector is less clear. The variable θ in PS should corresponds to spin fields with $\beta\gamma$ ghosts in RNS.[47]

4.3.2 Curved NS backgrounds

The heterotic action (4.49) on NS curved backgrounds is given by

$$S = \int d^2z d\kappa \left[\frac{1}{2} D\mathbb{X}^m \bar{\partial}\mathbb{X}^n \left(g_{mn}(X) + b_{mn}(X)\right) + \tilde{\Phi}_\alpha \bar{\nabla}\Theta'^\alpha \right.$$
$$\left. + (\bar{\lambda}\gamma_m \tilde{L}) \left(D\Theta'\gamma^m\Theta'\right) + \tilde{M}_m(\bar{\lambda}\gamma^m\tilde{\Phi}) \right], \quad (4.55)$$

where

$$\bar{\nabla}\Theta'^\alpha = \bar{\partial}\Theta'^\alpha + \bar{\partial}\mathbb{X}^m \, \omega_{m\beta}{}^\alpha(X)\Theta'^\beta. \quad (4.56)$$

We do not know how to put the action on Ramond backgrounds, because we do not know Ramond vertex operators.

Let us introduce the spacetime superfields $Z^M = (\mathbb{X}^m, \Theta^\alpha)$. The heterotic action is rewritten as

$$S = \int d^2z d\kappa \left[\frac{1}{2}\eta_{ab} E^a_M E^b_N \, DZ^M \bar{\partial} Z^N + \frac{1}{2} B_{MN} \, DZ^M \bar{\partial} Z^N + \Phi_\alpha E^\alpha_M \, \bar{\partial} Z^M \right.$$
$$\left. + (\bar{\lambda}\gamma_a \tilde{L}) \left(E^a_M \, DZ^M\right) + \tilde{M}_a(\bar{\lambda}\gamma^a\tilde{\Phi}) \right]. \quad (4.57)$$

In GS DZ^M was ∂Z^M. All ghosts come from the term with Φ. This action should be equivalent to the original PS action on curved backgrounds, which is

$$S_{\text{PS}} = \int d^2z d\kappa \left[\frac{1}{2}\eta_{ab} E^a_M E^b_N \, \partial Z^M \bar{\partial} Z^N + \frac{1}{2} B_{MN} \, \partial Z^M \bar{\partial} Z^N + d_\alpha E^\alpha_M \, \bar{\partial} Z^M + \omega_\alpha \bar{\nabla}\lambda^\alpha \right], \quad (4.58)$$

where

$$\bar{\nabla}\lambda^\alpha = \bar{\partial}\lambda^\alpha + \omega_{M\beta}{}^\alpha \, \lambda^\beta \, \bar{\partial} Z^M. \quad (4.59)$$

Around the flat background, it should behave as

$$S_{\text{curved}} = S_{\text{flat}} + \int d^2z \, V. \quad (4.60)$$

[47] The ghosts in PS were accounted by $\mathfrak{X} = (z_1 - z_2)(z_2 - z_3)(z_3 - z_1)$.

4.3.3 Type IIB

The type IIB action can be obtained by replacing $\bar{\partial}$ with $\bar{D}$ and adding right movers,

$$S = \int d^2 z d\kappa \Big[\frac{1}{2}\eta_{ab} E^a_M E^b_N \, DZ^M \bar{D} Z^N + \frac{1}{2} B_{MN} \, DZ^M \bar{D} Z^N \\ - \Phi_\alpha E^\alpha_M \, \bar{D} Z^M + \hat{\Phi}_{\hat{\alpha}} E^{\hat{\alpha}}_M \, DZ^M - \Phi_\alpha \hat{\Phi}_{\hat{\alpha}} F^{\alpha\hat{\alpha}} + (\text{Lagrange multipliers}) \Big]. \quad (4.61)$$

The term with $F^{\alpha\hat{\alpha}}$ is needed to reproduce vertex operators. The Lagrange multiplier contains $\bar{\lambda}$, which is defined patchwise unless the background has a Killing spinor. This action is simpler than the original PS action,

$$S_{\rm PS} = \int d^2 z d\kappa \Big[\frac{1}{2}\eta_{ab} E^a_M E^b_N \, \partial Z^M \bar{\partial} Z^N + \frac{1}{2} B_{MN} \, \partial Z^M \bar{\partial} Z^N \\ + d_\alpha E^\alpha_M \, \bar{\partial} Z^M + \bar{d}_{\hat{\alpha}} E^{\hat{\alpha}}_M \, \partial Z^M + \omega_\alpha \bar{\nabla} \lambda^\alpha + \bar{\omega}_{\hat{\alpha}} \nabla \bar{\lambda}^{\hat{\alpha}} + d_\alpha \bar{d}_{\hat{a}} F^{\alpha\hat{\alpha}} \\ + R_{abcd} (\lambda \gamma^{ab} \omega)(\bar{\lambda} \gamma^{cd} \bar{\omega}) + d_\alpha (\bar{\gamma}^{ab} \bar{\omega}) S^\alpha_{ab} + \bar{d}_\alpha (\gamma^{ab} \omega) \bar{S}^{\hat{\alpha}}_{ab} \Big]. \quad (4.62)$$

We conjecture

$$S = S_{\rm PS} \quad (4.63)$$

by integrating κ, imposing the constraints, and restricting to the sector of zero ghost number.

5 Conclusion

We summarized three formalisms of perturbative superstring theory, RNS, GS and PS. After the review of each method and relation in between, we introduced untwisting PS formalism, which realizes the manifest $\mathcal{N}=1$ worldsheet superconformal symmetry. This symmetry will be useful to study $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring.

We omit the last few lectures from this note, where twistor-string in $\mathbb{R}^4$, $\mathbb{R}^{10}$, and $\mathrm{AdS}_5 \times \mathrm{S}^5$ are discussed. Interested readers can consult [17, 16].

Mathematica codes

This lecture note is accompanied by two `Mathematica` codes. The first code computes the free-field OPE's relevant in RNS formailsm. The second implements Gamma matrices in $SO(1,9)$ which can be used to prove various identities.

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A Notation

In the flat spacetime, we use the following indices:

- $m, n, \ldots$ label vectors; X^m
- $\alpha, \beta, \ldots$ label spinors; θ^α
- $\hat{\alpha}, \hat{\beta}, \ldots$ label spinors of the opposite chirality; $\bar{\theta}^{\hat{\alpha}}$, $\hat{\theta}^{\hat{\alpha}}$ or $\tilde{\theta}^{\hat{\alpha}}$
- $A, B, \ldots$ label superspace; Z^A

In the curved spacetime,

- $m, n, \ldots$ label (locally Lorentz) coordinates; X^m
- $a, b, \ldots$ label tangent space; $e_a = e_a^m \frac{\partial}{\partial X^m}$

We also use $a, b = 1, 2, \ldots, 5$ for the $U(5)$ fundamental representation.

The indices are symmetrized as

$$F_{\{a_1 a_2 \ldots a_n\}} = \frac{1}{n!} \sum_{\sigma \in S_n} F_{a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}} , \quad F_{[a_1 a_2 \ldots a_n]} = \frac{1}{n!} \sum_{\sigma} \mathrm{sign}(\sigma) F_{a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}} . \quad (\mathrm{A.1})$$

A.1 OPE's

We summarize the OPE's of worldsheet conformal primaries.

Let us consider the action

$$S = \frac{1}{2\pi} \int dz d\bar{z} \left(\frac{1}{2} \partial X^m \bar{\partial} X_m - \frac{1}{2} \psi^m \bar{\partial} \psi_m + \beta \bar{\partial} \gamma + b \bar{\partial} c \right), \tag{A.2}$$

and take the variation such as $0 = \langle \frac{\delta}{\delta X^n} \left(X_m \, e^{-S} \right) \rangle$. This gives the OPE

$$X_m(z,\bar{z}) X_n(0) \sim -\eta_{mn} \log |z|^2, \qquad \gamma(z)\beta(0) \sim -\beta(z)\gamma(0) \sim \frac{1}{z}, \tag{A.3}$$

$$\psi_m(z)\psi_n(0) \sim \frac{-\eta_{mn}}{z}, \qquad c(z)b(0) \sim b(z)c(0) \sim \frac{1}{z}. \tag{A.4}$$

The stress-energy tensors are,

$$T_X = :-\frac{1}{2} \partial X^m \partial X_m:, \qquad T_\psi = :\frac{1}{2} \psi^m \partial \psi_m:, \tag{A.5}$$

$$T_{bc} = :-c(\partial b) + \lambda \partial (cb):, \qquad T_{\beta\gamma} = :\gamma(\partial\beta) - \frac{2\lambda - 1}{2} \partial(\gamma\beta):, \tag{A.6}$$

which satisfy the OPE's,

$$\begin{gathered} T(z)T(0) \sim \frac{c}{2z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z}, \\ c_X = D, \quad c_\psi = \frac{D}{2}, \quad c_{bc} = -12 \left(\lambda - \frac{1}{2} \right)^2 + 1, \quad c_{\beta\gamma} = 12\lambda(\lambda-1)^2 - 1. \end{gathered} \tag{A.7}$$

Here b, c and β, γ have the conformal dimensions

$$\dim\,(b,c) = (\lambda, 1-\lambda), \qquad \dim\,(\beta,\gamma) = \left(\lambda - \frac{1}{2}, \frac{3}{2} - \lambda \right). \tag{A.8}$$

We set $D = 10, \lambda = 2$ for the critical superstring in RNS.

When the worldsheet theory has $\mathcal{N} = 1$ supersymmetry, we define the stress-energy tensor superfield $\mathbb{T}(z) = G(z) + \kappa T(z)$. The superconformal current $G(z)$ satisfies

$$T(z)G(0) \sim \frac{3G(0)}{2z^2} + \frac{\partial G(0)}{z}, \qquad G(z)G(0) \sim \frac{c}{6z^3} + \frac{T(0)}{2z}. \tag{A.9}$$

In the case of $\mathcal{N} = 2$ supersymmetry, the stress-energy tensor multiplet (T, G^+, G^-, J) satisfies

$$\begin{aligned} & T(z)G^\pm(0) \sim \frac{3G^\pm(0)}{2z^2} + \frac{\partial G^\pm(0)}{z}, \qquad T(z)J(0) \sim \frac{J(0)}{z^2} + \frac{\partial J(0)}{z}, \\ & G^+(z)G^-(0) \sim \frac{2c}{3z^3} + \frac{2J(0)}{z^2} + \frac{2T(0) + \partial J(0)}{z}, \qquad J(z)G^\pm(0) \sim \frac{\pm G^\pm(0)}{z}, \\ & J(z)J(0) \sim \frac{c}{3z^2}. \end{aligned} \tag{A.10}$$

Let us study the OPE of a free chiral boson in detail,

$$\varphi(z)\,\varphi(w) = -\log(z-w) + :\varphi(z)\varphi(w):\,. \tag{A.11}$$

An exponential of φ satisfies the OPE

$$\varphi(z) :e^{n\varphi(w)}: = -n \log(z-w) :e^{n\varphi(w)}: + :\varphi(z)\, e^{n\varphi(w)}:, \tag{A.12}$$

$$:e^{m\varphi(z)}::e^{n\varphi(w)}: = (z-w)^{-mn} :e^{m\varphi(z)}\, e^{n\varphi(w)}: . \tag{A.13}$$

In particular, $e^{\pm\varphi}$ satisfy

$$:e^{\varphi(z)}::e^{\pm\varphi(w)}:= - :e^{\pm\varphi(w)}::e^{\varphi(z)}:= (z-w)^{\mp 1} :e^{\varphi(z)\pm\varphi(w)}: . \tag{A.14}$$

The exponential operators behave like free fermions ψ_m. However, they behave like bosons inside the normal-ordering, $:e^{\varphi(z)}e^{\pm\varphi(w)}: = :e^{\pm\varphi(w)}e^{\varphi(z)}: = :e^{\varphi(z)\pm\varphi(w)}:$.

In PS, we forget ψ_m in (A.2) and replace

$$b\bar\partial c \;\to\; p_\alpha \bar\partial\theta^\alpha, \qquad \dim(p_\alpha, \theta^\alpha) = (1,0), \quad c = -2\times 16, \tag{A.15}$$

$$\beta\bar\partial\gamma \;\to\; -\omega_\alpha \bar\partial\lambda^\alpha, \quad \dim(\omega_\alpha, \lambda^\alpha) = (1,0), \quad c = +2\times 11, \tag{A.16}$$

which satisfy the OPE

$$\begin{gathered} p_\alpha(z)\,\theta^\beta(0) \sim \theta^\beta(z)\, p_\alpha(0) \sim \frac{\delta^\alpha_\beta}{z}\,, \\ \lambda^\alpha(z)\,\omega_\beta(0) \sim \frac{\mathcal{P}^\alpha_\beta(z)}{z}\,, \qquad \omega_\beta(z)\,\lambda^\alpha(0) \sim -\frac{\mathcal{P}^\alpha_\beta(0)}{z}\,. \end{gathered} \tag{A.17}$$

Since λ^α is a PS, the metric $\mathcal{P}^\alpha_\beta$ should be a rank 11 matrix. We can construct $\mathcal{P}$ by using a fixed unconstrained spinor $\bar\lambda_\alpha$ as

$$\mathcal{P}^\alpha_\beta(z) = \delta^\alpha_\beta - \frac{1}{2}\,\frac{(\gamma_m\bar\lambda)^\alpha(\gamma^m\lambda)_b}{2\bar\lambda\lambda}(z), \tag{A.18}$$

which satisfies

$$\begin{gathered} \mathcal{P}^\alpha_\beta(z)\,\mathcal{P}^\beta_\gamma(z) = \mathcal{P}^\alpha_\gamma(z), \qquad \mathcal{P}^\alpha_\beta(z)\,\lambda^\beta(z) = \lambda^\alpha(z), \qquad \mathcal{P}^\alpha_\beta(z)\,(\lambda\gamma_n)_\alpha(z) = 0, \\ \lambda\gamma_m\lambda(z)\,\omega_\alpha(0) \sim \omega_\alpha(z)\,\lambda\gamma_m\lambda(0) \sim 0. \end{gathered} \tag{A.19}$$

If $\bar\lambda_\alpha$ is a fixed PS, then

$$(\bar\lambda\lambda)(z)\,\omega_\beta(0) \sim -\omega_\beta(z)\,(\bar\lambda\lambda)(0) \sim \frac{\bar\lambda_\beta(z)}{z}\,, \qquad \mathcal{P}^\alpha_\beta\,\bar\lambda_\alpha = \bar\lambda_\beta\,, \qquad \mathcal{P}^\alpha_\beta\,(\bar\lambda\gamma_n)^\beta = 0. \tag{A.20}$$

We need to be careful about the OPE between $T = -\omega_\alpha\partial\lambda^\alpha$ and ω_α,

$$T(z)\,\omega_\alpha(0) \sim \mathcal{P}^\alpha_\beta(0)\left(\frac{\omega_\alpha(0)}{z^2} + \frac{\partial\omega_\alpha(0)}{z}\right). \tag{A.21}$$

Since $\mathcal{P}^\alpha_\beta\,\partial\omega_\alpha \neq \partial(\mathcal{P}^\alpha_\beta\,\omega_\alpha)$, strictly speaking ω_α is not primary. This is not a serious problem, because ω_α appears only in the gauge-invariant combinations such as $\lambda^\alpha\omega_\alpha$ or $\mathcal{P}^\alpha_\beta\,\omega_\alpha$ and extra terms vanish.

For doing calculation, it is simpler to take the $U(5)$ basis $\omega_\alpha = (\omega_+ \,, \omega_{ab} \,, \omega_a)$ and fix the gauge by $\bar{\lambda}_\alpha = \bar{\lambda}_+$. From (A.53) it implies $\omega^a = 0$, thus

$$\omega_\alpha \bar{\partial} \lambda^\alpha \;\to\; \omega_+ \bar{\partial} \lambda^+ + \frac{1}{2} \omega_{ab} \bar{\partial} \lambda^{ab} \,. \tag{A.22}$$

where (u, t) are chiral scalars. They satisfy the OPE

$$\lambda^+(z)\, \omega_+(y) \sim \frac{1}{z-y} \,, \qquad \lambda^{ab}(z)\, \omega_{cd}(y) \sim \frac{\delta^a_c \delta^b_d - \delta^a_d \delta^b_c}{z-y} \,. \tag{A.23}$$

After the untwisting in Section 4, the equation (A.16) becomes

$$\beta \bar{\partial} \gamma \;\to\; -\omega_\alpha \bar{\partial} \lambda^\alpha, \quad \dim\left(\omega_\alpha, \lambda^\alpha\right) = \left(\tfrac{1}{2} \,, \tfrac{1}{2}\right), \quad c = -1 \times 11. \tag{A.24}$$

The difference of the central charges between (A.16) and (A.24) is -33. To cancel it, we add the following non-minimal PS,

$$b \bar{\partial} c \;\to\; s^\alpha \bar{\partial} r_\alpha, \quad \dim\left(s^\alpha, r_\alpha\right) = \left(\tfrac{1}{2} \,, \tfrac{1}{2}\right), \quad c = +1 \times 11, \tag{A.25}$$

$$\beta \bar{\partial} \gamma \;\to\; \bar{\omega}^\alpha \bar{\partial} \bar{\lambda}_\alpha \,, \quad \dim\left(\bar{\omega}^\alpha, \bar{\lambda}_\alpha\right) = (1, 0), \quad c = +2 \times 11, \tag{A.26}$$

with the PS conditions[48]

$$\bar{\lambda}_\alpha (\gamma_m)^{\alpha\beta} \bar{\lambda}_\beta = 0, \qquad \bar{\lambda}_\alpha (\gamma_m)^{\alpha\beta} r_\beta = 0. \tag{A.27}$$

A.2 Curved spacetime with torsion

There are two ways to represent tensors in the curved spacetime,

$$\eta_{ab} = e^m_a e^n_b \, g_{mn} \,, \qquad B_{ab} = e^m_a e^n_b \, B_{mn} \,. \tag{A.28}$$

Covariant derivatives can be written in two ways,

$$\nabla_m v^n = \partial_m v^n + \Gamma^n{}_{mp} v^p \,, \qquad \nabla_a = e^m_a \nabla_m = e^m_a \partial_m + \omega_{abc} M^{bc} \,, \tag{A.29}$$

with $\omega_{abc} = e^m_a e^n_b \nabla_m (e_c)_n$.

The torsion is defined by

$$\left(\nabla_m \nabla_n - \nabla_n \nabla_m\right) f = T_{nm}{}^p \nabla_p f \,. \tag{A.30}$$

The covariant derivative satisfies

$$[\nabla_m, \nabla_n] = R_{mnp}{}^q \, \nabla_q + T_{mn}{}^p \, \nabla_p \,. \tag{A.31}$$

[48] The quantity $r_\alpha (\gamma_m)^{\alpha\beta} r_\beta$ is trivially zero since r_α are fermionic.

Note that $T_{mn}{}^p = 0$ does not imply $T_{ab}{}^c = 0$, because ∇_a acts on vielbeins. The differential form is given by

$$(d\Omega)_{[m_1 \dots m_{p+1}]} = (\nabla\Omega)_{[m_1 \dots m_{p+1}]} + \frac{p}{2}\,(-1)^p\, T_{[m_1 m_2}{}^n\, \Omega_{m_3 \dots m_{p+1}]n}\,. \tag{A.32}$$

In string theory it is useful to introduce the generalized metric $\mathcal{G}_{mn} = G_{mn} + B_{mn}$ to maintain $O(D,D)$ covariance. If we require $\nabla_m \mathcal{G}_{np} = 0$, the Christoffel symbol is determined uniquely as

$$\Gamma^p{}_{mn} = \frac{1}{2}\, g^{pq}\, (\partial_m G_{qn} + \partial_n G_{mq} - \partial_q G_{mn})\,. \tag{A.33}$$

By comparing it with $\Gamma^p{}_{[mn]} = -\frac{1}{2}\, T_{mn}{}^p$, we find

$$H_{mnp} = \partial_{[m} B_{np]} = T_{mnp}\,. \tag{A.34}$$

A.3 Spinors and Gamma matrices

The Majorana-Weyl spinors exist in ten-dimensional Lorentzian spacetime. By setting $\Gamma^0\Gamma^1 \dots \Gamma^9 = \mathrm{diag}\,(\mathbf{1}_{16}, -\mathbf{1}_{16})$, one can decompose a general 32-component spinor into a pair of 16-component chiral and anti-chiral spinors as

$$\begin{pmatrix} \lambda^\alpha \\ \chi_\alpha \end{pmatrix} \tag{A.35}$$

We use the Majorana-Weyl representation of 32×32 $SO(1,9)$ Gamma matrices satisfying

$$\Gamma^m = \begin{pmatrix} 0 & (\gamma^m)^{\alpha\beta} \\ (\gamma^m)_{\alpha\beta} & 0 \end{pmatrix}, \qquad (\gamma^m)_{\alpha\beta} = (\gamma^m)_{\beta\alpha}\,, \tag{A.36}$$

$$(\gamma^m)_{\alpha\beta}(\gamma^n)^{\beta\gamma} + (\gamma^n)_{\alpha\beta}(\gamma^m)^{\beta\gamma} = 2\eta^{mn}\delta_\alpha^\gamma\,, \tag{A.37}$$

where η^{mn} has the signature $(-,+,\dots,+)$ with $\eta^{00} = -1$. From (A.37) it follows that

$$\begin{gathered} (\gamma^m)_{\alpha\beta}(\gamma_m)^{\beta\gamma} = 10\,\delta_\alpha^\gamma\,, \qquad (\gamma^m\gamma^n\gamma_m)_{\alpha\delta} = -8\,(\gamma^n)_{\alpha\delta}\,, \\ (\gamma^p\gamma^m\gamma^n\gamma_p)^\beta{}_\alpha = 4\,\eta^{mn}\,\delta_\alpha^\beta + 6\,(\gamma^{mn})^\beta{}_\alpha\,, \\ \gamma^m\gamma^{np} = \gamma^{mnp} + \eta^{mn}\gamma^p - \eta^{mp}\gamma^n\,, \qquad \gamma^{mn}\gamma^p = \gamma^{mnp} + \eta^{np}\gamma^m - \eta^{mp}\gamma^n\,. \end{gathered} \tag{A.38}$$

We also find

$$\mathrm{tr}\,\left(\gamma^{m_1 m_2 \dots m_\ell}\gamma_{n_k n_{k-1} \dots n_1}\right) = \begin{cases} 16\,\delta^{m_1}_{[n_1} \dots \delta^{m_\ell}_{n_\ell]} & (\ell = k) \\ 0 & (\ell \neq k), \end{cases} \tag{A.39}$$

where $\gamma^{m_1 m_2 \dots m_\ell} = \gamma^{[m_1}\gamma^{m_2} \dots \gamma^{m_\ell]}$.

The Majorana-Weyl representation matrix (A.36) can be constructed explicitly as[49]

$$
\begin{aligned}
&\Gamma^1 = \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_3\,, && \Gamma^2 = \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_3 \otimes \sigma_2\,, \\
&\Gamma^3 = \sigma_1 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_2 \otimes \sigma_1\,, && \Gamma^4 = \sigma_1 \otimes \sigma_1 \otimes \sigma_0 \otimes \sigma_1 \otimes \sigma_3\,, \\
&\Gamma^5 = \sigma_1 \otimes \sigma_1 \otimes \sigma_3 \otimes \sigma_0 \otimes \sigma_1\,, && \Gamma^6 = \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_3 \otimes \sigma_0\,, \\
&\Gamma^7 = \sigma_1 \otimes \sigma_1 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3\,, && \Gamma^8 = \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1\,, \\
&\Gamma^9 = \sigma_1 \otimes \sigma_3 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0\,, && \Gamma^{10} = \sigma_2 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0\,, \\
&\Gamma^0 = -i\,\Gamma^{10}\,.
\end{aligned}
\tag{A.40}
$$

The tensor product is defined as

$$
A \otimes B = \begin{pmatrix} a_{11}\,B & a_{12}B & \dots \\ a_{21}\,B & a_{22}B & \\ \vdots & & \ddots \end{pmatrix}. \tag{A.41}
$$

All matrix elements of $\Gamma^0, \Gamma^1, \dots, \Gamma^9$ are real and hermitian, and satisfies

$$
\begin{gathered}
(\gamma^0)_{\alpha\beta} = -(\gamma^0)^{\alpha\beta}, \qquad (\gamma^i)_{\alpha\beta} = (\gamma^i)^{\alpha\beta} \quad (i = 1, 2, \dots, 9) \\
(\gamma^m)^{\alpha\beta} = (\gamma_m)_{\alpha\beta} \qquad (m = 0, 1, \dots, 9).
\end{gathered}
\tag{A.42}
$$

The light-cone components satisfy

$$
\frac{\gamma_0 + \gamma_9}{2} = \begin{pmatrix} \mathbf{1}_8 & \mathbf{0}_8 \\ \mathbf{0}_8 & \mathbf{0}_8 \end{pmatrix}, \qquad \frac{\gamma_0 - \gamma_9}{2} = \begin{pmatrix} \mathbf{0}_8 & \mathbf{0}_8 \\ \mathbf{0}_8 & \mathbf{1}_8 \end{pmatrix}. \tag{A.43}
$$

The matrices γ^{mn} for $m, n = 0, 1, \dots, 9$ are real, anti-symmetric and have the eigenvalues $\pm i$. The matrices γ^{mnp} for $m, n, p = 0, 1, \dots, 9$ are again real and anti-symmetric.

From (A.40) one can derive various identities [18, 20],

$$
0 = \eta_{mn}\,\gamma^m_{\alpha(\beta}\gamma^n_{\gamma\delta)} \tag{A.44}
$$

$$
(\gamma^{mn})_\alpha{}^\gamma\,(\gamma_{mn})_\beta{}^\delta = 4(\gamma^m)_{\alpha\beta}(\gamma_m)^{\gamma\delta} - 2\delta_\alpha^\gamma\delta_\beta^\delta - 8\delta_\alpha^\delta\delta_\beta^\gamma\,. \tag{A.45}
$$

Note that RHS of (A.45) is not manifestly symmetric with respect to $\alpha \leftrightarrow \beta$ and $\gamma \leftrightarrow \delta$. By multiplying $\lambda^\beta\lambda^\gamma$ to (A.44) and using the pure spinor condition for λ, we obtain

$$
(\gamma_m\lambda)_\alpha\,(\lambda\gamma^m)_\delta = -\frac{1}{2}(\gamma^m)_{\alpha\delta}\,(\lambda\gamma_m\lambda) = 0 \tag{A.46}
$$

The $U(5)$ basis is suitable for solving the pure spinor condition. Let us write a 32-component Dirac spinor by $\lambda^{s_1,s_2,s_3,s_4,s_5}$ with $s_k = \pm 1$. We introduce creation-annihilation operators by

$$
\gamma_{k,\pm}\,\lambda^{s_1,s_2,s_3,s_4,s_5} = \left(\prod_{j=1}^{k-1} s_j\right) \frac{1 \mp s_k}{2} \left[\lambda^{s_1,s_2,s_3,s_4,s_5}\right]_{s_k \to -s_k}, \tag{A.47}
$$

[49] Another construction based on the representation of Spin(8) in [18] can be found in [19].

so that $\gamma_{k,\pm}$ kills $\lambda^{s_1,s_2,s_3,s_4,s_5}$ if $s_k = \pm 1$. The first factor in RHS is needed to make γ's anti-commute. One can construct the $SO(10)$ Gamma matrices by

$$\Gamma_{2k-1} = \gamma_{k,+} + \gamma_{k,-}\,, \qquad \Gamma_{2k} = -i\left(\gamma_{k,+} - \gamma_{k,-}\right)\,. \tag{A.48}$$

A shortcut to obtain a complex Weyl representation of $SO(10)$ Gamma matrices is to use (A.40). Then, the state

$$\lambda^{+++++} = \{0^{16}, -i, 1, i, 1, i, 1, -i, 1, 0, 0, 0, 0, 0, 0, 0, 0\} \tag{A.49}$$

is annihilated by all $\gamma_{k,+}$. The 32 components of $\lambda^{s_1,s_2,s_3,s_4,s_5}$ split into two groups, whether the number of $-$'s is even or odd. Each group has a definite chirality. Suppose $(\lambda^\alpha, \chi_\alpha)$ is a pair of 16-component spinors, chiral and anti-chiral.

Consider the PS condition $\lambda\gamma^m\lambda = 0$ in the $U(5)$ basis. We write the chiral fermion as

$$\lambda = u\,\Lambda^+ + u_{ab}\left([\gamma_-^a\,,\gamma_-^b]\Lambda^+\right) + u_{abcd}\left([\gamma_-^a\,,\gamma_-^b][\gamma_-^c\,,\gamma_-^d]\Lambda^+\right), \tag{A.50}$$

where $\Lambda^+ = \lambda^{+++++}/|\lambda^{+++++}|$ and $a, b, c, d = 1, \ldots, 5$.[50] The PS condition is equivalent to the bilinear conditions

$$\lambda\gamma_-^e\lambda = 0 \quad \Leftrightarrow \quad u\,u_{abcd} = u_{ab}\,u_{cd} - u_{ac}\,u_{bd} + u_{ad}\,u_{bc}\,, \tag{A.51}$$

$$\lambda\gamma_+^e\lambda = 0 \quad \Leftrightarrow \quad 0 = \epsilon_{defgh}\,u^{de}\,u^{efgh}\,. \tag{A.52}$$

The first line solves the second line trivially. We are left with $16 - 5 = 11$ degrees of freedom.

For a pair of chiral spinors, the conditions $\lambda\gamma_\mp^e\tilde\lambda = 0$ are equivalent to

$$\begin{aligned} u\,\tilde u_{abcd} + u_{abcd}\,\tilde u &= u_{ab}\,\tilde u_{cd} - u_{ac}\,\tilde u_{bd} + u_{ad}\,\tilde u_{bc} + u_{cd}\,\tilde u_{ab} - u_{bd}\,\tilde u_{ac} + u_{bc}\,\tilde u_{ad} \\ 0 &= \epsilon^{defgh}\left(u_{de}\,\tilde u_{efgh} + u_{efgh}\,\tilde u_{de}\right). \end{aligned} \tag{A.53}$$

The general solution of these equations is complicated. When $\tilde\lambda = \tilde\lambda^{+++++}$, or equivalently $(\tilde u, \tilde u_{ab}, \tilde u_{abcd}) = (1, 0, 0)$, we have $u_{abcd} = 0$.

B Literature

Below are the references relevant to the lectures. This list is by no means comprehensive, because this article is a lecture note rather than a review.

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RNS. Review of RNS and GS [25]. Picture changing was introduced in [6, 7]. The picture-changing at loop level has been recently discussed in [26].

GS. Textbooks on GS [18, 27]. Relation between κ symmetry, generalized supergravity equations of motion and Weyl invariance. [28, 29, 30] Review of GS on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and integrability. The construction of σ-model action using $\mathbb{Z}_4$ grading [31].

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