Refined Counting of Anomalous Dimensions

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based on 1703.05798

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$$Z_{\mathcal{N}=4\ \text{SYM}}\left[S^1 \times S^3; \beta, \vec{\mu}\ ; N_c, \lambda\right]$$

β = Radius of S^1 = Inverse temperature

(Radius of S^3 = 1)

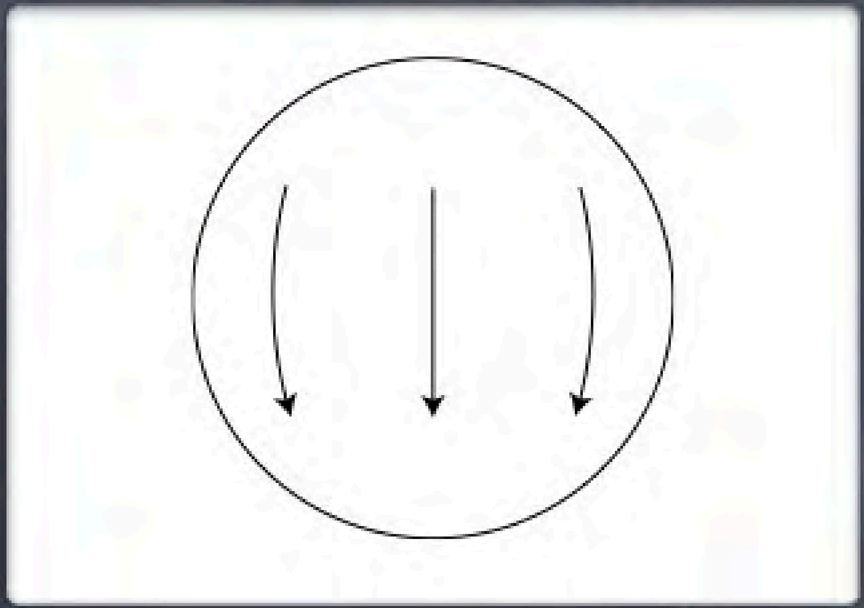
μ = Chemical potentials

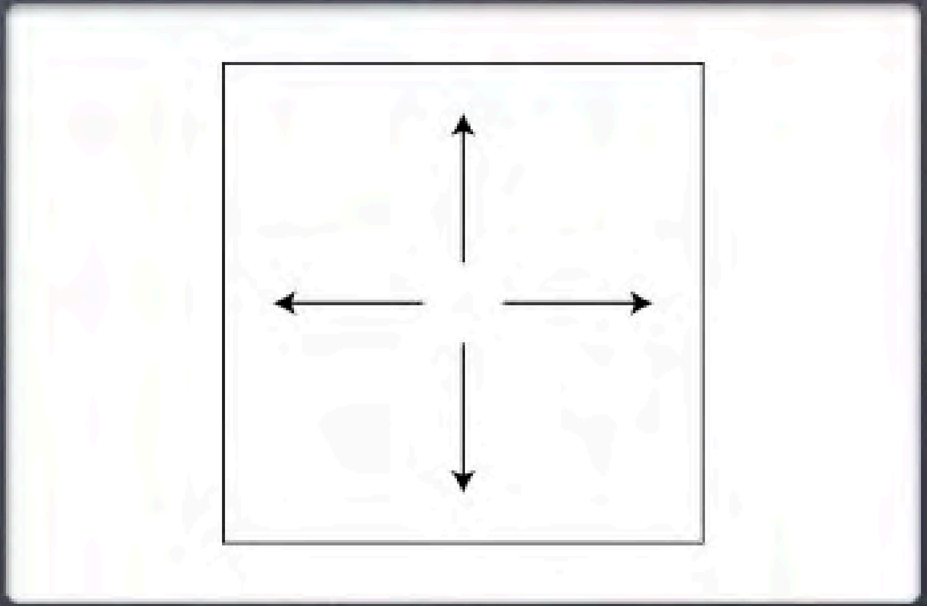
N_c = rank of gauge group (mostly $U(N_c)$)

λ = 't Hooft coupling

$$Z_{\mathcal{N}=4\ \mathrm{SYM}}\left[S^1 \times S^3; \beta, \vec{\mu}\ ; N_c, \lambda\right]$$

Put color charges in S^3 and $\mathbb{R}^3$





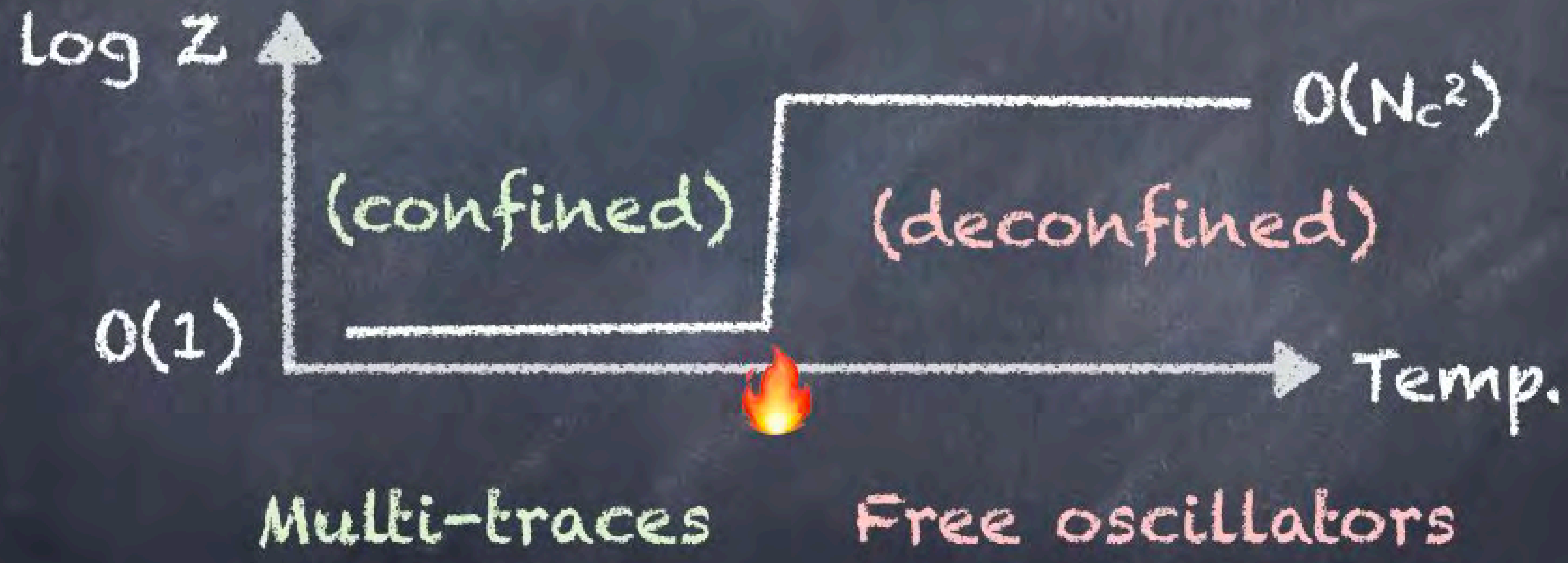
$\int DA_i \Rightarrow$ Gauss Law

Only color-singlets appear
"Confinement"

- In real QCD on R^3, strong dynamics causes confinement
- The 3/4 problem

Phase transition

"Deconfinement" from Gauss-Law occurs at $T = T_H$ if $N_c = \infty$

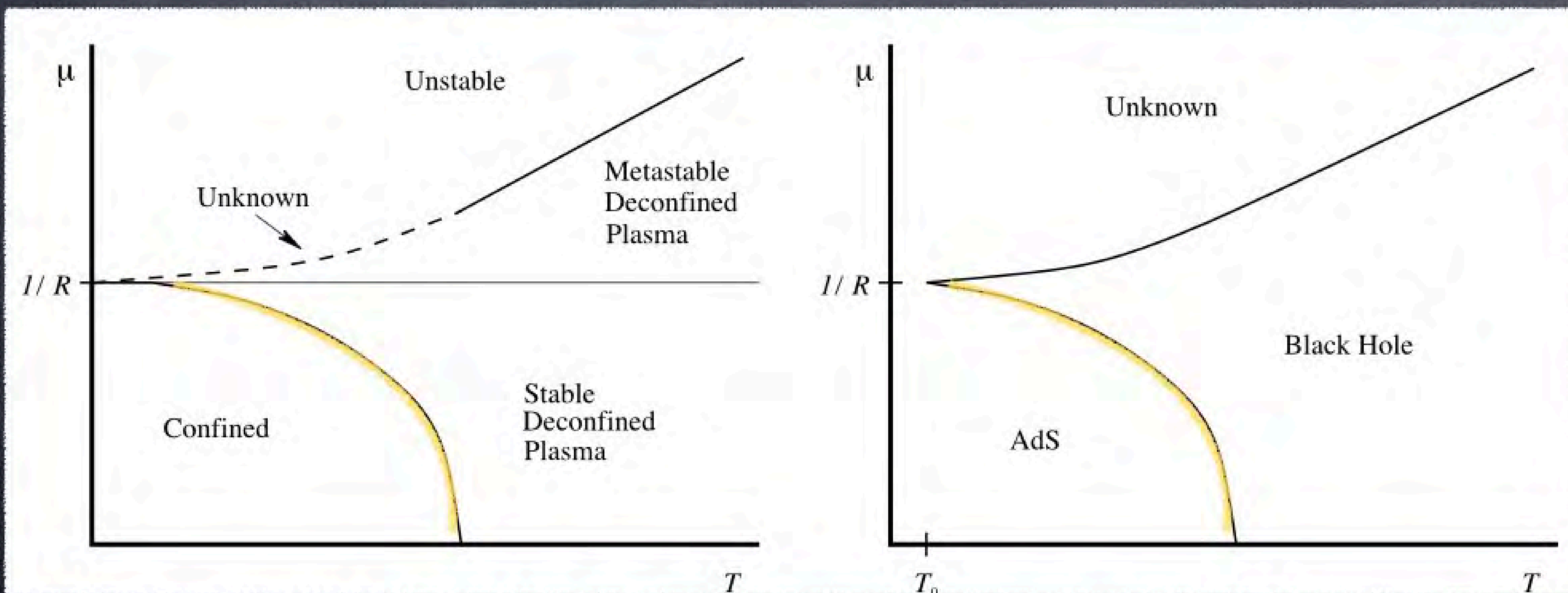


Hagedorn : driven by entropy $\rho(E) \sim \exp(cE)$

Hagedorn transition

<-> Confinement/Deconfinement transition

<-> Hawking-Page transition



gauge gravity

[Aharony, Marsano, Minwalla, Papadodimas, van Raamsdonk (2003, 2005)]
[Yaffe, Yamada (2006)]

Mathematical motivation

Q: How to sum over Bethe Ansatz solutions?

$$F \equiv \sum_{u_*:\mathrm{BAE}} (\text{Multiplicity})\, f(u_*)$$

where $\{u*\}$ are the BAE solns at fixed (L,M)

$$-1 = \left(\frac{u+\frac{i}{2}}{u-\frac{i}{2}}\right)^L \prod_{j=1}^{M} \frac{u-u_j-i}{u-u_j+i}\Bigg|_{u=u_k} \qquad (k=1,2,\ldots,M)$$

Such questions show up in various SYM's

$$F \equiv \sum_{u_*:\mathrm{BAE}} (\mathrm{Multiplicity})\, f(u_*)$$

- D=4 N=4 SYM at $N_c = \infty$ (this talk)

  f = Spin chain energy

  $\rightarrow$ $F \sim$ (Part of) $Z_{\mathcal{N}=4\ \mathrm{SYM}}\left[S^1 \times S^3; \beta\right]$

- D=2 N=2 SYM [Nekrasov-Shatashvili (2014)]

  f = (Norm of Bethe wave-function)^g

  $\rightarrow$ $F \sim Z_{\mathcal{N}=2\ \mathrm{SYM}}[\Sigma_g; (-1)^F]$

Today I look for NON-integrable methods

Tools

Finite group theory

Multi-trace operators <- Permutations

® Tony Fisher

Tools

Finite group theory

Multi-trace operators <- Permutations

!! : extends to finite N_c

!? : so far perturbative in g_{YM}

: one-loop mixing "simple" but not diagonal

-> useful for statistical quantities, e.g.

$$Z_{\mathcal{N}=4\ \text{SYM}}\Big|_{\text{one-loop}} \sim \text{tr (mixing matrix)}$$

Main results

Partition fn in the SU(2) sector at small λ

$$Z(\beta, x, y) = Z_0(x, y) - 2\lambda\beta Z_2(x, y) + O(\lambda^2)$$

$$= \sum_{m,n\geq 0} M_{m,n} x^m y^n$$

$$M_{m,n} = \sum_{\mathcal{O}} (\text{1-loop dim. of ops. } \mathcal{O} \sim W^m Z^n)$$

$$\frac{Z_2^{MT}(x,y)}{N_c} = 6x^2y^2 + \left(10x^3y^2 + 10x^2y^3\right)$$
$$+ \left(26x^4y^2 + 36x^3y^3 + 26x^2y^4\right)$$
$$+ \left(44x^5y^2 + 84x^4y^3 + 84x^3y^4 + 44x^2y^5\right) + \dots$$

MT: sum over all multi-traces

"plethystic exponential" of single-trace gen. fn.

$$Z_2^{MT}(x,y) = Z_0^{MT}(x,y) \sum_{k=1}^{\infty} Z_2^{ST}(x^k, y^k)$$

Single-trace sum written by Euler Totient and GCD

$$\frac{Z_2^{MT}(x,y)}{N_c} = 2 \prod_{h=1}^{\infty} \frac{1}{1 - x^h - y^h} \sum_{k=1}^{\infty} \times$$

$$\left\{ \sum_{d=1}^{\infty} \mathrm{Tot}(d) \frac{x^{kd} y^{kd}}{1 - x^{kd} - y^{kd}} - \sum_{L=2}^{\infty} \sum_{m=1}^{L-1} x^{km} y^{k(L-m)} \delta(\gcd(m,L),1) \right\}$$

Position of pole
= Hagedorn temp. T_H

Residue
= Corrections to T_H

[RS (2017)]

Plan

- ~~Introduction~~
- N=4 Partition functions
- Permutation basis
- One-loop dimensions
- Hagedorn temperature
- Conclusion

§ N=4 Partition functions

Partition fn of N=4 SYM

$$Z = \int DA\, e^{-S_{\mathcal{N}=4}[A]}$$ (Path integral)

$$= \int [dU]\, e^{-S_{eff}}, \quad U \equiv P\exp\left(\oint_{S^1} A_0\right)$$ (Matrix model)

$$= \mathrm{tr}_{\mathcal{H}}\left(e^{-\beta D + \mu_i J_i}\right),$$ (radial quant.) (Hamiltonian)

$D = D_0 + \lambda D_2 + \lambda^2 D_4 + \ldots$, dilatation

J_i = other global charges of $psu(2,2|4)$

$\mathcal{H}$ = Hilbert sp. of multi-trace ops.

Tree-level ($\lambda=0$) -> Counts operators

Define # of length-L ops, $N(L, N_c) = N_L^{(p)} - N_L^{(np)}$

Anti-symmetrization identity:

$$W_1^{[i_1} W_2^{i_2} \dots W_{N+1}^{i_{N+1}]} = 0 \quad \text{if} \quad i_1, i_2, \dots \in \{1, 2, \dots, N\}$$

Tree-level partition fn with $\mu = 0$ is

$$Z[\beta, N_c] = \mathrm{tr}(e^{-\beta D_0})$$

$$= \underbrace{\sum_{L \geq 0} N_L^{(p)} e^{-\beta L}}_{\text{constant}} - \underbrace{\sum_{L > N_c} N_L^{(np)} e^{-\beta L}}_{\text{finite } N_c}$$

$O(e^{-\beta N_c}) \Rightarrow$ non-perturbative in $1/N_c$

$$Z[\beta, N_c] = \mathrm{tr}\left(e^{-\beta D_0}\right)$$

$$= \sum_{L \geq 0} N_L^{(p)} e^{-\beta L} + \boxed{\sum_{L > N_c} N_L^{(np)} e^{-\beta L}}$$

I neglect finite-N_c terms, but there can be subtlety:

Hagedorn transition happens at a finite temperature only if $N_c = \infty$

Hagedorn temperature: $Z\left[\beta = \beta_H\right] = \infty$

$\rightarrow$ growth of N_L $(L \rightarrow \infty)$ is important

$N_L^{(p)} \sim e^{\beta_H L}$, $N_L^{(np)} \sim 1$, $(L \rightarrow \infty, N_c \rightarrow \infty)$??

Truncation to SU(2) sector

Matrix-model formula for the partition fn:

$$Z = \int [dU] \exp\left(\sum_{n=1}^{\infty} \frac{\chi_{adj}(U^n)}{n} \zeta_{SYM}(e^{-\beta n})\right)$$

$\zeta(\omega)$ for the PSU(2,2|4) sector of N=4 SYM

$$\zeta_{SYM}(e^{-\beta}) = \frac{2e^{-\beta}(3 - e^{-\beta/2})}{(1 - e^{-\beta/2})^3}$$

[Sundborg (1999)]

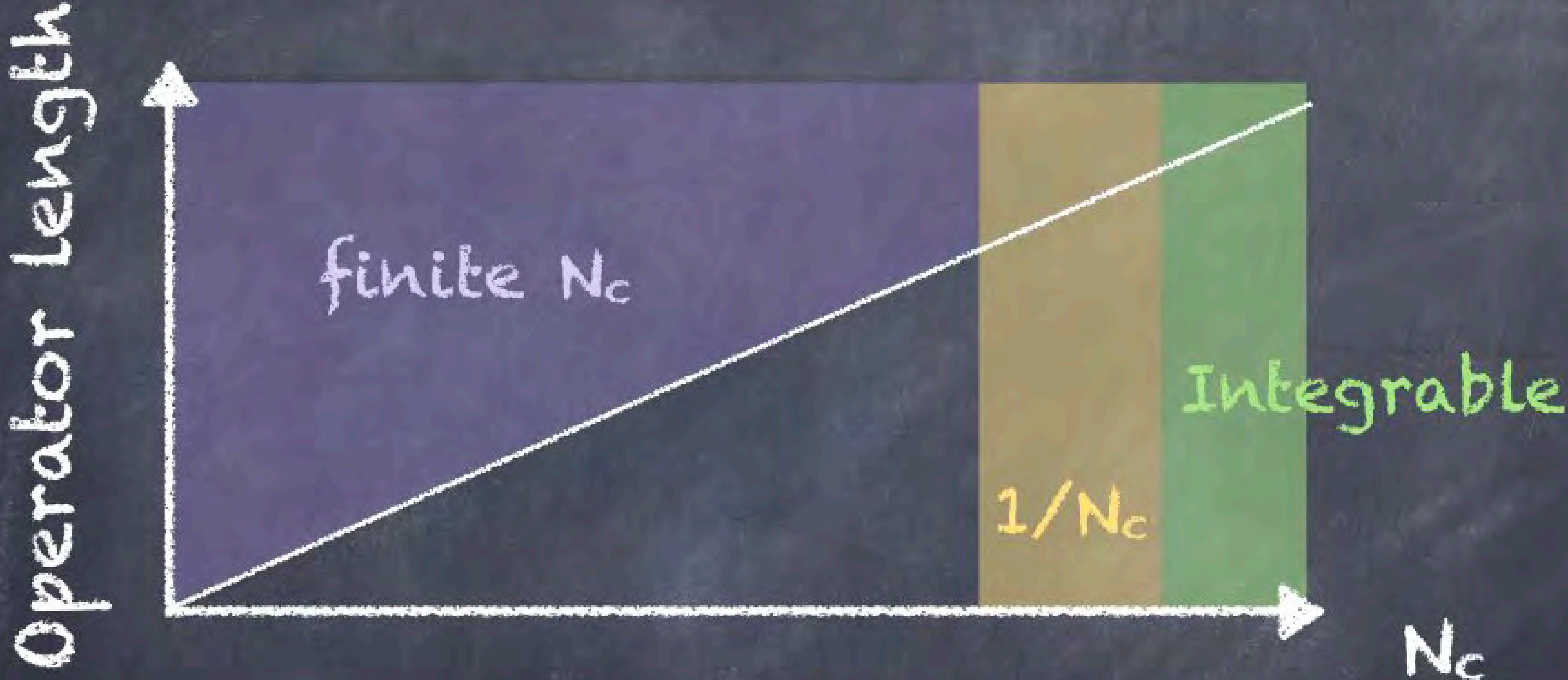
Truncate to the SU(2) sector, introduce fugacity

$$\zeta_{SU(2)} = x + y$$

Hagedorn transition still exists, at different T_H

§ Permutation basis

Operator Bases



(Representation basis)

e.g. restricted Schur

Permutation basis

Single-traces

Simple question

Count $N^{MT}_{m,n} \equiv$ # of multi-trace ops $\sim W^m Z^n$

$m=n=2$:

$\mathrm{tr}(ZZWW)$, $\mathrm{tr}(ZWZW)$

$\mathrm{tr}(ZZW)\mathrm{tr}(W)$, $\mathrm{tr}(Z)\mathrm{tr}(ZWW)$

$\mathrm{tr}(ZZ)\mathrm{tr}(WW)$, $\mathrm{tr}(ZW)^2$

$\mathrm{tr}(ZZ)\mathrm{tr}(W)^2$, $\mathrm{tr}(Z)^2\mathrm{tr}(WW)$, $\mathrm{tr}(ZW)\mathrm{tr}(Z)\mathrm{tr}(W)$

$\mathrm{tr}(Z)^2\mathrm{tr}(W)^2$

$\# = 2+2+2+3+1 = 10$

How to get it?

Gauge-invariant Operators

$\mathcal{N}=4$ alphabet : $\mathcal{W}^A \in \{\nabla^s\Phi^I, \nabla^s\psi, \nabla^s F, \ldots\}$

Permutation basis $(\alpha \in S_L)$

$$O_\alpha^{A_1\ldots A_L} = \mathrm{tr}_L\left(\alpha W^{A_1}\otimes W^{A_2}\otimes \ldots W^{A_L}\right)$$

$$= \sum_{a_1,\ldots,a_L=1}^{N_c} (W^{A_1})^{a_1}_{a_{\alpha(1)}}(W^{A_2})^{a_2}_{a_{\alpha(2)}}\ldots(W^{A_L})^{a_L}_{a_{\alpha(L)}}$$

Relabelling $(a_i, A_i) \rightarrow (a_{\gamma(i)}, A_{\gamma(i)})$

$$\Rightarrow O_\alpha^{A_1\ldots A_L} = O_{\gamma\alpha\gamma^{-1}}^{A_{\gamma(1)}\ldots A_{\gamma(L)}}, \quad \forall\gamma\in S_L$$

Claim: Equivalence class = Unique multi-trace

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SU(2) sector: $\mathcal{O}_\alpha^{m,n} = \mathrm{tr}_{m+n}(\alpha W^m Z^n)$, $(\alpha \in S_{m+n})$

Relabelling = permutations inside W^m or Z^n

$$\mathcal{O}_\alpha^{m,n} = \mathcal{O}_{\gamma^{-1}\alpha\gamma}^{m,n} \quad (\forall \gamma \in S_m \times S_n)$$

Equivalence class $= S_{m+n}/(S_m \times S_n)$

$$N_{m,n}^{MT} = \frac{1}{m!n!} \sum_{\alpha \in S_{m+n}} \sum_{\gamma \in S_m \times S_n} \delta_{m+n}(\alpha^{-1}\gamma^{-1}\alpha\gamma)$$

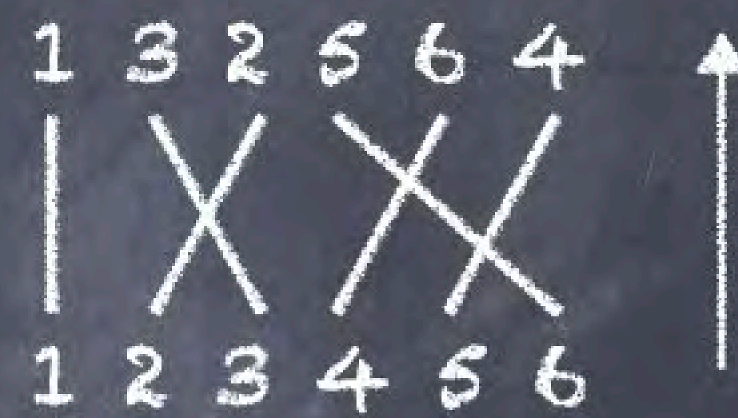
$$\delta_L(\sigma) = 1 \quad \text{iff} \quad \sigma = 1 \in S_L$$

Solving $\alpha^{-1}\gamma^{-1}\alpha\gamma = 1$

- Each permutation has a cycle type
- The cycle type is invariant under conjugation

e.g. $S_6 \ni \sigma = (1)(23)(456)$

Cycle type of: $\sigma = [1^1, 2^1, 3^1] \vdash 6$



Identity: $\alpha^{-1}\sigma\alpha = \big(\alpha(1)\big)\big(\alpha(2)\alpha(3)\big)\big(\alpha(4)\alpha(5)\alpha(6)\big)$

Redundancy: $(123) = (231) = (312)$, $(45)(67) = (67)(45)$

Counting redundancy = # of solutions

Generating fn

Formula for $N_{m,n}$ ->

Tree-level part. fn of N=4 SYM in SU(2) sector

$$Z_0^{MT}(\beta, \mu_i) = tr_{MT}\left(e^{-\beta D_0 + \mu_i J_i}\right)$$

Redefine (β, μ_i) to (x, y) s.t. the operator $\mathcal{O} \sim W^m Z^n$

is counted with the weight $x^m y^n$

$$Z_0^{MT}(x,y) = \sum_{m,n=0}^{\infty} N_{m,n}^{MT}\, x^m y^n = \cdots = \prod_{k=1} \frac{1}{1 - x^k - y^k}$$

$$= 1 + (x+y) + \cdots + \left(5x^4 + 7x^3 y + \boxed{10x^2y^2} + 7xy^3 + 5y^4\right) + \ldots$$

Generating fn of # of multi-traces in SU(2) sector

$$Z_0^{MT}(x,y) = \prod_{k=1} \frac{1}{1-(x^k+y^k)}$$

1) Hagedorn temperature @ SU(2) sector, tree-level

$$x + y = 1 \quad \text{if } x \geq 0 \text{ and } y \geq 0$$

2) Plethysitc exponential of single-trace gen. fn

$$Z_0^{MT}(x,y) = \exp\left(\sum_{m=1} \frac{Z_0^{ST}(x^m, y^m)}{m}\right)$$

3) Valid in large N_c

$$Z_0^{MT}(x,y) = 1 + (x+y) + 2(x^2+xy+y^2) + \underbrace{(3x^3+4x^2y+4xy^2+3y^3)}_{\text{wrong if } N_c=2} + \ldots$$

Partition fn at finite N_c

No transition at finite T at finite N_c
in (the scalar sector of) N=4 SYM

($\because$ matrix model with finite d.o.f.)

($\because$ checked by Molien-Weyl formula)

$Z^{\text{exact}}_{N_c}(x,y) =$ Hilbert-Poincare series of $GL(N_c)$ invariants

[Feng, Hanany, He (2007)], [Djokovic (2006)]

$$P_N(x,y) = \frac{(2\pi i)^{1-n}}{(1-x)^N(1-y)^N} \int_U \frac{dt_1}{t_1} \cdots \int_U \frac{dt_{N-1}}{t_{N-1}} \prod_{1 \le k \le r \le N-1} \frac{f^{(+)}_{k,r}(1)}{f^{(+)}_{k,r}(x)\, f^{(-)}_{k,r}(x)\, f^{(+)}_{k,r}(y)\, f^{(-)}_{k,r}(y)}$$

$$f^{(\pm)}_{k,r}(u) = 1 - u\,(t_k t_{k+1} \dots t_r)^{\pm 1}$$

U = Counterclockwise contour with unit radius

Partition fn at finite N_c

No transition at finite T at finite N_c
in (the scalar sector of) N=4 SYM

In string theory, the density of states
usually grows exponentially

e.g. in flat space, $T_H = \dfrac{1}{4\pi\sqrt{\alpha'}}$

How do they match in view of AdS/CFT?

§ One-loop dimensions

One-loop partition fn

Expansion of N=4 partition fn at small λ:

$$Z = \mathrm{tr}\left(e^{-\beta(D_0+\lambda D_2+\dots)+\mu_i J_i}\right)$$

$$= Z_0 - 2\lambda\beta Z_2 + \dots$$

$$Z_2^{MT} \equiv \mathrm{tr}_{MT}\left(D_2 e^{-\beta D_0+\mu_i J_i}\right) = \sum_{m,n=0} \langle M_2\rangle_{m,n} x^m y^n$$

the sum of one-loop dimensions
over all multi-traces at fixed charges (m,n)

$$\langle M_2\rangle_{m,n} = \sum_{\alpha,\beta} (M_2)_\alpha^\beta \, \delta_\beta^\alpha$$

One-loop mixing

Notation:

Dilatation operator in the SU(2) sector

$$\mathfrak{D} = \sum_{n=0} \lambda^n \mathfrak{D}_{2n} = \mathrm{tr}(W\check{W} + Z\check{Z}) - \frac{2\lambda}{N_c} :\mathrm{tr}[W, Z][\check{W}, \check{Z}]: + \dots$$

One-loop mixing matrix: $\mathfrak{D}_2 \mathcal{O}_\alpha \equiv \frac{2}{N_c} (M_2)_\alpha{}^\beta \mathcal{O}_\beta$

[Beisert Kristjansen Staudacher (2003)]

Mixing matrix in the permutation basis

$$(M_2)_\alpha{}^\beta = \frac{1}{m!n!} \sum_{i \neq j}^{L} \sum_{\mu \in S_m \times S_n} \delta_L\left(\mu\beta^{-1}\mu^{-1}\{\alpha - (ij)\alpha(ij)\}[\![i\alpha(j)]\!]\right)$$

[Bellucci Casteill Morales Sochichiu (2004)]

$$[\![ij]\!] = (ij) \qquad (i \neq j)$$
$$= N_c \qquad (i = j)$$

<- N_c dependence

Trace of mixing matrix

$$\langle M_2 \rangle_{m,n} = \frac{1}{m!\,n!} \sum_{i \neq j}^{L} \sum_{\alpha \in S_L} \sum_{\mu \in S_m \times S_n} \delta_L\Big(\mu\alpha^{-1}\mu^{-1}\{\alpha - (ij)\alpha(ij)\}[\![i\alpha(j)]\!]\Big)$$

Parity of permutations

-> any odd powers of transpositions cannot become identity (unless N_c is finite)

-> Only the $O(N_c)$ terms survive

The sum of one-loop dimensions is

$$\langle M_2 \rangle_{m,n} = \frac{N_c}{m!\,n!} \sum_{i \neq j}^{L} \sum_{\alpha \in S_L} \sum_{\mu \in S_m \times S_n} \delta_L(i\alpha(j)) \times$$

$$\Big\{\delta_L(\mu\alpha^{-1}\mu^{-1}\alpha) - \delta_L\Big(\mu\alpha^{-1}\mu^{-1}(ij)\alpha(ij)\Big)\Big\}$$

Compute the generating fn

$$Z_2^{MT}(x,y) \equiv \sum_{m,n=0}^{\infty} \langle M_2 \rangle_{m,n} x^m y^n$$

The sum of one-loop dimensions is

$$\langle M_2 \rangle_{m,n} = \frac{N_c}{m!\,n!} \sum_{i\neq j}^{L} \sum_{\alpha\in S_L} \sum_{\mu\in S_m\times S_n} \delta_L(i\alpha(j)) \times$$

$$\Big\{ \underbrace{\delta_L(\mu\alpha^{-1}\mu^{-1}\alpha)}_{\text{1st term}} - \underbrace{\delta_L\Big(\mu\alpha^{-1}\mu^{-1}(ij)\alpha(ij)\Big)}_{\text{2nd term},\ \mu_0=(ij)\mu} \Big\}$$

This can be done in 2 ways

$\alpha \in S_L \Rightarrow$ Partition form, $\alpha \in \mathbb{Z}_L \Rightarrow$ Totient form

Strategy: 1st term

Solve: $\sum_{i\neq j}^{L} \sum_{\alpha\in S_L} \sum_{\mu\in S_m\times S_n} \delta_L(i\alpha(j))\delta_L(\mu\alpha^{-1}\mu^{-1}\alpha)$

1) Choose cycle type of $\mu \in S_m \times S_n$
2) Choose which cycles of μ the (i,j) belong to
3) Solve the δ-function constraints simultaneously

$$\mu^{-1} = \prod_{k=1}^{L} \prod_{h=1}^{p_k+q_k} \left(m_{h,1}^{(k)} m_{h,2}^{(k)} \dots m_{h,k}^{(k)}\right)$$

$$\alpha^{-1}\mu^{-1}\alpha = \prod_{k=1}^{L} \prod_{h=1}^{p_k+q_k} \left(\alpha(m_{h,1}^{(k)})\alpha(m_{h,2}^{(k)}) \dots \alpha(m_{h,k}^{(k)})\right)$$

$i = \alpha(j)$

Strategy: 2nd term

Solve: $$\sum_{i\neq j}^{L}\sum_{\alpha\in S_L}\sum_{\mu\in S_m\times S_n}\delta_L(i\alpha(j))\,\delta_L\left(\mu_0\alpha^{-1}\mu_0^{-1}\alpha\right),\quad \mu_0=(ij)\mu$$

1) Choose cycle type of $\mu\in S_m\times S_n$
2) Choose which cycles of μ the (i,j) belong to
3) Generate various μ_0 by $\mu_0=(ij)\,\mu$
4) Solve the δ-function constraints simultaneously

$$\mu_0^{-1}=\prod_{k=1}^{L}\prod_{h=1}^{r'_k}\left(\tilde{m}_{h,1}^{(k)}\tilde{m}_{h,2}^{(k)}\ldots\tilde{m}_{h,k}^{(k)}\right)$$

$$\alpha^{-1}\mu_0^{-1}\alpha=\prod_{k=1}^{L}\prod_{h=1}^{r'_k}\left(\alpha(\tilde{m}_{h,1}^{(k)})\alpha(\tilde{m}_{h,2}^{(k)})\ldots\alpha(\tilde{m}_{h,k}^{(k)})\right)$$

$i=\alpha(j)$

Results in Partition form

Written as a sum over partitions

$$\frac{Z_2^{MT}(x,y)}{N_c} = \sum_{L=0}^{\infty} \sum_{r \vdash L} \prod_{k=1}^{\infty} (x^k + y^k)^{r_k} \times$$

$$\Big\{ L - \sum_{a=1}^{L} \theta_>(r_a) - \sum_{a=1}^{L/2} a\,(r_a + 1)\theta_>(r_{2a})$$

$$-2 \sum_{a<b}^{L} \theta_>(L + 1 - a - b)\theta_>(r_a)\theta_>(r_b) - \sum_{a=1}^{L/2} \theta_>(r_a - 1) \Big\}$$

$$\theta_>(x) = 1 \ (\text{if } x > 0), \quad \theta_>(x) = 0 \ (\text{if } x \leq 0)$$

Results in Totient form

"plethystic exponential" of single-trace gen. fn.

$$Z_2^{MT}(x,y) = Z_0^{MT}(x,y) \sum_{k=1}^{\infty} Z_2^{ST}(x^k, y^k)$$

$$\frac{Z_2^{MT}(x,y)}{N_c} = 2 \prod_{h=1}^{\infty} \frac{1}{1 - x^h - y^h} \sum_{k=1}^{\infty} \times$$

$$\left\{ \sum_{d=1}^{\infty} \mathrm{Tot}(d) \frac{x^{kd} y^{kd}}{1 - x^{kd} - y^{kd}} - \sum_{L=2}^{\infty} \sum_{m=1}^{L-1} x^{km} y^{k(L-m)} \delta(\gcd(m,L),1) \right\}$$

If x=y, it agrees with Polya's theorem [Spradlin Volovich (2004)]

The agreement of

Partition form = Totient form

is non-trivial (not proven directly)

$$\frac{Z_2^{MT}(x,y)}{N_c} = 6x^2y^2 + \left(10x^3y^2 + 10x^2y^3\right)$$
$$+ \left(26x^4y^2 + 36x^3y^3 + 26x^2y^4\right)$$
$$+ \left(44x^5y^2 + 84x^4y^3 + 84x^3y^4 + 44x^2y^5\right)$$
$$+ \left(84x^6y^2 + 176x^5y^3 + 254x^4y^4 + 176x^3y^5 + 84x^2y^6\right) + \ldots$$

Compare with Bethe Ansatz

Single-trace Operators in SU(2) sector at 1-loop

<-> $XXX_{1/2}$ spin chain with level-matching

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = -\prod_{j=1}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}, \quad \prod_{k=1}^{M} \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} = 1$$

Compare with Bethe Ansatz

Single-trace Operators in SU(2) sector at 1-loop
<-> $XXX_{1/2}$ spin chain with level-matching

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = -\prod_{j=1}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}, \quad \prod_{k=1}^{M} \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} = 1$$

Complete list of the solutions of $XXX_{1/2}$ model
-> (Extended) Q-system

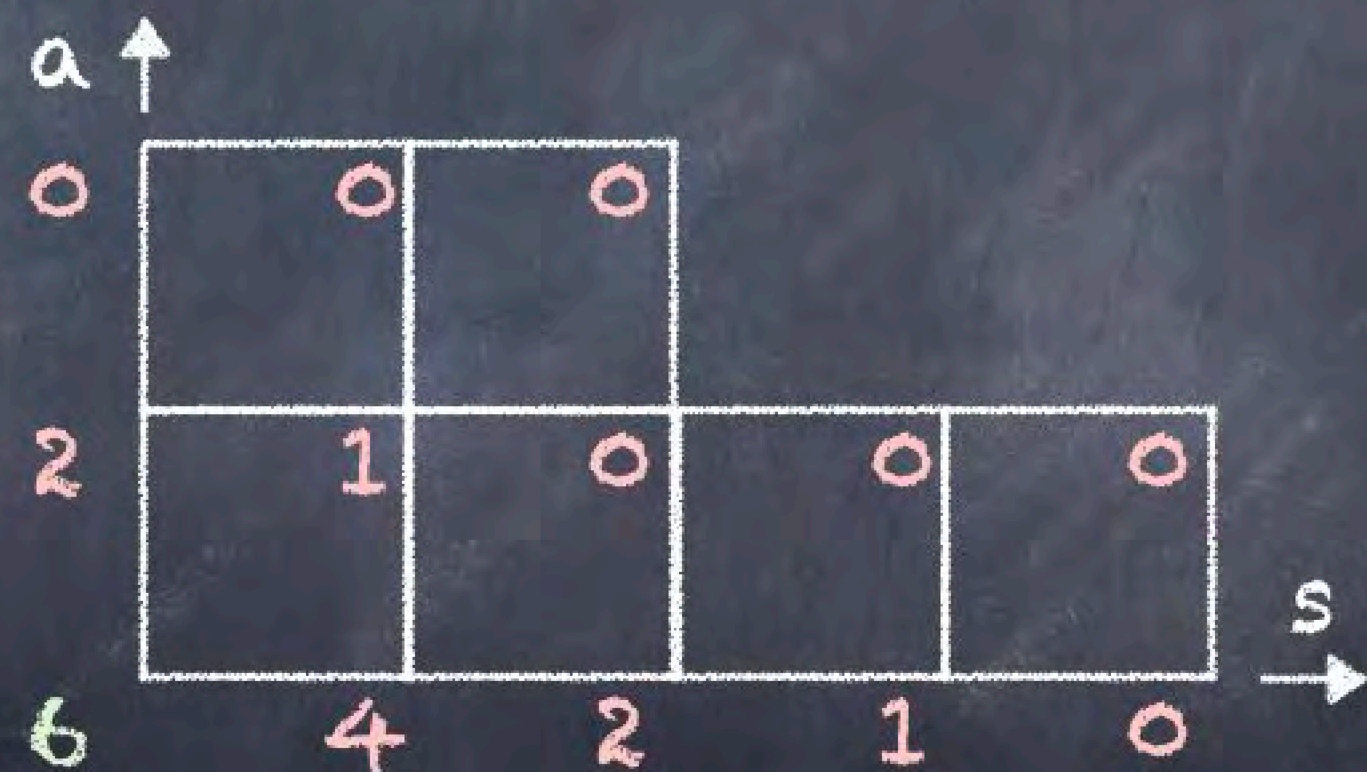
$$Q_{a+1,s}\,Q_{a-1,s} = Q^{+}_{a+1,s+1}\,Q^{-}_{a,s} - Q^{-}_{a+1,s+1}\,Q^{+}_{a,s}$$

$Q_{a,s}(u)$= polynomial of u of degree $M_{a,s}$

[Hao, Nepomechie, Sommese (2013)] [Marboe, Volin (2016)]

Extended Q-system

- Representation of global symmetry (SU(2))
-> Young diagram of L boxes on (a,s)-plane
- Polynomial degree $M_{a,s} \neq$ Hook length



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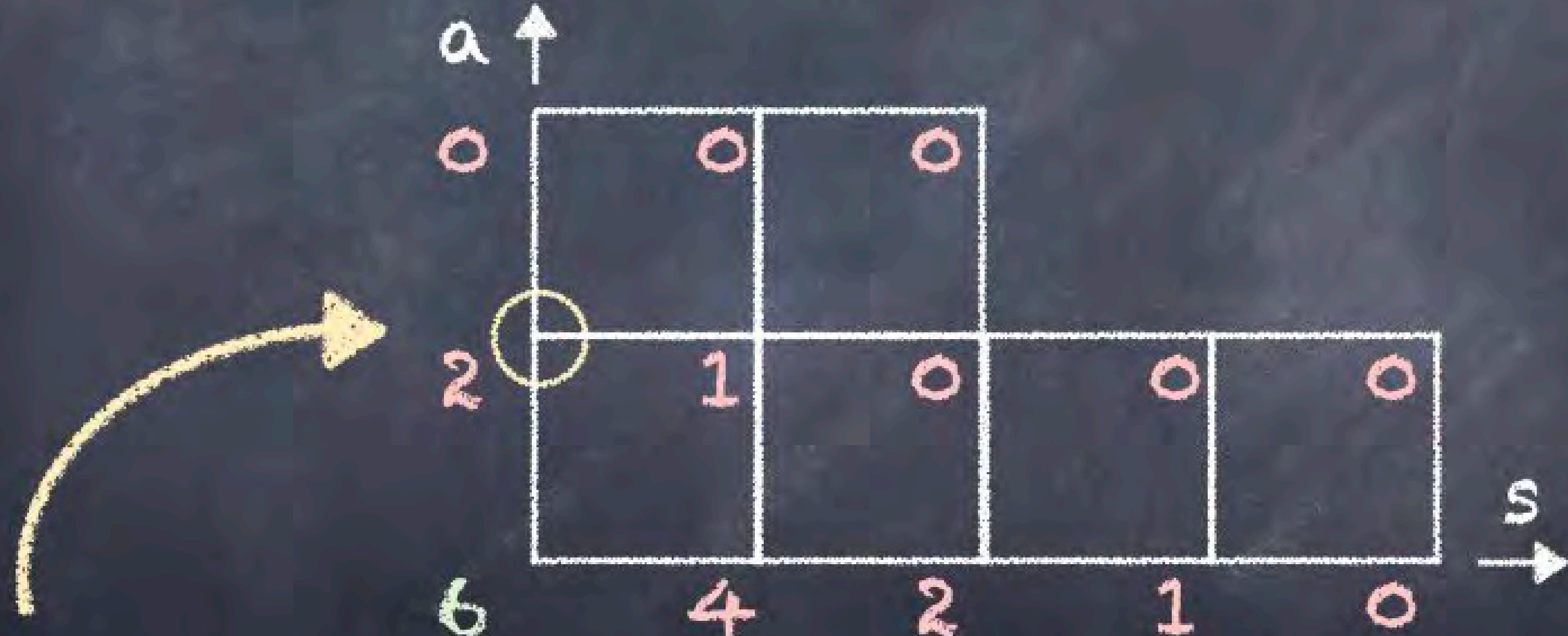

Momentum-carrying root at (a,s)=(1,0)

Energy: $E \propto \frac{Q'_{1,0}(i/2)}{Q_{1,0}(i/2)} - \frac{Q'_{1,0}(-i/2)}{Q_{1,0}(-i/2)}$

Comparison

Cautions about Q-system:

- Level-matching
- Exceptional solutions -> regularize by twist

Exceptional solutions have Bethe roots at $u=i/2$ or $u=-i/2$

Comparison

Cautions about Q-system:

- Level-matching
- Exceptional solutions -> regularize by twist

Cautions about comparison:

- Single-trace -> Multi-trace
- SU(2) highest weight states -> all states

Bethe Ansatz with regular Bethe roots describe the highest weight states only

Comparison

Cautions about Q-system:

- Level-matching
- Exceptional solutions -> regularize by twist

Cautions about comparison:

- Single-trace -> Multi-trace
- SU(2) highest weight states -> all states

n \ m	1	2	3	4	5
1	0	0	0	0	0
2		6	10	26	44
3			36	84	176

Permutation

n \ m	1	2	3	4	5
1	0	0	0	0	0
2		6	4	10	8
3			6	10	14

Q-system

Agreement!

§ Hagedorn temperature

Grand partition function

Grand partition fn at weak coupling:

$$\mathrm{tr}_{MT}\left(e^{-\beta D(\lambda)\; +\; \omega_i J_i}\right) = \boxed{Z_0^{MT}(x,y)} - \frac{2\lambda}{N_c}\,\beta\, Z_2^{MT}(x,y) + \ldots$$

Set: $x = e^{-\beta}\tilde{x}, \; y = e^{-\beta}\tilde{y}$

$$Z_0^{MT}(\beta,\tilde{x},\tilde{y}) = \prod_{k=1} \frac{1}{1 - e^{-k\beta}(\tilde{x}^k + \tilde{y}^k)}$$

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Poles at: $T_k^* = \dfrac{k}{\log\left(\tilde{x}^k + \tilde{y}^k\right)},\quad (k = 1, 2, \ldots)$

One-loop part has the double-pole at the same position

Singularity of Partition fn

Poles at: $T_k^* = \dfrac{k}{\log\left(\tilde{x}^k + \tilde{y}^k\right)}$, $(k = 1, 2, \dots)$

Hagedorn temperature depends on chem. pot.

Look for the smallest value of $T_k^* > 0$
for general chemical potentials $(\tilde{x}, \tilde{y})$

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Look for the smallest value of $T_k^* > 0$
for general chemical potentials $(\tilde{x}, \tilde{y})$

$k = 1$ is the smallest for $\mathcal{R}_+ = \{\tilde{x} \geq 0 \text{ and } \tilde{y} \geq 0,\ \tilde{x} + \tilde{y} \geq 1\}$

$k = 2$ is the smallest for $\mathcal{R}_- = \{\tilde{x} \leq 0 \text{ or } \tilde{y} \leq 0,\ \tilde{x}^2 + \tilde{y}^2 \geq 1\}$

$k = p$ is the smallest for

$$\mathrm{Arg}(\tilde{x}) = \frac{2\pi}{p_1},\ \mathrm{Arg}(\tilde{y}) = \frac{2\pi}{p_2},\ p = \mathrm{lcm}(p_1, p_2),\ (p \ll N_c^2)$$

Singularity of Partition fn

Poles at: $T^*_k = \dfrac{k}{\log\left(\tilde{x}^k + \tilde{y}^k\right)}$, $(k = 1, 2, \ldots)$

Hagedorn temperature depends on chem. pot.

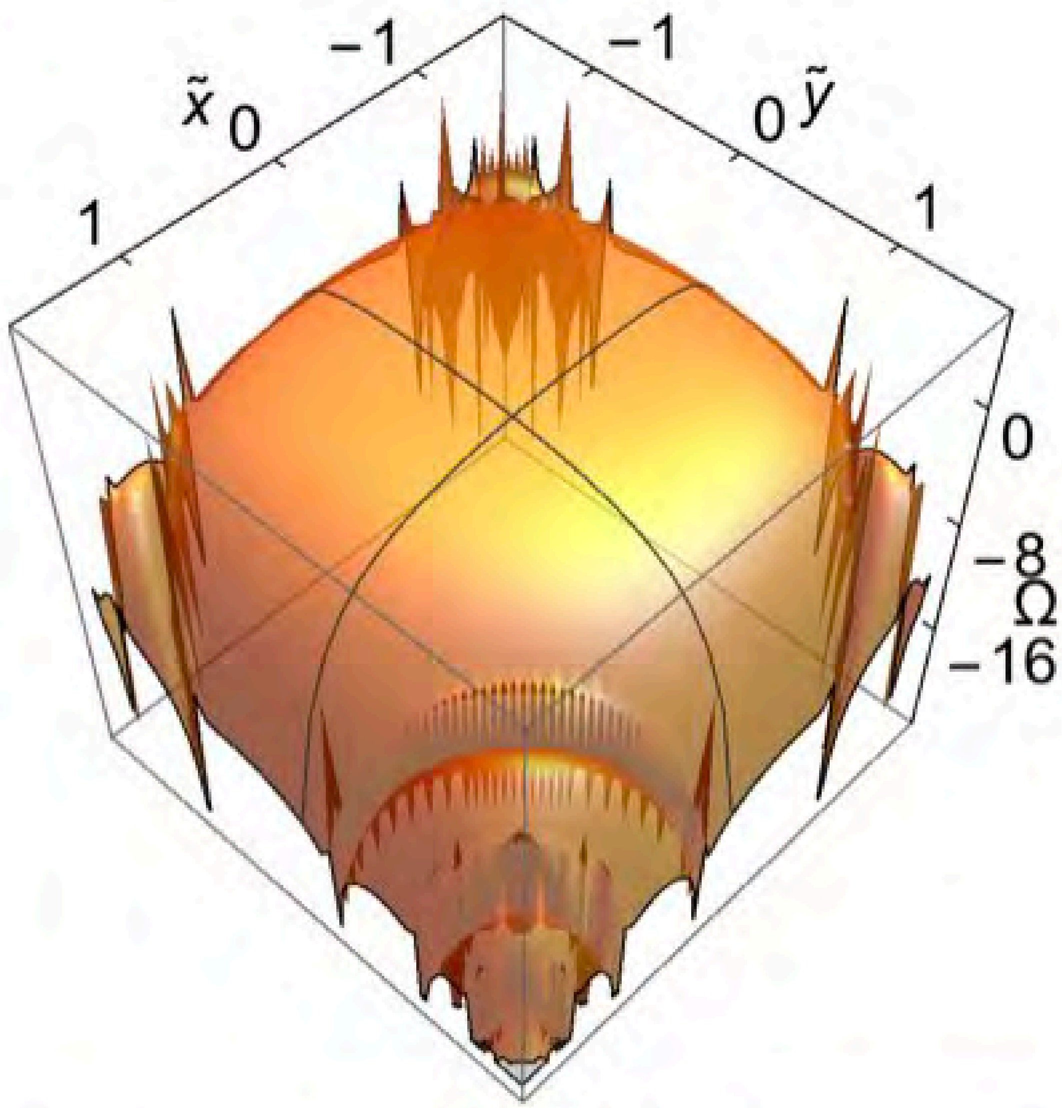
Adding one-loop corrections:

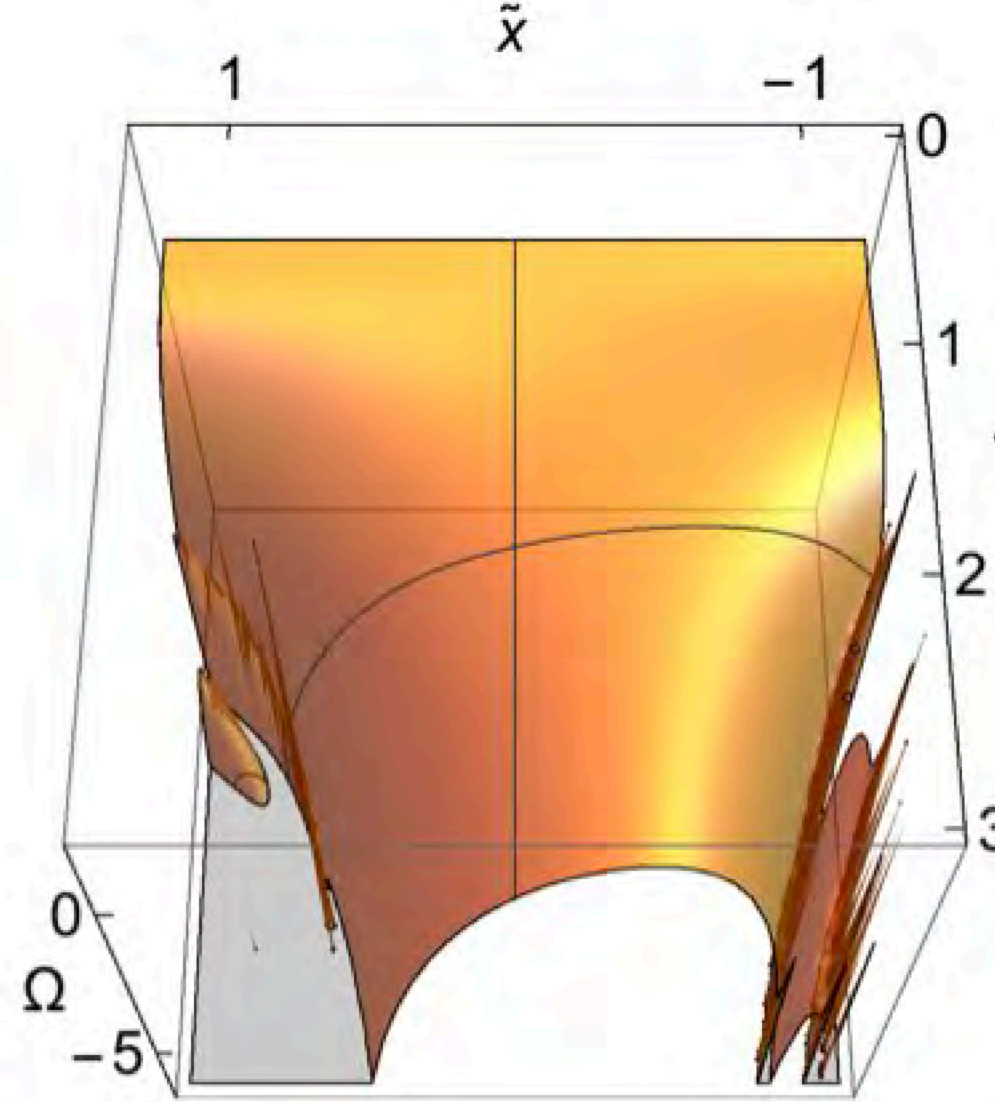
$$T_H(\lambda) = \frac{1}{\log(\tilde{x} + \tilde{y})}\left[1 + \frac{4\lambda\tilde{x}\tilde{y}}{\left(\tilde{x} + \tilde{y}\right)^2}\right], \quad (\tilde{x}, \tilde{y}) \in \mathcal{R}_+$$

$$= \frac{2}{\log(\tilde{x}^2 + \tilde{y}^2)}\left[1 + \frac{4\lambda\tilde{x}^2\tilde{y}^2}{(\tilde{x}^2 + \tilde{y}^2)^2}\right], \quad (\tilde{x}, \tilde{y}) \in \mathcal{R}_-$$

Plots of $\Omega = -T \log Z$

Plot of Ω at fixed T (Left) or along $\tilde{x} = \tilde{y}$ (Right)





Plots of $\Omega = -T \log Z$

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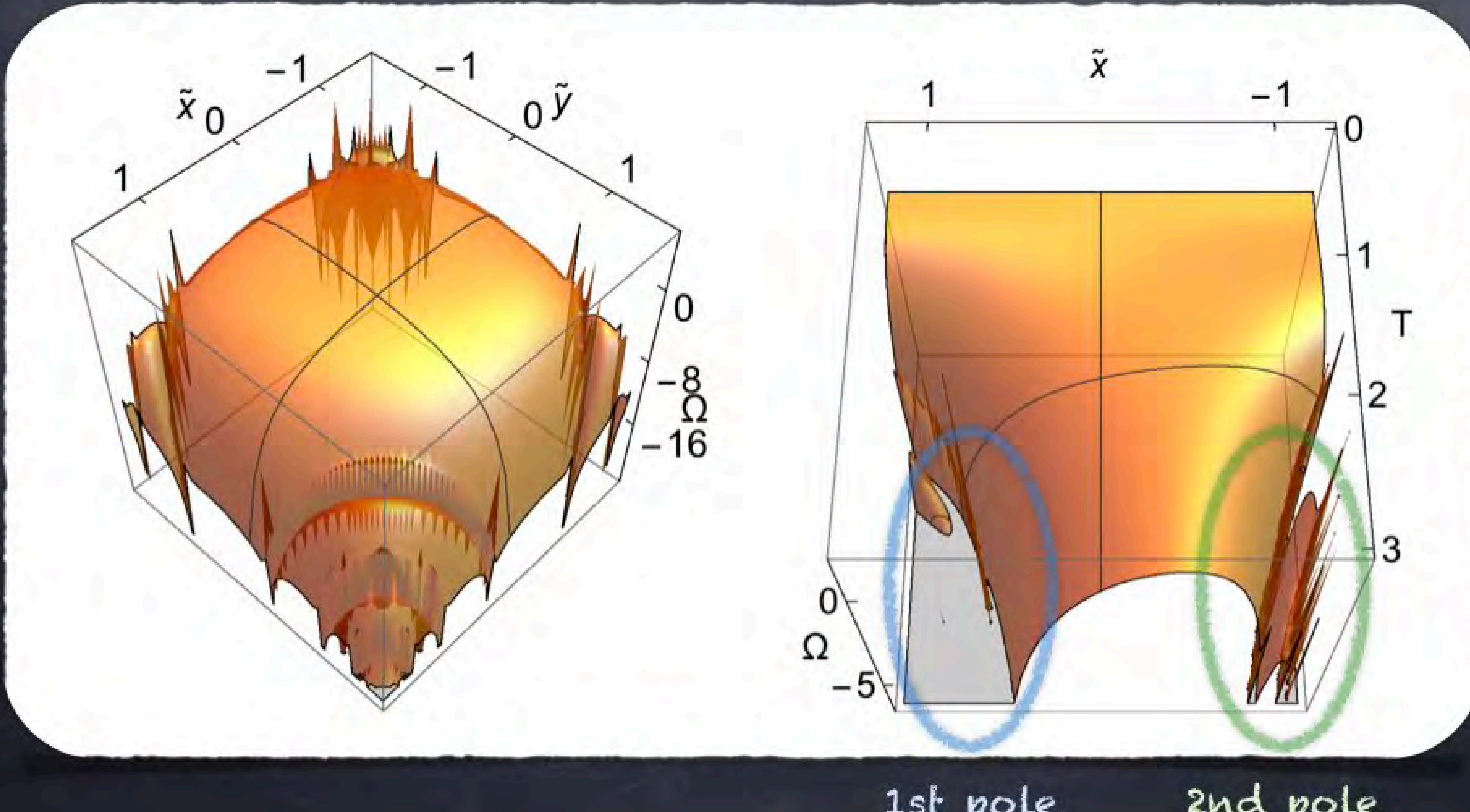

§Conclusion

Conclusion:

- Permutation basis of operators
- Tree-level counting formula
- Sum of one-loop dimensions
- Hagedorn temperature

Outlook:

- Higher loop order
- Larger sector

  Hagedorn TBA of [Harmark, Wilhelm (2017)]
- OPE limit of 4-pt functions

Thank you for your attention