Exact tachyon spectrum in AdS/CFT

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Based on JHEP03(2014)055 [arXiv:1312.3900] in collaboration with
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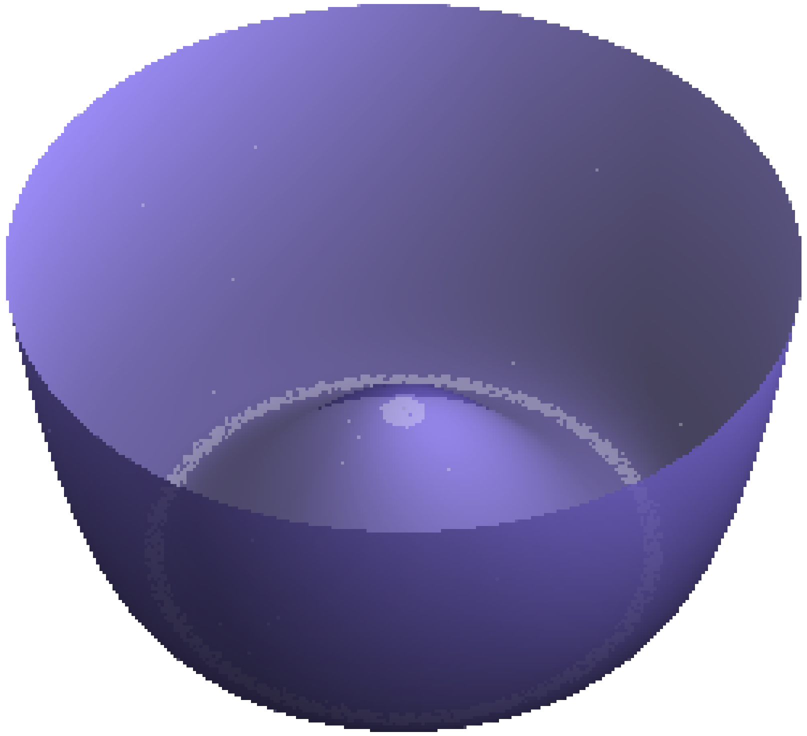
Tachyon and instability

Lagrangian density of a complex-scalar QFT

$$\mathcal{L} = |\partial_\mu \phi|^2 - V(\phi, \bar{\phi})$$

The 1st derivative defines the vacuum, the 2nd the mass

When the mass is pure imaginary, the corresponding particle is called tachyon, and the extremum is unstable



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(mass)$^2 > 0$

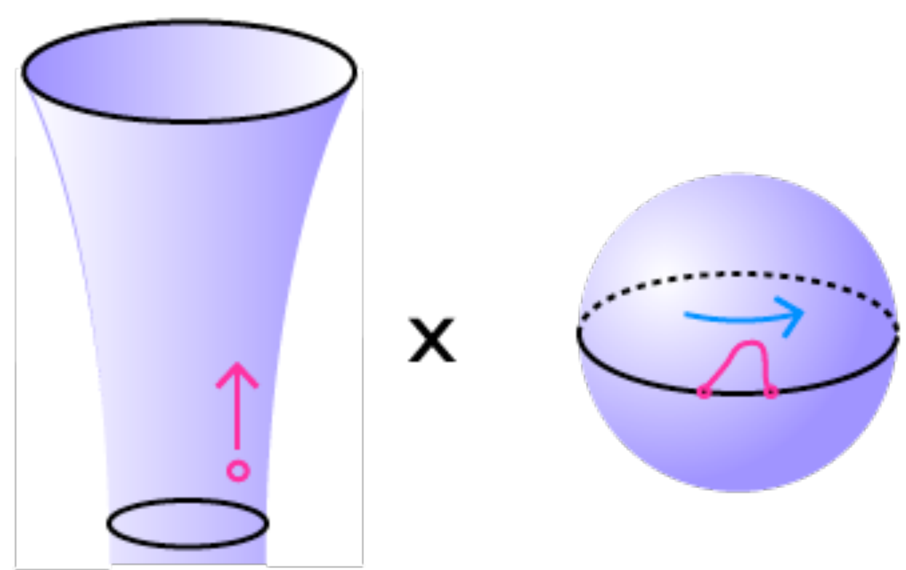
(mass)$^2 < 0$

http://tohbidou.com/2_275.html

Brane-antibrane system

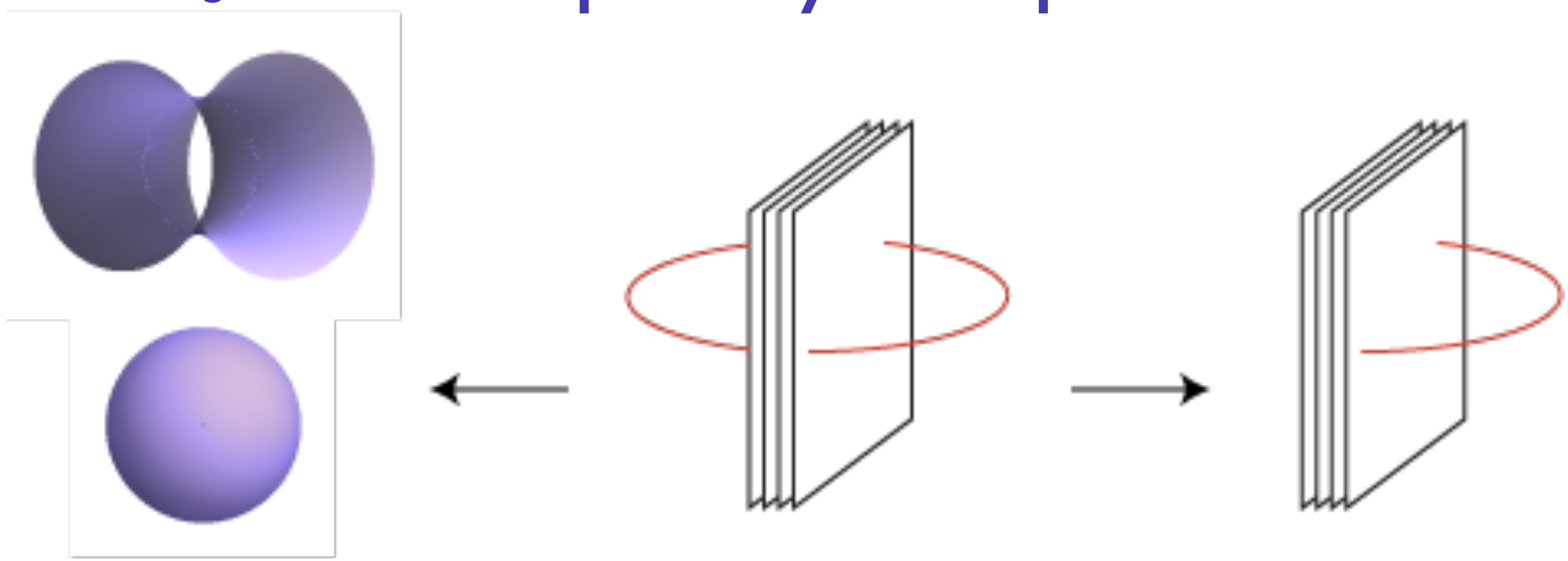
D-brane & D-antibrane (D-$\overline{\text{D}}$) system in the flat spacetime
is an example of unstable state in string theory

D-brane & D-antibrane and open strings in between
in the curved spacetime ($AdS_5 \times S^5$) are less well-understood



AdS/CFT correspondence

$AdS_5 \times S^5$ is the primary example of AdS/CFT



Superstring theory on $AdS_5 \times S^5$

$$N \to \infty, \; g_s \to 0$$

$$\lambda = N g_s$$

Stack of N D3-branes

$$\stackrel{?}{=}$$

$\mathcal{N}$=4 4dim SU(N) super Yang-Mills

$$N \to \infty, \; g_{\rm YM} \to 0$$

$$\lambda = N g_{\rm YM}^2$$

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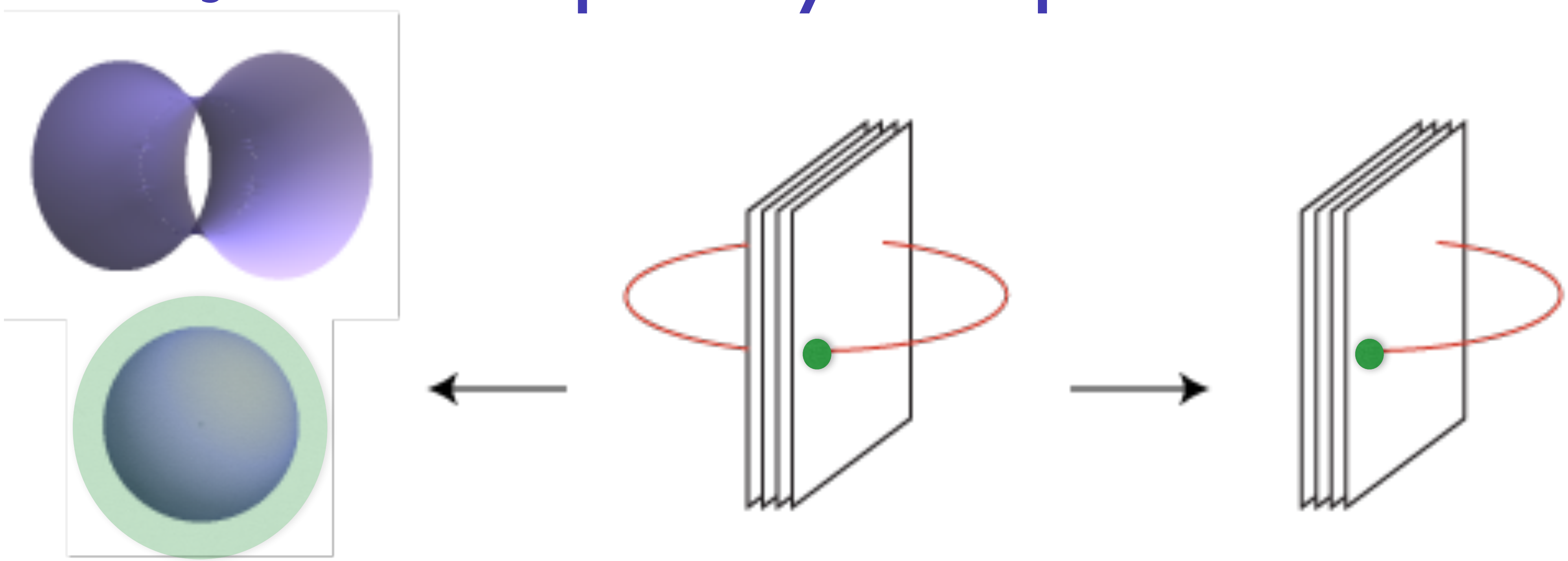

<u>Our setup</u> Add another "giant graviton" D3-brane which extends in the transversal directions to stack branes. On the left figure, it wraps on $R \times S^3$ inside $AdS_5 \times S^5$.

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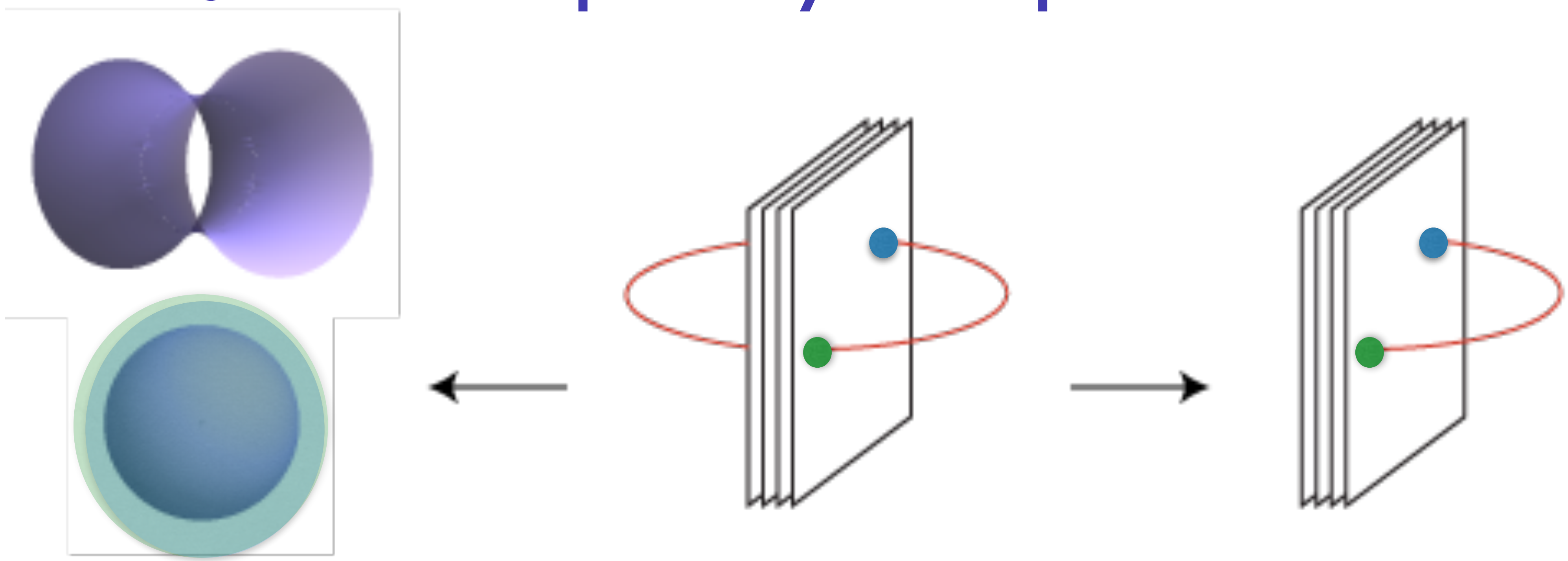

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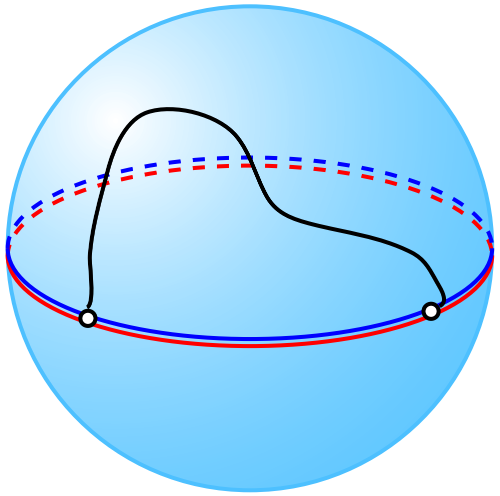
Add a charge-conjugate of the "giant graviton" D3-brane

AdS/CFT correspondence

The energy of an open string in $AdS_5 \times S^5$ ending on a pair of "giant-graviton" D-$\overline{\text{D}}$ branes



should be dual to the dimension of a determinant-like operator in 4D $SU(N)$ $\mathcal{N}$=4 super Yang-Mills theory

$$\mathcal{O}_{Y,\overline{Y}}[V,W] \sim \epsilon^{i_1 \cdots i_N}\, \epsilon_{j_1 \cdots j_N}\, \epsilon^{k_1 \cdots k_N}\, \epsilon_{l_1 \cdots l_N} \times$$
$$Y^{j_1}_{i_1} \cdots Y^{j_{N-1}}_{i_{N-1}}\, V^{l_N}_{i_N}\, \overline{Y}^{l_1}_{k_1} \cdots \overline{Y}^{l_{N-1}}_{k_{N-1}}\, W^{j_N}_{k_N}$$

[Balasubramanian, Berkooz, Naqvi, Strassler (2001)] [Balasubramanian, Huang, Levi, Naqvi (2002)]

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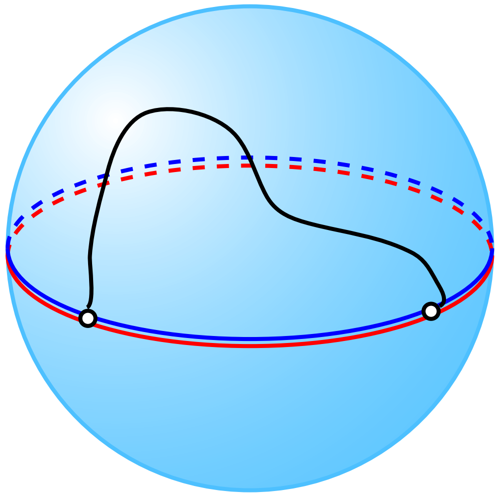

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$$Y^{j_1}_{i_1}\cdots Y^{j_{N-1}}_{i_{N-1}}\,V^{l_N}_{i_N}\,\overline{Y}^{l_1}_{k_1}\cdots\overline{Y}^{l_{N-1}}_{k_{N-1}}\,W^{j_N}_{k_N}$$

$\boxed{\text{Hope: demonstrate the duality using integrability}}$

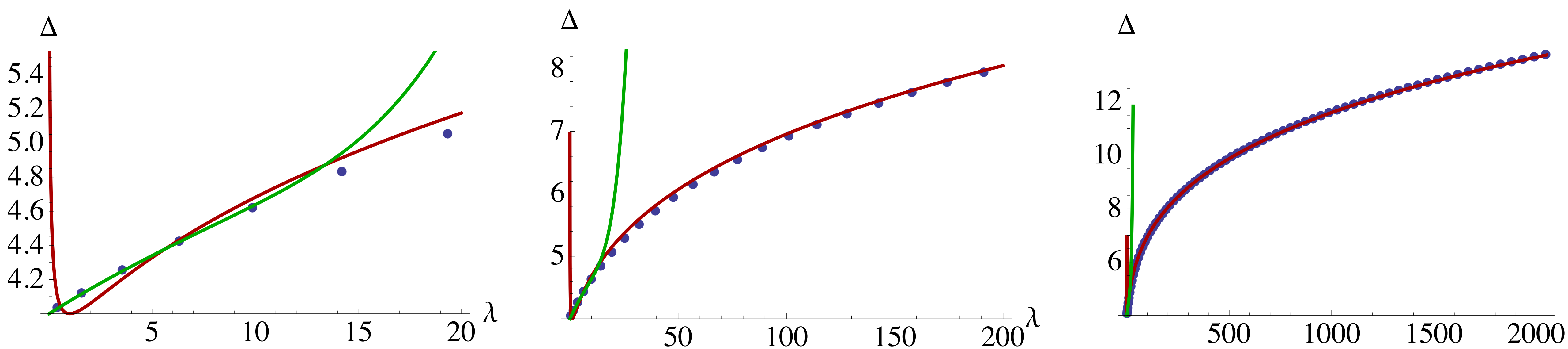
Integrability Predictions

The spectral problem at large N is now "solvable" through (Asymptotic/Thermodynamic) Bethe Ansatz

$$E_{\text{string}}(\lambda) \xleftarrow{\sim} E_{\text{ABA}}(\lambda) \;\text{ or }\; E_{\text{TBA}}(\lambda) \xrightarrow{\sim} \Delta_{\text{SYM}}(\lambda)$$

We want to solve TBA; i.e. obtain $E_{\text{TBA}}(\lambda)$

$\boxed{\text{Example: the exact dimension of Konishi operator}}$



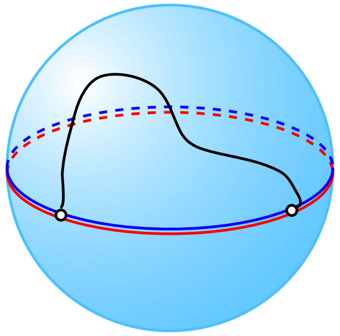
Green: SYM, weak 5-loop Blue: TBA, numerics Red: String, strong 1-loop

[Gromov, Kazakov, Vieira (2009)] [Frolov (2010)] and others

To do

$$E_{\text{string}}(\lambda) \overset{\sim}{\longleftarrow} E_{\text{ABA}}(\lambda) \text{ or } E_{\text{TBA}}(\lambda) \overset{\sim}{\longrightarrow} \Delta_{\text{SYM}}(\lambda)$$

We propose BTBA equations
(Boundary Thermodynamic Bethe Ansatz)
and solve them numerically



$$\mathcal{O}_{Y,\overline{Y}}[V,W] \sim \epsilon^{i_1\cdots i_N}\,\epsilon_{j_1\cdots j_N}\,\epsilon^{k_1\cdots k_N}\,\epsilon_{l_1\cdots l_N}\,Y_{i_1}^{j_1}\cdots Y_{i_{N-1}}^{j_{N-1}}\,V_{i_N}^{l_N}\,\overline{Y}_{k_1}^{l_1}\cdots\overline{Y}_{k_{N-1}}^{l_{N-1}}\,W_{k_N}^{j_N}$$

However, the DDbar system should contain open tachyons.
Indeed, integrability method *apparently* predicts singularity

cf. closed tachyons and non-conformality, [Dymarsky, Klebanov, Roiban] hep-th/0509132 [Fokken, Sieg, Wilhelm] 1308.4420
cf. examples of a singular TBA energy [Frolov, RS] 0906.0499 [de Leeuw, van Tongeren] 1201.1451

Plan of Talk

✓ Introduction

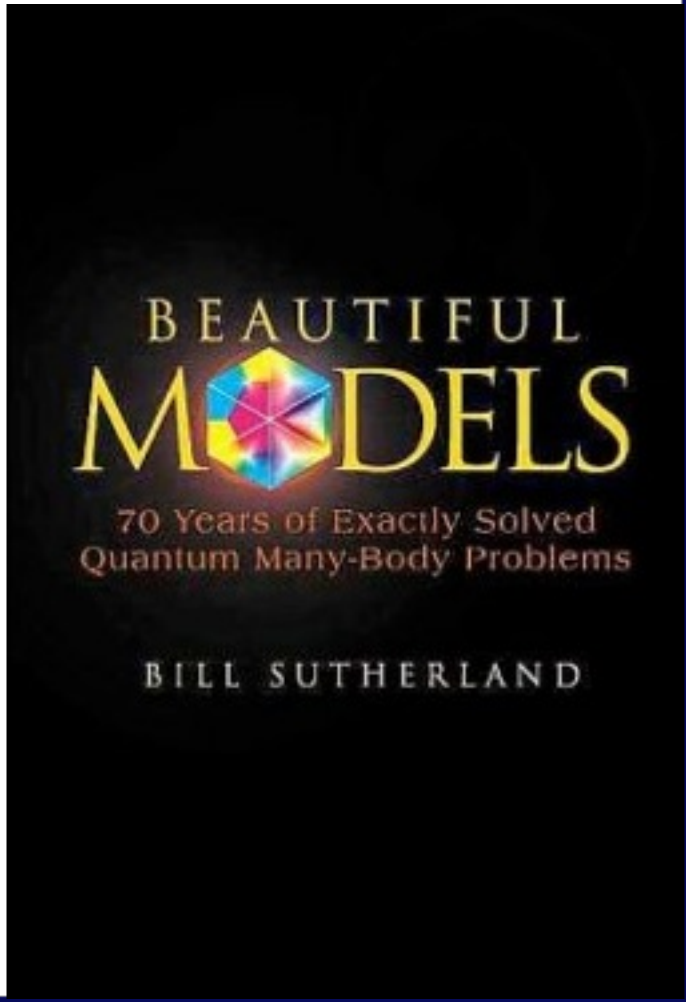
- Integrability and AdS/CFT
- Determinants and giant-gravitons
- BTBA equations and energy bound
- Summary and outlook

Integrability and AdS/CFT

What is integrability?

Textbook definition

Infinitely many conserved charges, S-matrix factorization, Yang-Baxter relation, …



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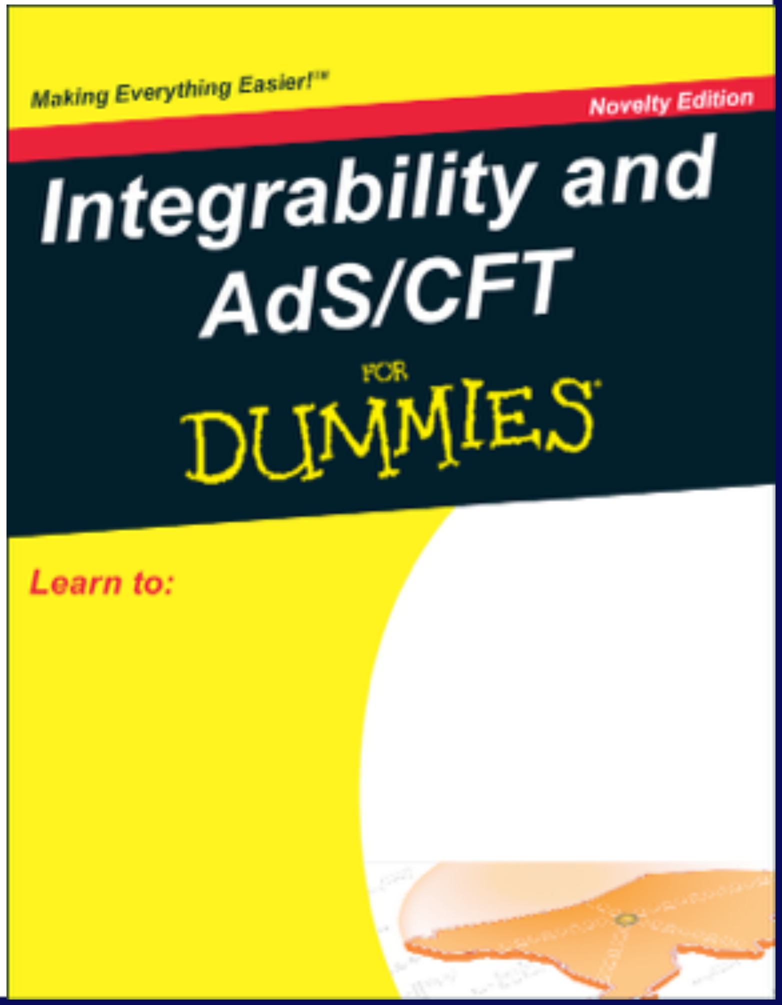
Working definition

1. Compute physical quantities
2. Find infinite-dimensional symmetry
3. Conjecture "Bethe-Ansatz" formula
4. Check your conjecture -- agreement!



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4. Check your conjecture -- agreement!

Integrability in string theory

The integrability method is an alternative to the RNS formalism when the background spacetime contains D-branes.

Integrability in the σ-model on AdS$_5$ x S^5

String theory reduces to a 2d σ-model at small g_s and large N

- Ramond-Neveu-Schwarz formalism (worldsheet susy manifest)

✓ Green-Schwarz formalism (spacetime susy manifest)

$$S_{\rm GS} = -\frac{\sqrt{\lambda}}{4\pi}\int d^2\sigma\, G_{MN}\,\partial Z^M \partial Z^N + \ldots, \quad Z^M = (x^m, \theta^I_\alpha)$$

On AdS$_5$×S^5, GS action has the susy completion as a supercoset σ-model

$$\mathrm{AdS}_5 \times \mathrm{S}^5 + \text{fermions} = \frac{PSU(2,2|4)}{SO(4,1)\times SO(5)} \curvearrowleft \mathbb{Z}_4$$

$$S_{\rm coset} = -\frac{\sqrt{\lambda}}{4\pi}\int d^2\sigma\, \mathrm{Str}\left[\gamma^{\alpha\beta} A^{(2)}_\alpha A^{(2)}_\beta \pm \epsilon^{\alpha\beta} A^{(1)}_\alpha A^{(3)}_\beta\right]$$

$$\mathfrak{g} \in SU(2,2|4), \quad A = -\mathfrak{g}^{-1} d\mathfrak{g} = A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)},$$

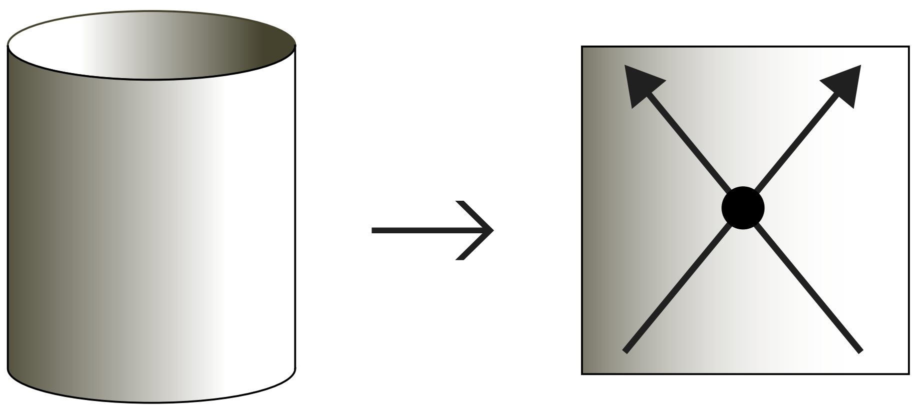
[Metsaev Tseytlin] (1998)

Integrability in the σ-model on $AdS_5 \times S^5$

The supercoset σ-model is classically integrable;

We determine the (asymptotic) spectrum via the S-matrix bootstrap assuming quantum integrability

- First, break worldsheet conformal symmetry by a gauge choice (worldsheet circumference = string angular momentum on S^5)
- Second, take the large-volume (asymptotic) limit; we can define asymptotic states and their worldsheet S-matrix



[Bena Polchinski Roiban] (2003) [Hofman Maldacena] (2006) and others

$\mathcal{N}$=4 SU(N) Super Yang-Mills

$$S^{\mathcal{N}=4}_{\text{bare}} = \frac{1}{g^2_{\text{YM}}} \int d^4x \, \text{Tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \Phi_m D^\mu \Phi_m - \frac{1}{4} [\Phi_m, \Phi_n]^2 + \textbf{fermions} \right]$$

As a SCFT, an interesting local observable of $\mathcal{N}$=4 SYM is

the anomalous dimension of gauge-invariant (single-trace) non-BPS operators

$$\langle \mathcal{O}_a(x) \mathcal{O}_b(0) \rangle = \frac{Z_{ab}}{|x|^{2\Delta_0}} \quad \to \quad \langle \mathcal{O}_{a'}(x) \mathcal{O}_{b'}(0) \rangle = \frac{\delta_{a'b'}}{|x|^{2\Delta_{a'}}} \,, \qquad \Delta_{a'} = \Delta_0 + \gamma_{a'}$$

Scaling transformation: $(x, \Lambda_{\text{UV}}) \to (\lambda x, \lambda^{-1} \Lambda_{\text{UV}}), \quad Z(\Lambda_{\text{UV}}) \to \lambda^{-\gamma} \, Z(\Lambda_{\text{UV}})$

The one-loop dilatation operator in the scalar sector is

$$\mathcal{D}_{\text{1-loop}} = \frac{dZ}{d \log \Lambda_{\text{UV}}} Z^{-1} = \frac{-\lambda}{16\pi^2 N} \left(\text{tr} \, : [\Phi_m, \Phi_n] \, [\check{\Phi}_m, \check{\Phi}_n] : + \frac{1}{2} \text{tr} \, : [\Phi_m, \check{\Phi}_n]^2 : \right)$$

This produces the operator mixing through algebraic rules of Φ, Φ-check's

$$\text{tr}\big(A \, \check{\Phi}_m B \, \Phi_n\big) = \delta_{mn} \, \text{tr} A \, \text{tr} B, \quad \text{tr}\big(A \, \check{\Phi}_m\big) \, \text{tr}\big(\Phi_n B\big) = \delta_{mn} \, \text{tr}\big(AB\big)$$

[Minahan Zarembo (2002)] [Beisert Kristjansen Staudacher (2003)]

$\mathcal{N}$=4 SYM and spin chain

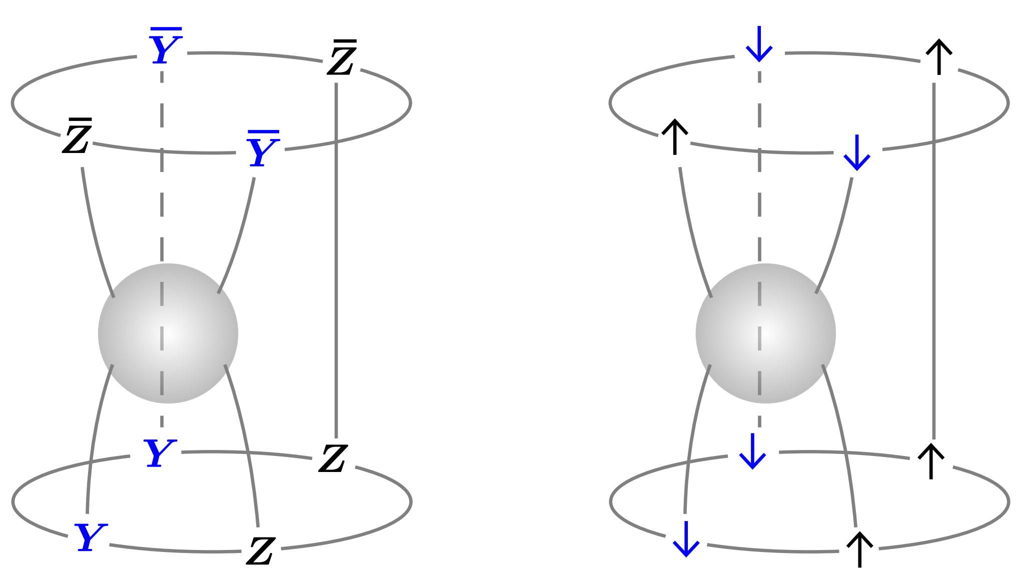
In the large N limit, the $\mathcal{N}$=4 SYM dilatation operator reduces to the Hamiltonian of an integrable spin chain

tr Z^L (half-BPS operator) = Spin-chain ground state

$$\mathcal{D}_{\text{1-loop}} = \frac{-\lambda}{16\pi^2 N}\left(\text{tr} \ :[\Phi_m,\Phi_n]\,[\check{\Phi}_m,\check{\Phi}_n]: + \frac{1}{2}\,\text{tr} \ :[\Phi_m,\check{\Phi}_n]^2:\right)$$

$$\mathcal{D}_{\text{1-loop}}\big|_L = \frac{\lambda}{16\pi^2}\boxed{\sum_{l=1}^{L}\left(2 - 2P_{l,l+1} + K_{l,l+1}\right)}, \qquad \begin{cases} P_{l,l+1} &= \delta_{m_l}^{n_{l+1}}\delta_{m_{l+1}}^{n_l} \\ K_{l,l+1} &= \delta_{m_l,m_{l+1}}\delta^{n_l,n_{l+1}} \end{cases}$$

Can be diagonalized by Bethe Ansatz



$$Z = \Phi_5 + i\Phi_6$$
$$Y = \Phi_3 + i\Phi_4$$

[Minahan Zarembo (2002)]

$\mathcal{N}$=4 SYM and spin chain

Symmetry almost determines the dispersion and S-matrix, and allows us to propose all-loop (asymptotic) Bethe Ansatz

- Global symmetry of $\mathcal{N}$=4 SYM = $\mathfrak{psu}(2,2|4)$
- The choice of vacuum as tr Z^L breaks it to $\mathfrak{psu}(2|2)^2 \times$ R

$$\mathfrak{psu}(2,2|4) \;\to\; \mathfrak{psu}(2|2)^2 \ltimes \mathbb{R} \sim (E = \Delta, S_1, S_2, J_1, J_2, L)$$

- The residual global symmetry enhances to
  $\mathfrak{psu}(2|2)^2 \times \mathrm{R}^3 = \mathfrak{su}(2|2)^2 \times \mathrm{R}$ in the asymptotic limit

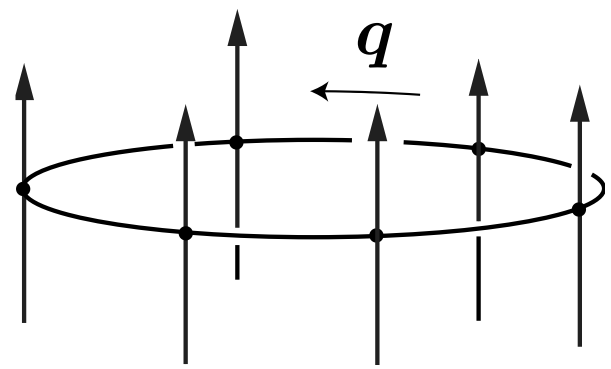
$$\mathrm{tr}\left(Z^{L-m}\chi Z^m\right) \;\to\; (\ldots ZZ \ldots Z\chi Z \ldots ZZ \ldots)$$

$$\mathfrak{psu}(2|2)^2 \ltimes \mathbb{R} \;\to\; \mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}^3 = \mathfrak{su}(2|2)^2 \ltimes \mathbb{R}$$

[Beisert (2005)] and others

Finite L spectrum

The large L (but finite) spectrum is governed by transfer matrix

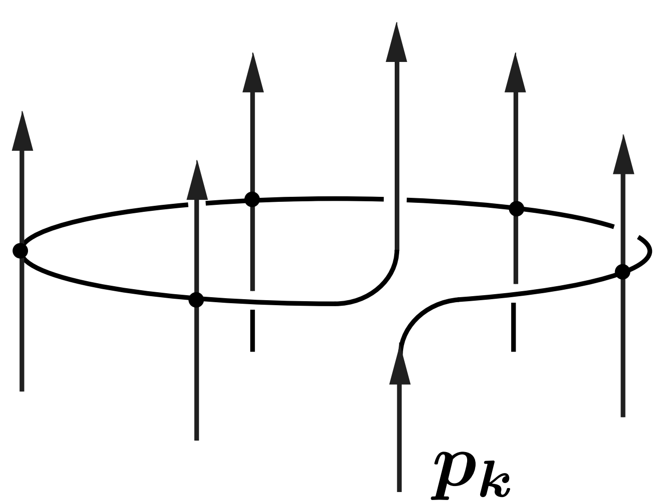


$$T_a(q|\vec{p}) \equiv (\mathrm{s})\mathrm{tr}_{V_a}\left[\mathbb{S}_{a1}(q,p_1)\cdots\mathbb{S}_{aN}(q,p_N)\right]$$

Yang-Baxter relation for integrable S-matrices $\Rightarrow \quad [T_a(q_a|\vec{p}), T_b(q_b|\vec{p})] = 0$

By taking q as one of the momentum of physical excitations,

we obtain the Bethe Ansatz equations



$$T(q|\vec{p})\, e^{-iLq}\Big|_{q=p_k} = \prod_{j=1}^{M} S(p_j,p_k)\, e^{-iLp_k} = -1$$

$$\Delta - L = \sum_{j=1}^{M}\sqrt{1+4g^2\sin^2\frac{p_j}{2}} + \boxed{\mathcal{O}(e^{-cL}),} \quad g = \frac{\sqrt{\lambda}}{2\pi}$$

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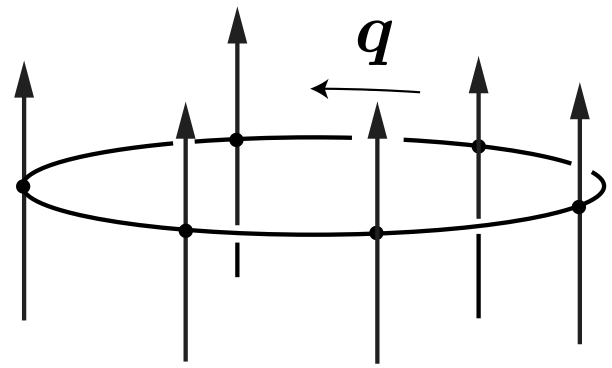

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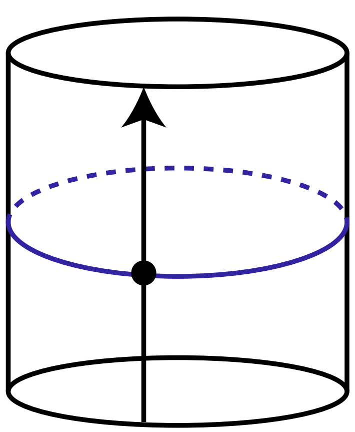
Yang-Baxter relation for integrable S-matrices $\Rightarrow \quad [T_a(q_a|\vec{p}), T_b(q_b|\vec{p})] = 0$

By taking q as the "mirror" momentum of virtual excitations, we obtain the Lüscher formula

$$\Delta_{\text{Lüscher}} \sim \sum_Q \int_{-\infty}^{\infty} d\widetilde{p}_Q \, e^{-\widetilde{\mathcal{E}}_Q(\widetilde{p}_Q)J}$$



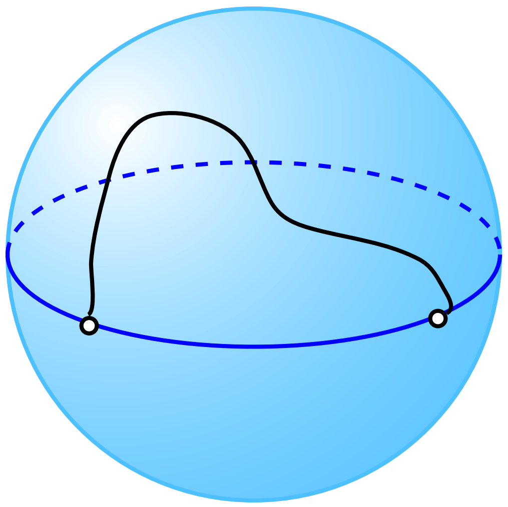
$$(\mathcal{E}_Q, p_Q) = (-i\widetilde{p}_Q, -i\widetilde{\mathcal{E}}_Q), \quad \widetilde{\mathcal{E}}_Q = 2\mathrm{arcsinh}\left(\sqrt{Q^2+\widetilde{p}_Q^2}/(2g)\right)$$

Determinants and giant-gravitons

Spherical Maximal Giant Gravitons (SMGG's)

Giant graviton = Half-BPS, D3-brane solution on $\mathrm{AdS}_5 \times \mathrm{S}^5$

carrying a large angular momentum $L = \mathcal{O}(N)$



Spherical $\Leftrightarrow$ **"wrap" on $\mathbf{S^3 \subset S^5}$**

bound on the angular momentum $L \leq N$

Maximal $\Leftrightarrow$ $L = N$

SMGG's are classified by the choice:

$$\mathbf{S^3 \subset S^5} = \{|X|^2 + |Y|^2 + |Z|^2 = R^2_{\text{sphere}}\}$$

$$X = 0 \ \text{ or } \ Y = 0 \ \text{ or } \ Z = 0 \ \cdots$$

$\overline{Y} = 0$ **brane** $\Leftrightarrow$ **Carrying negative angular momentum compared to $Y = 0$**

[McGreevy, Susskind, Toumbas (2000)]

Giant graviton is determinant

SMGG's are dual to determinants

$$\det \Phi^N = \epsilon^{i_1\cdots i_N}\,\epsilon_{j_1\cdots j_N}\,\Phi^{j_1}_{i_1}\cdots\Phi^{j_N}_{i_N}$$

Open strings on the Y=0 brane are dual to det-like operator

$$\det\left(Y^{N-1}V\right) = \epsilon^{i_1\cdots i_N}\,\epsilon_{j_1\cdots j_N}\,Y^{j_1}_{i_1}\cdots Y^{j_{N-1}}_{i_{N-1}}\,V^{j_N}_{i_N}$$

A pair of open strings on Y=0 and Ybar=0 should be dual to:

$$\mathcal{O}_{Y,\overline{Y}}[V,W] \sim \epsilon^{i_1\cdots i_N}\,\epsilon_{j_1\cdots j_N}\,\epsilon^{k_1\cdots k_N}\,\epsilon_{l_1\cdots l_N}\,Y^{j_1}_{i_1}\cdots Y^{j_{N-1}}_{i_{N-1}}\,V^{l_N}_{i_N}\,\overline{Y}^{l_1}_{k_1}\cdots\overline{Y}^{l_{N-1}}_{k_{N-1}}\,W^{j_N}_{k_N}$$

[Balasubramanian, Berkooz, Naqvi, Strassler (2001)] [Balasubramanian, Huang, Levi, Naqvi (2002)]

SMGG as boundary condition

SMGG is an integrable boundary condition for
an asymptotic open spin chain / open string

Y=0 brane: $\epsilon^{i_1 \cdots i_N} \epsilon_{j_1 \cdots j_N} \boxed{Y^{j_1}_{i_1} \cdots Y^{j_{N-1}}_{i_{N-1}}} \boxed{(ZZ \ldots ZZ)^{j_N}_{i_N}}$

Dilatation operator = Open spin chain Hamiltonian

(discussed later)

- Ground state $\quad (ZZ \ldots ZZ) \sim |0\rangle$
- One-particle state

$$\sum_x e^{ipx} (Z \ldots Z\chi Z \ldots Z) \sim A^\dagger_\chi(p)\,|0\rangle$$

- Two-particle state

$$\sum_{x<x'} e^{ip_1 x + ip_2 x'} (Z \ldots Z\chi ZZ\chi' Z \ldots Z) \sim A^\dagger_\chi(p_1) A^\dagger_\chi(p_2)\,|0\rangle$$

[Hofman, Maldacena (2007)] [Chen, Correa (2007)] [Dekel, Oz (2011)]

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Y=0 brane: $\epsilon^{i_1\cdots i_N}\epsilon_{j_1\cdots j_N}\boxed{Y^{j_1}_{i_1}\cdots Y^{j_{N-1}}_{i_{N-1}}}\boxed{(ZZ\ldots ZZ)^{j_N}_{i_N}}$

The Y=0 preserves the symmetry psu(1|2)^2

which determines *the reflection matrix*,

a solution of the boundary Yang-Baxter relation

$$\mathbb{S}(-p_2,-p_1)\mathbb{R}_Y(p_1)\mathbb{S}(p_1,-p_2)\mathbb{R}_Y(p_2)=\mathbb{R}_Y(p_2)\mathbb{S}(p_2,-p_1)\mathbb{R}_Y(p_1)\mathbb{S}(p_1,p_2)$$

$$\mathbb{R}^-_Y(p)=R^-_0(p)^2\begin{pmatrix}e^{-ip/2}&&&\\&-e^{ip/2}&&\\&&1&\\&&&1\end{pmatrix}^{\otimes 2}$$

$R^-_0(p)^2=-e^{-ip}\,\sigma(p,-p)$ **obeys boundary crossing relation**

[Hofman, Maldacena (2007)] [Chen, Correa (2007)] [Dekel, Oz (2011)]

The Y_θ=0 boundary condition

New reflection amplitudes can be found by rotating R_Y

- $\mathcal{N}$=4 SYM: **Field redefinition:** $\det Y^N \to \det\left(Y\cos\theta + \overline{Y}\sin\theta\right)^N$
- Integrable system:

$$\textbf{Rotation } T: \begin{pmatrix}1\\2\end{pmatrix} \to \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}, \quad \textbf{same for } (\dot{1},\dot{2})$$

$$\mathbb{R}_\theta^-(p) \equiv T R_Y^- T^{-1} = R_0^-(p)^2 \begin{pmatrix} \cos^2\theta\, e^{-ip/2} - \sin^2\theta e^{ip/2} & \sin\theta\cos\theta\left(e^{-ip/2}+e^{ip/2}\right) & & \\ \sin\theta\cos\theta\left(e^{-ip/2}+e^{ip/2}\right) & \sin^2\theta\, e^{-ip/2} - \cos^2\theta e^{ip/2} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}^{\otimes 2}$$

- R_θ still solves boundary Yang-Baxter relation!

$$\mathbb{S}(-p_2,-p_1)\,\mathbb{R}(p_1)\,\mathbb{S}(p_1,-p_2)\,\mathbb{R}(p_2) = \mathbb{R}(p_2)\,\mathbb{S}(p_2,-p_1)\,\mathbb{R}(p_1)\,\mathbb{S}(p_1,p_2)$$

- $\theta = \pi/2$ corresponds to the Ybar=0 brane

Asymptotic Bethe Ansatz (and Lüscher formula, etc)
can be generalized to boundary integrable models

Dilatation on det-like operators

One-loop dilatation operator acting on the Y=0 det-like operators

= Hamiltonian of an integrable open spin-chain

$$\mathcal{O}_{Y,Y}[V] = \epsilon^{i_1 \ldots i_N} \epsilon_{j_1 \ldots j_N} Y^{j_1}_{i_1} \ldots Y^{j_{N-1}}_{i_{N-1}} V^{j_N}_{i_N}, \quad V \sim Z^L$$

$$\mathcal{D}_{\text{1-loop}} = \frac{\lambda}{8\pi^2} Q^Y_1 Q^Y_L \left[\sum_{l=1}^{L-1} \left(I_{l,l+1} - P_{l,l+1} + \frac{1}{2} K_{l,l+1} \right) + (1 - Q^Y_1) + (1 - Q^Y_L) \right] Q^Y_L Q^Y_1$$

$$\textbf{Projector:} \quad Q^Y_\ell \, (\Phi_{m_1} \ldots \Phi_{m_L}) = (1 - \delta_{Y,m_\ell}) \, (\Phi_{m_1} \ldots \Phi_{m_L})$$

[Berenstein Vazquez] (2005) [Hofman Maldacena] (2007)

Dilatation on the Y=0 and Ybar=0 det-like operators *should* look like

$$\mathcal{O}_{Y,\overline{Y}}[V,W] \sim \epsilon^{i_1 \cdots i_N} \epsilon_{j_1 \cdots j_N} \epsilon^{k_1 \cdots k_N} \epsilon_{l_1 \ldots l_N} Y^{j_1}_{i_1} \cdots Y^{j_{N-1}}_{i_{N-1}} V^{l_N}_{i_N} \overline{Y}^{l_1}_{k_1} \cdots \overline{Y}^{l_{N-1}}_{k_{N-1}} W^{j_N}_{k_N}$$

$$\mathcal{D}_{\text{1-loop}} = \mathcal{D}^{(L)}_{\text{1-loop}} + \mathcal{D}^{(R)}_{\text{1-loop}}$$

$$\mathcal{D}^{(L)}_{\text{1-loop}} = \frac{\lambda}{8\pi^2} \boxed{Q^{\bar{Y}}_1} Q^Y_L \left[\sum_{l=1}^{L-1} \left(I_{l,l+1} - P_{l,l+1} + \frac{1}{2} K_{l,l+1} \right) + (1 - Q^Y_1) + (1 - \boxed{Q^{\bar{Y}}_L}) \right] Q^Y_L \boxed{Q^{\bar{Y}}_1}$$

Caveat!

Actually the representative state is not a dilatation eigenstate

$$\mathcal{O}_{Y,\overline{Y}}[V,W] \sim \epsilon^{i_1\cdots i_N}\,\epsilon_{j_1\cdots j_N}\,\epsilon^{k_1\cdots k_N}\,\epsilon_{l_1\cdots l_N}\,Y^{j_1}_{i_1}\cdots Y^{j_{N-1}}_{i_{N-1}}\,V^{l_N}_{i_N}\,\overline{Y}^{l_1}_{k_1}\cdots\overline{Y}^{l_{N-1}}_{k_{N-1}}\,W^{j_N}_{k_N}$$

An example of mixings

$$\mathcal{O}'_{Y,\overline{Y}}[V,W] \sim \epsilon^{i_1\cdots i_N}\,\epsilon_{j_1\cdots j_N}\,\epsilon^{k_1\cdots k_N}\,\epsilon_{l_1\cdots l_N}\,Y^{j_1}_{i_1}\cdots (Y\overline{Y})^{j_{N-1}}_{i_{N-1}}\,V^{l_N}_{i_N}\,\overline{Y}^{l_1}_{k_1}\cdots\delta^{l_{N-1}}_{k_{N-1}}\,W^{j_N}_{k_N}$$

$$+\,\epsilon^{i_1\cdots i_N}\,\epsilon_{j_1\cdots j_N}\,\epsilon^{k_1\cdots k_N}\,\epsilon_{l_1\cdots l_N}\,Y^{j_1}_{i_1}\cdots \overline{Y}^{j_{N-1}}_{i_{N-1}}\,V^{l_N}_{i_N}\,\overline{Y}^{l_1}_{k_1}\cdots Y^{l_{N-1}}_{k_{N-1}}\,W^{j_N}_{k_N}$$

We must use the true eigenstate before computing anomalous dimensions.
However, the classification of dilatation eigenstate at finite *N* is difficult particularly when the length of operator exceeds *N*.

Heuristic arguments

- The Y=0 and YYbar operators should differ only by boundary interaction, ie. wrapping corrections starting at the order ~$O(L)$ (actually $2L$)
- The wrapping computation seem to be insensitive to the details of Y,Ybar

Degeneracy of 2pt functions

Consider the two-point function of a YbarY operator

$$\langle \mathcal{O}_{\mathbf{Y},\overline{\mathbf{Y}}}[Z^L, Z^{L'}](x)\, \mathcal{O}_{\overline{\mathbf{Y}},\mathbf{Y}}[\bar{Z}^{L'}, \bar{Z}^L](0)\rangle \sim |x|^{-2\Delta}$$

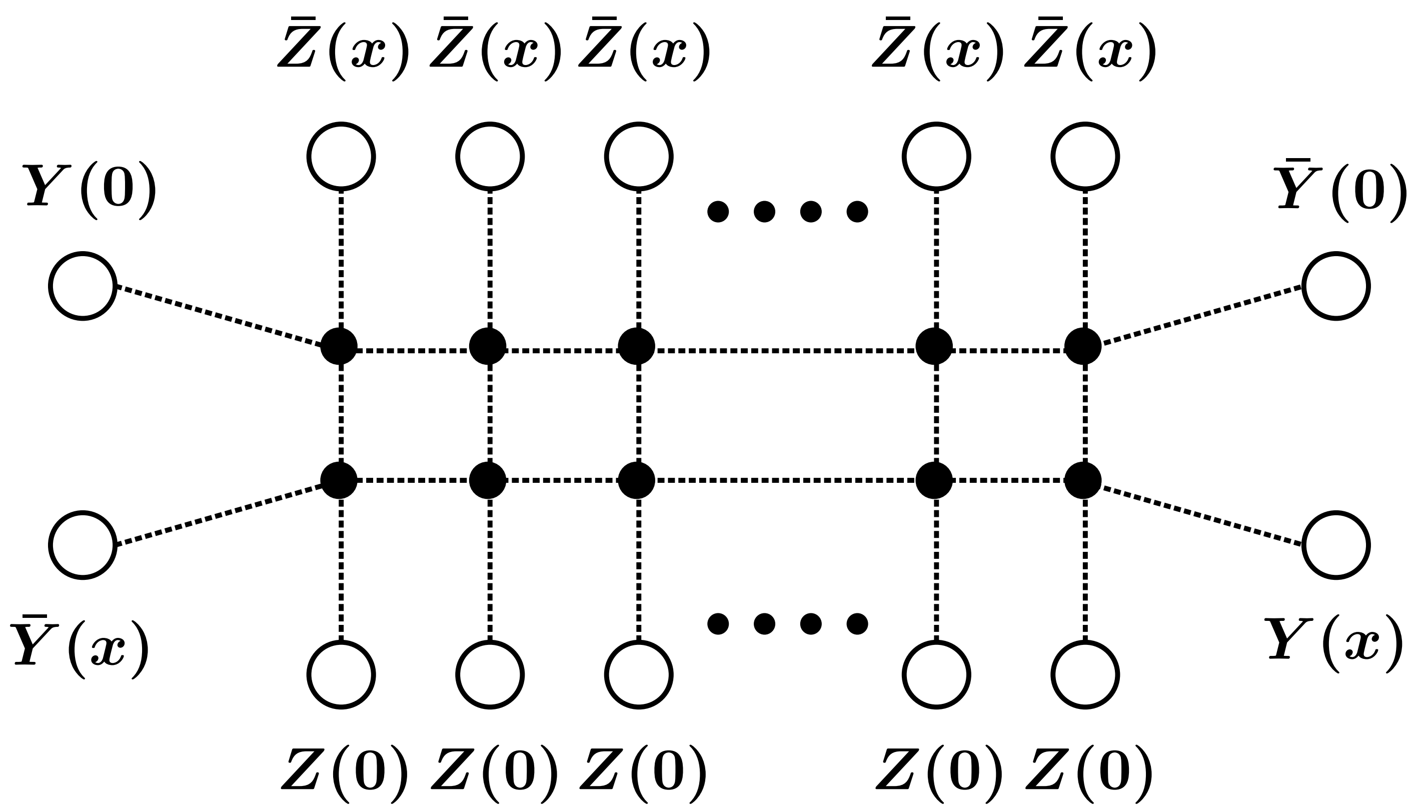
$$\mathcal{O}_{\mathbf{Y},\overline{\mathbf{Y}}}[Z^L, Z^{L'}] \sim \epsilon^{i_1\cdots i_N}\,\epsilon_{j_1\cdots j_N}\,\epsilon^{k_1\cdots k_N}\,\epsilon_{l_1\cdots l_N}\times$$
$$Y^{j_1}_{i_1}\cdots Y^{j_{N-1}}_{i_{N-1}}\,(Z^L)^{l_N}_{i_N}\,\overline{Y}^{l_1}_{k_1}\cdots\overline{Y}^{l_{N-1}}_{k_{N-1}}\,(Z^{L'})^{j_N}_{k_N}$$

Computation goes in almost the same way as on a YY operator

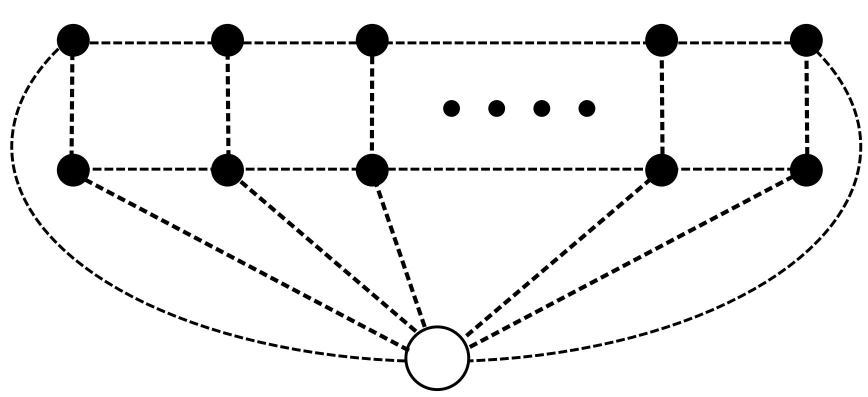
$$\mathcal{O}_{\mathrm{BPS}}[Z^L, Z^{L'}] \sim \epsilon^{i_1\cdots i_N}\,\epsilon_{j_1\cdots j_N}\,\epsilon^{k_1\cdots k_N}\,\epsilon_{l_1\cdots l_N}\times$$
$$Y^{j_1}_{i_1}\cdots Y^{j_{N-1}}_{i_{N-1}}\,(Z^L)^{l_N}_{i_N}\,Y^{l_1}_{k_1}\cdots Y^{l_{N-1}}_{k_{N-1}}\,(Z^{L'})^{j_N}_{k_N}$$

After a lot of tree-level contractions between Y-$\overline{Y}$, we obtain the following diagrams

Wrapping diagram



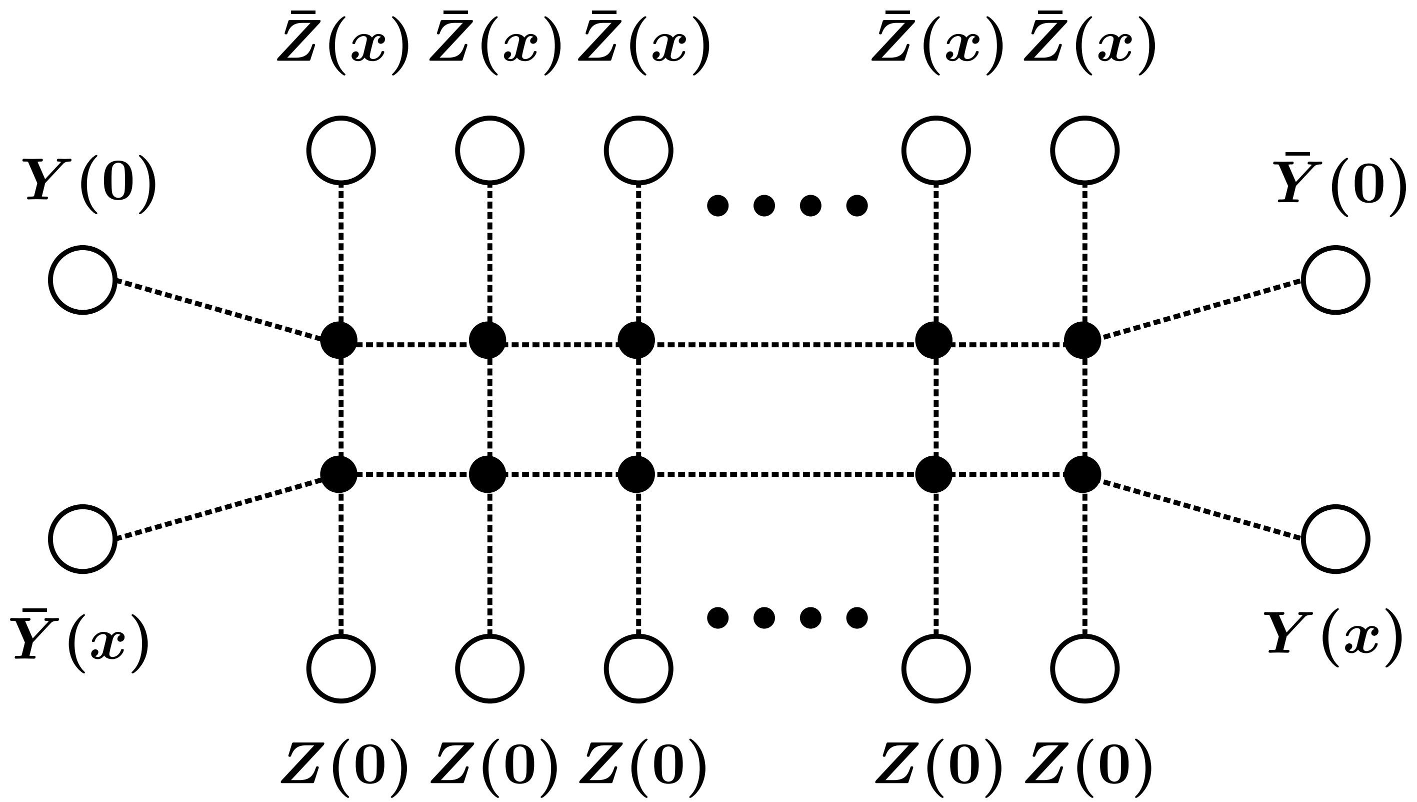
Spacetime structure
(amputated)



this is same as the so-called zig-zag diagram

cf. [Brown, Schnetz] arXiv:1208.1890, [Schnetz] arXiv:1210.5376

Wrapping diagram



The result is $\left(g = \frac{\sqrt{\lambda}}{2\pi} \right)$

$$\delta\Delta_L = -\frac{4(g/2)^{4L}}{4L-1}\binom{4L}{2L}\zeta(4L-3) + \mathcal{O}(g^{4L+2}), \quad g \ll 1$$

Agree with the boundary Lüscher formula for $L > 1$

Our heuristic argument should be improved at $L = 1$

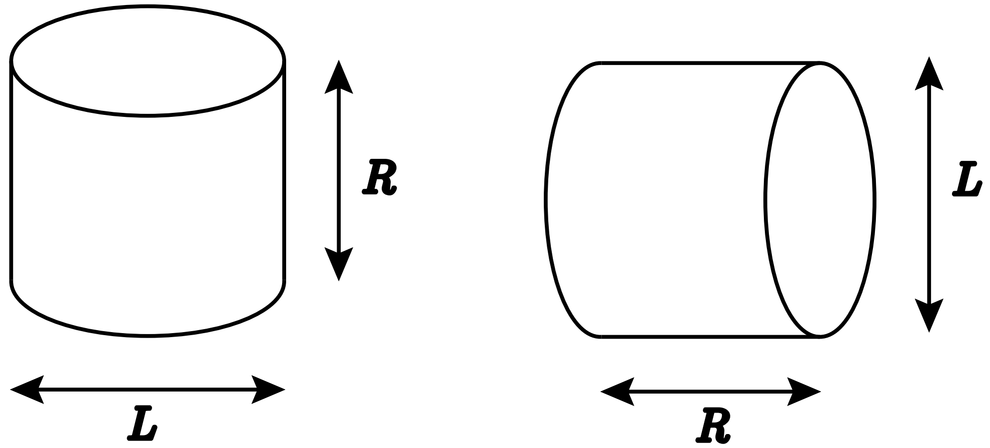
cf. [Brown, Schnetz] arXiv:1208.1890, [Schnetz] arXiv:1210.5376

BTBA equations
and energy bound

Exact dimension/energy

Begin with the equivalence of Euclidean worldsheet partition functions

[Zamolodchikov (1990)]
[Arutyunov, Frolov (2007)]



$$Z_E(L,R) = \int [dX]\, e^{-S_E} = \int [d\tilde{X}]\, e^{-\tilde{S}_E} = \tilde{Z}(R,L)$$

In Hamiltonian formalism, $\mathrm{tr}\, e^{-RH(L)} = \mathrm{tr}\, e^{-L\tilde{H}(R)}$

Take the large R limit, $e^{-RE_0(L)} = \lim_{R\to\infty} e^{-\boxed{\tilde{\mathcal{F}}(\mathcal{R})}}$

The "mirror" free energy can be computed by the "mirror" asymptotic Bethe Ansatz equations in the thermodynamic limit

⇨ Thermodynamic Bethe Ansatz equations (TBA)

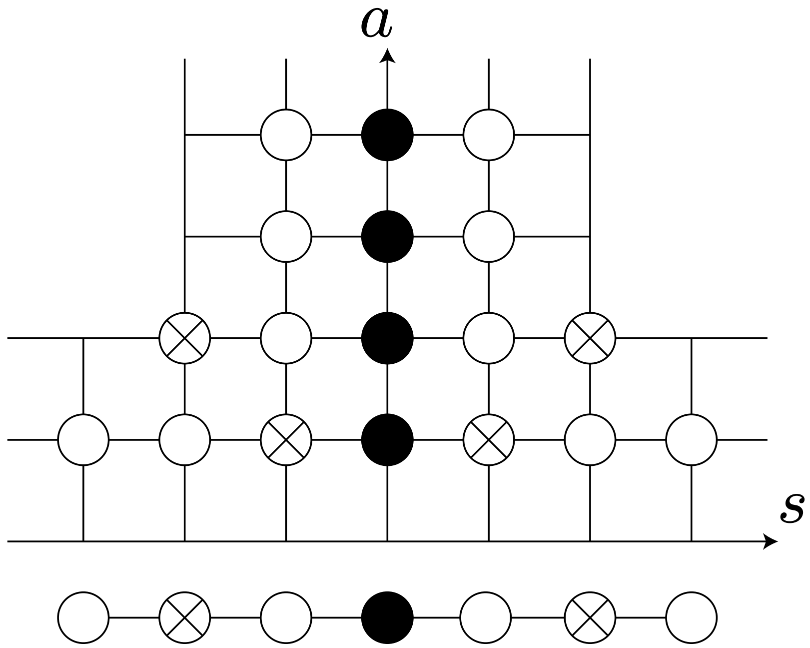
TBA in $AdS_5 \times S^5$ = Y-system + discontinuity

TBA (schematically): $$\log Y_A = V_A + \boxed{\sum_B} \log(1 \pm Y_B) \star K_{BA}$$

$$\log(1+Y) * K(v) = \int dt\, \log(1+Y(t)) \frac{1}{2\pi i} \frac{\partial}{\partial t} \log S(t,v)$$

Exact energy: $$E - L = -\sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{d\tilde{p}_Q}{2\pi} \log(1+Y_Q), \quad Y_Q = Y_{Q,0}$$

$\mathfrak{psu}(2,2|4)$-hook



The hook is related to functional equations called Y-system, which can be derived from the TBA equations.

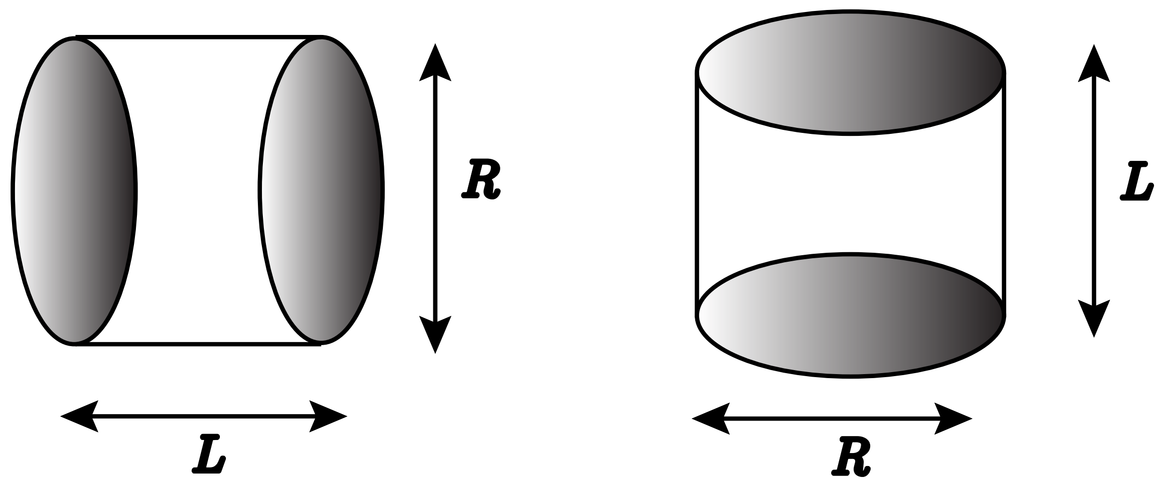
$$\frac{Y_{a,s}^+ Y_{a,s}^-}{Y_{a-1,s} Y_{a+1,s}} = \frac{(1+Y_{a,s-1})(1+Y_{a,s+1})}{(1+Y_{a-1,s})(1+Y_{a+1,s})}$$

$$Y^{\pm}(v) = Y(v \pm i/g)$$

[Gromov, Kazakov, Vieira (2009)] [Bombardelli, Fioravanti, Tateo (2009)] [Gromov, Kazakov, Kozak, Vieira (2009)] [Arutyunov, Frolov (2009)] and others

Mirror trick with boundary

A simple generalization is to change boundary conditions



$$Z_E^{(\alpha\beta)}(L,R) = \int [dX]_{\alpha\beta}\, e^{-S_E} = \int [d\tilde{X}]_{\alpha\beta}\, e^{-\tilde{S}_E} = \tilde{Z}^{(\alpha\beta)}(R,L)$$

$$\mathrm{tr}\, e^{-RH_{\alpha\beta}(L)} = \langle B_\alpha |\, e^{-L\tilde{H}(R)}\, | B_\beta \rangle = \sum_\psi \frac{\langle B_\alpha | \psi \rangle \langle \psi | B_\beta \rangle}{\langle \psi | \psi \rangle}\, e^{-L\tilde{\mathcal{E}}_\psi(R)}$$

Take the large R limit, $e^{-RE_{\alpha\beta,0}(L)} = \lim_{R\to\infty} e^{-\tilde{\mathcal{F}}(\mathcal{R}) + \boxed{B_{\alpha\beta}(R)}}$

Difficult to derive the boundary factor $B_{\alpha\beta}$
in integrable models with non-diagonal S-matrix

BTBA for YbarY

- Notice that the boundary just introduces a momentum-dependent chemical potential which just changes the source term V_a
- We define the source term V_a by asymptotic Y-functions, and conjecture the BTBA of the Y=0 & Ybar=0 as follows:

$$\log Y_a = \log(1 \pm Y_b) \star K_{ba} + V_a$$

$$V_a \equiv \log Y_a^\circ - \log(1 \pm Y_b^\circ) * K_{ba}$$

$$Y_{\mathrm{aux}}^\circ = \text{asymptotitc Y-functions}, \quad Y_Q^\circ = 0$$

The asymptotic source term for the ground-state BTBA should be exact

- The asymptotic ground-state Y's have double zeroes or poles at the origin.
- Those zeroes are correlated to $Y=(-1)^F$ at $v=\pm i/g$
- It follows that the singularities at the origin cannot move as long as all Y-functions are real and parity-even.

BTBA for YbarY

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$$Y_{\mathrm{aux}}^\circ = \text{asymptotitc Y-functions}, \quad Y_Q^\circ = 0$$

Our ground-state BTBA takes the form

$$\log \frac{Y_a}{Y_a^\circ} = \log \left(\frac{1 \pm Y_b}{1 \pm Y_b^\circ} \right) \star K_{ba} \quad \text{for auxiliary Y}$$

$$\log \frac{Y_Q}{Y_Q^\bullet} = \log \left(\frac{1 \pm Y_b}{1 \pm Y_b^\circ} \right) \star K_{bQ}$$

Summary of YbarY energy

YbarY BTBA: $$\log \frac{Y_a}{Y_a^\circ} = \log\left(\frac{1 \pm Y_b}{1 \pm Y_b^\circ}\right) \star K_{ba}$$

(∞ nonlinear integral equations can be solved by numerical iteration)

BTBA energy: $$E_{\mathrm{BTBA}}(L, g) = -\sum_{Q=1}^{\infty} \int_0^\infty \frac{d\tilde{p}_Q}{2\pi} \log(1 + Y_Q)$$

Our BTBA describes Δ of the determinant-like operator:

$$\mathcal{O}_{Y,\overline{Y}}[Z^L, Z^{L'}] \sim \epsilon^{i_1 \cdots i_N}\, \epsilon_{j_1 \cdots j_N}\, \epsilon^{k_1 \cdots k_N}\, \epsilon_{l_1 \cdots l_N} \times$$
$$Y_{i_1}^{j_1} \cdots Y_{i_{N-1}}^{j_{N-1}} (Z^L)_{i_N}^{l_N}\, \overline{Y}_{k_1}^{l_1} \cdots \overline{Y}_{k_{N-1}}^{l_{N-1}} (Z^{L'})_{k_N}^{j_N}$$

$$\Delta = 2N - 2 + L + L' + \underbrace{E_{\mathrm{BTBA}}(L, g) + E_{\mathrm{BTBA}}(L', g)}_{\text{all wrapping corrections, negative values}}$$

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$$Y_{i_1}^{j_1} \cdots Y_{i_{N-1}}^{j_{N-1}} (Z^L)_{i_N}^{l_N}\, \overline{Y}_{k_1}^{l_1} \cdots \overline{Y}_{k_{N-1}}^{l_{N-1}} (Z^{L'})_{k_N}^{j_N}$$

$$\Delta = \underbrace{2N}_{\text{Energy of D-branes}} \underbrace{-\, 2 + L + L' + E_{\mathrm{BTBA}}(L,g) + E_{\mathrm{BTBA}}(L',g)}_{\text{Energy of a pair of open strings}}$$

$$E_{\mathrm{open}}[Z^L] = -1 + L + E_{\mathrm{BTBA}}(L,g)$$

Summary of YbarY energy

YbarY BTBA: $$\log \frac{Y_a}{Y_a^\circ} = \log\left(\frac{1 \pm Y_b}{1 \pm Y_b^\circ}\right) \star K_{ba}$$

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$$Y_{i_1}^{j_1} \cdots Y_{i_{N-1}}^{j_{N-1}}\, (Z^L)_{i_N}^{l_N}\, \overline{Y}_{k_1}^{l_1} \cdots \overline{Y}_{k_{N-1}}^{l_{N-1}}\, (Z^{L'})_{k_N}^{j_N}$$

$$\Delta = 2N - 2 + L + L' + E_{\mathrm{BTBA}}(L,g) + E_{\mathrm{BTBA}}(L',g)$$

Interestingly, there exists a lower bound for the (B)TBA energy

YQ(v) at large v

BTBA equation for YQ in the large v limit

$$\log \frac{Y_Q(v)}{Y_Q^{\bullet}(v)} = -2\int_{-\infty}^{\infty} dt\, \log(1+Y_{Q'}(t))\, \boxed{K_{\Sigma}^{Q'Q}(t,v)} + \dots$$

$$\sim -4E_{BTBA}\log(v), \quad v \gg 1$$

$$\Leftrightarrow \quad \log Y_Q(v) \sim -\left(4L + 4E_{\mathrm{BTBA}}\right)\log(v)$$

$$\boxed{K_{Q'Q}^{\Sigma}(t,v) = \frac{1}{2\pi i}\frac{\partial}{\partial t}\log \Sigma^{Q'Q}(t,v)}$$

$$\begin{aligned}
\frac{1}{i}\log \Sigma^{Q'Q}(t,v) = {}& \Phi(y_1^+,y_2^+) - \Phi(y_1^+,y_2^-) - \Phi(y_1^-,y_2^+) + \Phi(y_1^-,y_2^-) \\
&+\frac{1}{2}\Big(\Psi(y_2^+,y_1^+) + \Psi(y_2^-,y_1^+) - \Psi(y_2^+,y_1^-) - \Psi(y_2^-,y_1^-)\Big) \\
&-\frac{1}{2}\Big(\Psi(y_1^+,y_2^+) + \Psi(y_1^-,y_2^+) - \Psi(y_1^+,y_2^-) - \Psi(y_1^-,y_2^-)\Big) \\
&+\frac{1}{i}\log \frac{i^{Q'}\,\Gamma\big[Q - \frac{i}{2}g\big(y_1^+ + \frac{1}{y_1^+} - y_2^+ - \frac{1}{y_2^+}\big)\big]}{i^{Q}\,\Gamma\big[Q' + \frac{i}{2}g\big(y_1^+ + \frac{1}{y_1^+} - y_2^+ - \frac{1}{y_2^+}\big)\big]}\, \frac{1 - \frac{1}{y_1^+ y_2^-}}{1 - \frac{1}{y_1^- y_2^+}}\sqrt{\frac{y_1^+ y_2^-}{y_1^- y_2^+}}
\end{aligned}$$

$$\Phi(x_1,x_2) = i\oint \frac{dw_1}{2\pi}\oint \frac{dw_2}{2\pi}\frac{1}{(w_1-x_1)(w_2-x_2)}\log \frac{\Gamma[1+\frac{ig}{2}\left(w_1+\frac{1}{w_1}-w_2-\frac{1}{w_2}\right)]}{\Gamma[1-\frac{ig}{2}\left(w_1+\frac{1}{w_1}-w_2-\frac{1}{w_2}\right)]}$$

$$\Psi(x_1,x_2) = i\oint \frac{dw}{2\pi}\frac{1}{w-x_2}\log \boxed{\frac{\Gamma[1+\frac{ig}{2}\left(x_1+\frac{1}{x_1}-w-\frac{1}{w}\right)]}{\Gamma[1-\frac{ig}{2}\left(x_1+\frac{1}{x_1}-w-\frac{1}{w}\right)]}}$$

$$x(v) = \frac{1}{2}\left(v - i\sqrt{4-v^2}\right), \quad y_1^{\pm} = x\Big(t \pm \frac{iQ'}{g}\Big), \quad y_2^{\pm} = x\Big(v \pm \frac{iQ}{g}\Big)$$

YQ(v) at large v

BTBA equation for YQ in the large v limit

$$\log \frac{Y_Q(v)}{Y_Q^{\bullet}(v)} = -2 \int_{-\infty}^{\infty} dt\ \log(1 + Y_{Q'}(t))\, K_{\Sigma}^{Q'Q}(t, v) + \dots$$

$$\sim -4E_{BTBA}\ \log(v), \quad v \gg 1$$

$$\Leftrightarrow \quad \log Y_Q(v) \sim -\left(4L + 4E_{\rm BTBA}\right) \log(v)$$

However, the integrals in BTBA energy diverges if $Y_Q(v) \sim 1/v$

$$\int_0^{\infty} \frac{dv}{2\pi} \frac{d\widetilde{p}_Q}{dv}\ \log(1 + Y_Q(v)) \sim (\text{const}) \int^{\infty} dv\, v^{-4L-4E_{\rm BTBA}}$$

The BTBA energy cannot be negative and large

$$4L + 4E_{\rm BTBA} > 1 \quad \Leftrightarrow \quad E_{\rm BTBA} > 1/4 - L$$

YQ(v) at large Q

BTBA equation for YQ in the large Q limit

$$\Leftrightarrow \quad \log Y_Q(v) \sim (3 - 4L - 4E_{\mathrm{BTBA}}) \log(Q)$$

However, the sum in BTBA energy diverges if $Y_Q(v) \sim 1/Q$

$$E_{\mathrm{BTBA}} = -\sum_{Q=1}^{\infty} \int_0^{\infty} \frac{d\widetilde{p}_Q}{2\pi} \log(1 + Y_Q)$$

$$\sim \sum_{Q=1}^{\infty} (\text{const})\, Q^{3-4L-4E_{\mathrm{BTBA}}}$$

The BTBA energy cannot be negative and large

$$4L + 4E_{\mathrm{BTBA}} > 4 \quad \Leftrightarrow \quad E_{\mathrm{BTBA}} > 1 - L$$

Closer look at the bound

The stronger bound is

$$E_{\mathbf{open}}[Z^L] = L - 1 + E_{\mathbf{BTBA}}(L, g) > 0$$

It is impossible to saturate this lower bound.

Suppose $E_{\mathrm{BTBA}} = 1 - L$

then BTBA dictates $Y_Q(v) \sim 1/Q$

This implies E_{BTBA} diverges, which is a contradiction

A sign of divergences can also be seen at numerical analysis

(ie. indeed TBA energy seems to “hit” the bound)

CPU resources

120

Mars Beowulf cluster
(Utrecht University)

4

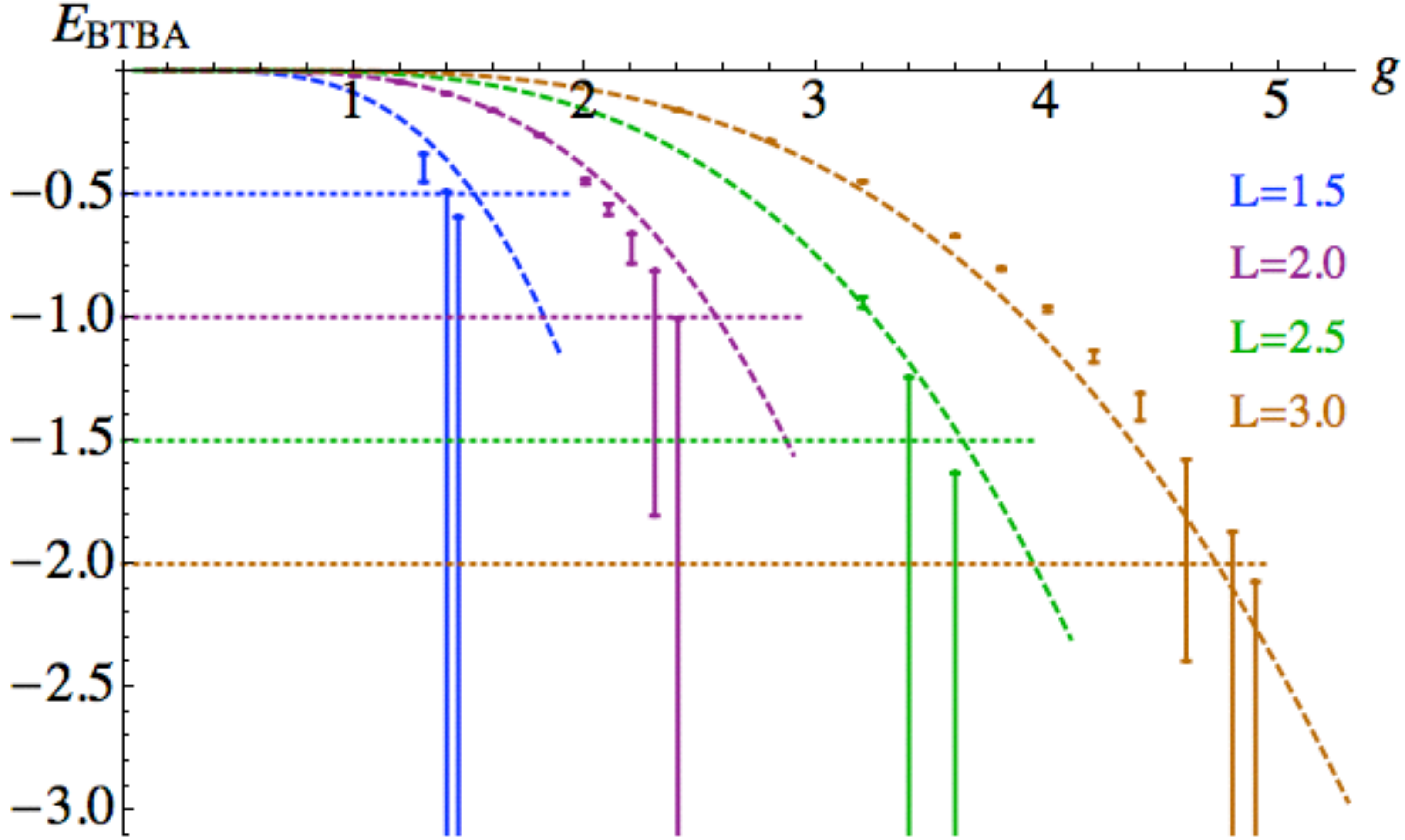
Laptop
(Personal)

Sushiki server
(Yukawa Institute)

24

Mathematica

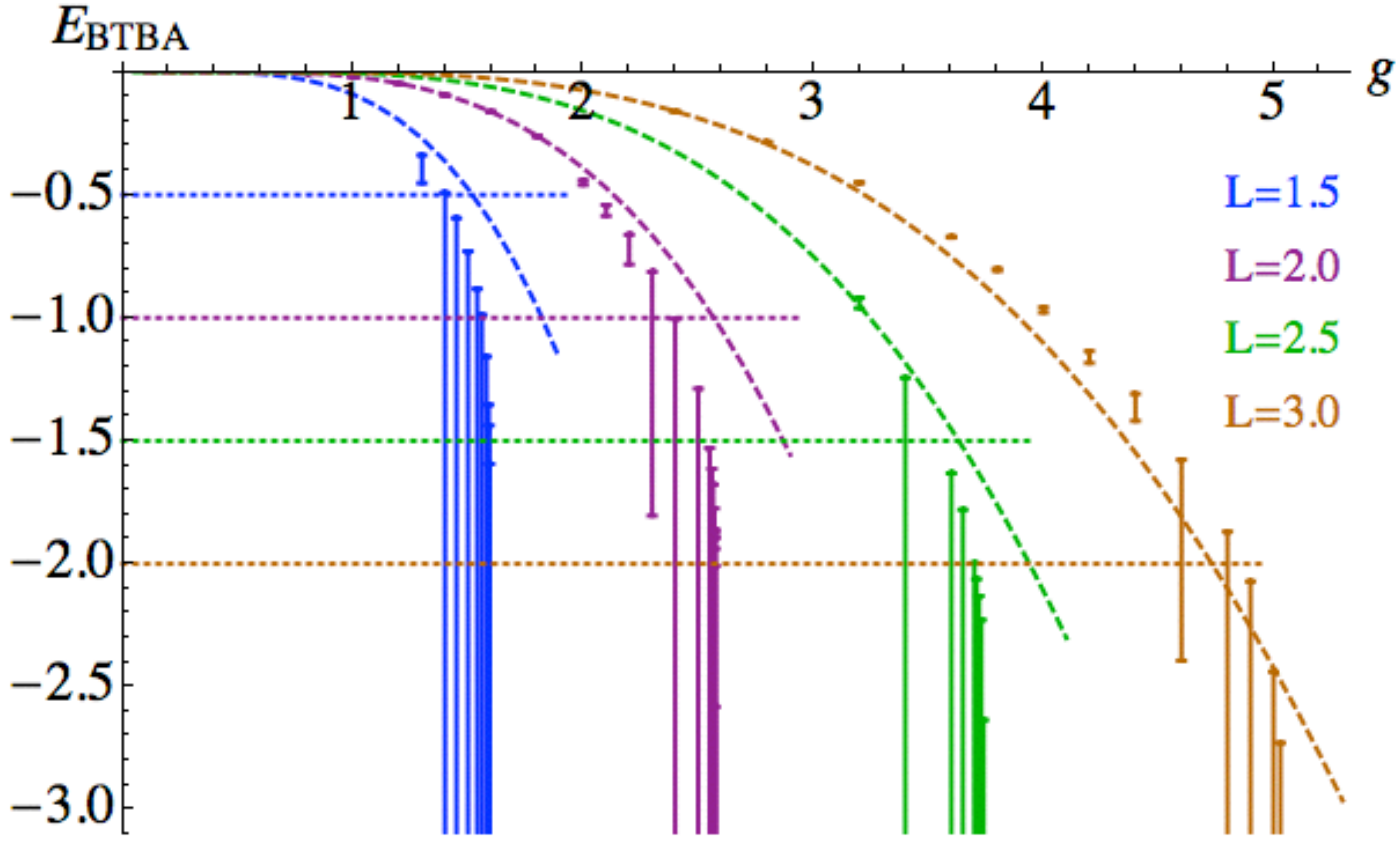
Numerical Results



Solid: BTBA solution, Dashed: Lüscher formula, Dotted: Lower bound

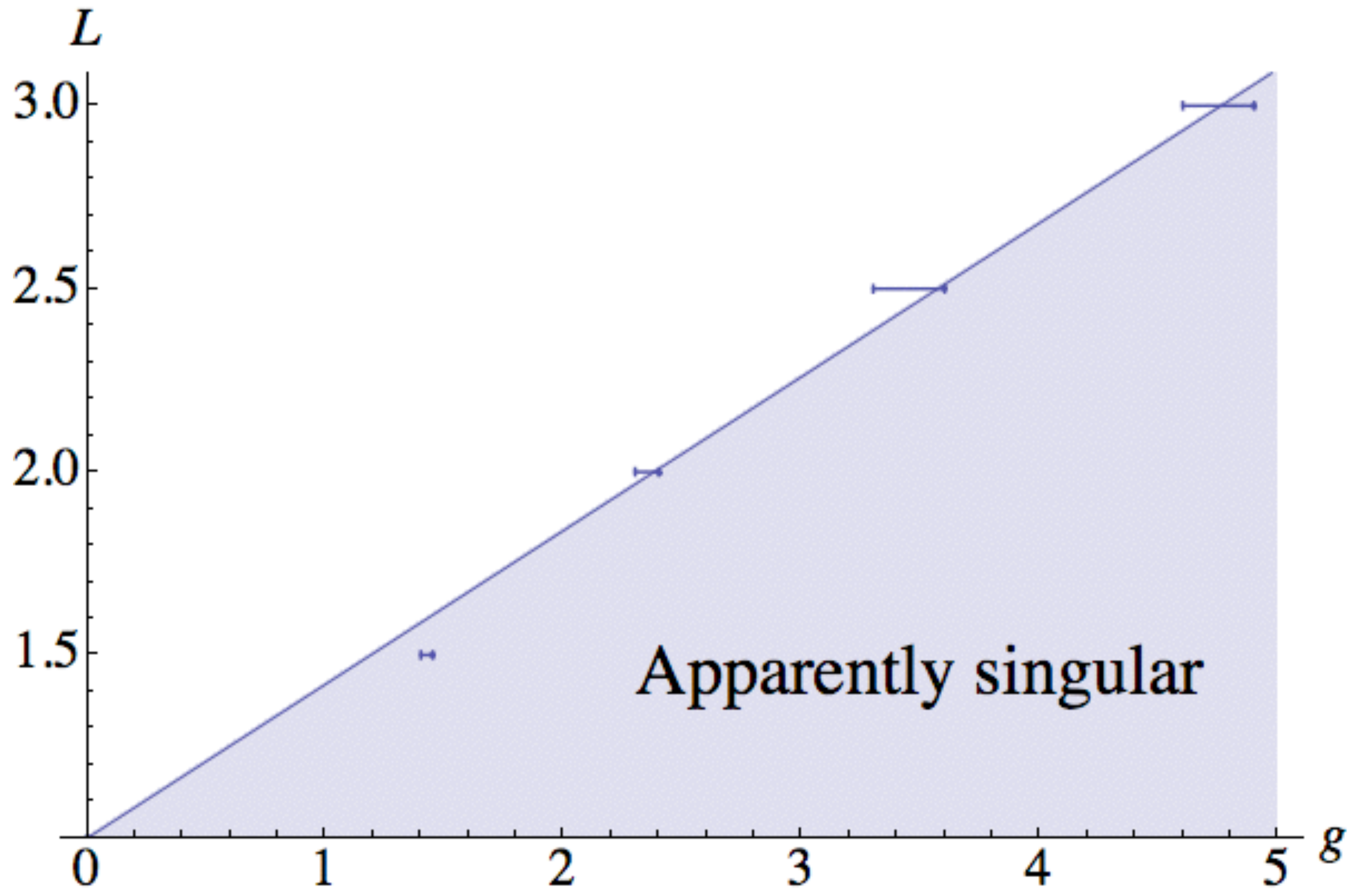
$$E^{(\mathrm{num})}_{\mathrm{BTBA}}(L,g) = -\sum_{Q=1}^{Q_{\max}} \int_{-\infty}^{\infty} \frac{d\tilde{p}_Q}{2\pi} \log(1+Y_Q) - \sum_{Q=Q_{\max}+1}^{100} \int_{-\infty}^{\infty} \frac{d\tilde{p}_Q}{2\pi} \log(1+Y_Q^{\bullet})$$

Numerical Results



- Cannot go further just by a brute-force computation
- Not clear how to go beyond the critical coupling analytically

Phase diagram



under the assumption that the $L = 1$ energy diverges at $g = 0$

Physical interpretation?

✦ The breakdown may indicate open string tachyon at strong coupling via AdS/CFT

- In string theory, the classical energy of short string is zero, but the quantum zero-point energy can be complex
- No gauge-theory description of $O(N)$ operators for $g > g_{\mathrm{cr}}$

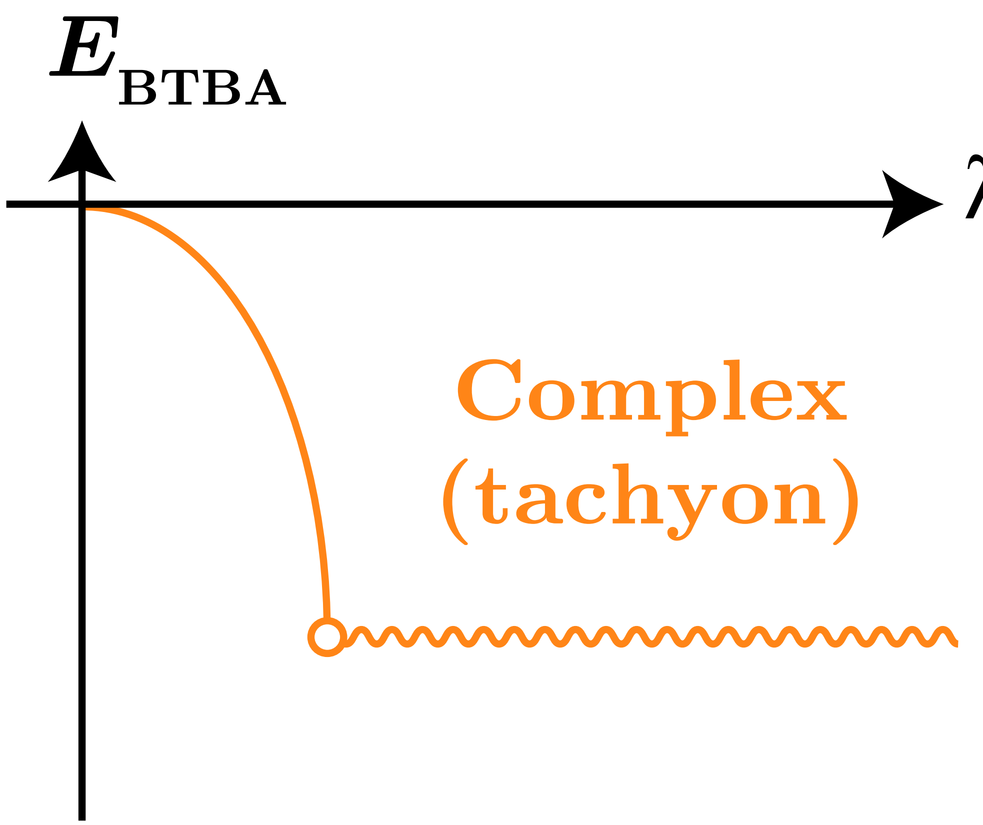
$$\Delta \sim 2N + 2E_{\mathrm{open}} \xrightarrow{g \to \infty} \text{complex}$$

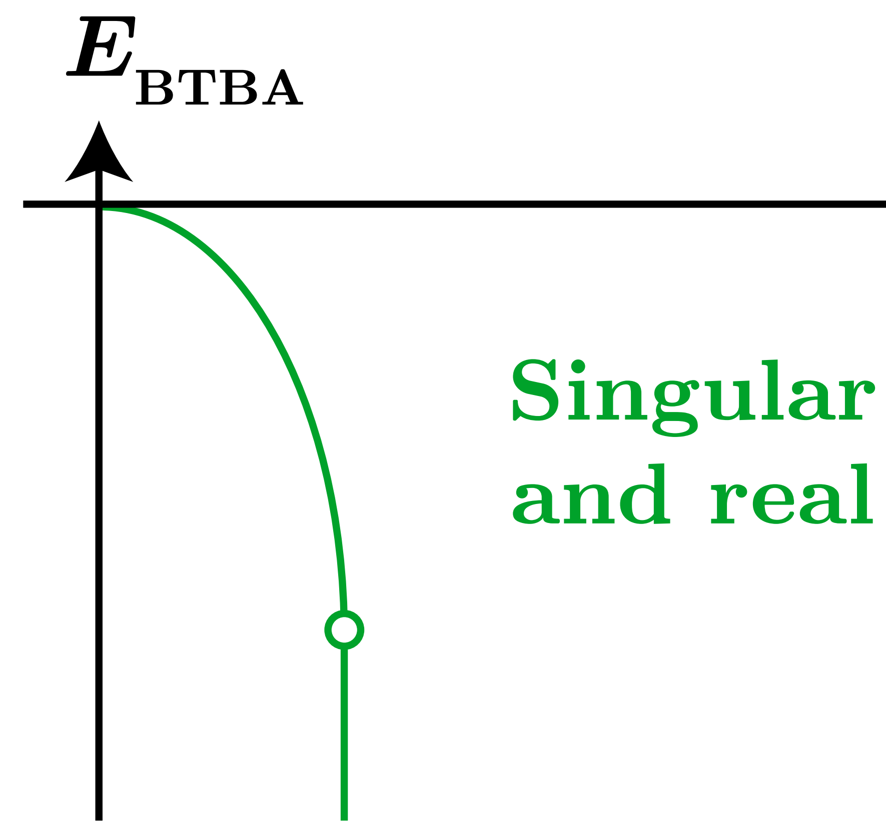
✦ Unitarity of $\mathcal{N}$=4 SYM requires Δ to be real at any g

- The energy of the string-brane system after tachyon condensation should be real
- Then, g_{cr} may be related to the radius of convergence in gauge theory

$$\Delta \sim 2N + 2E_{\mathrm{BTBA}} = \infty - \infty$$

Physical interpretation?





$\Delta \sim 2N + 2E_{\rm open} \xrightarrow{g\to\infty}$ **complex**

tachyonic open string

tachyon condensation

$$\Delta \sim 2N + 2E_{\rm BTBA} = \infty - \infty$$

$$\xrightarrow{g\to\infty} \mathcal{O}(N^0)?$$

Summary and outlook

Summary

- Studied the spectrum of determinant-like operators
  dual to open strings ending on giant gravitons
- Wrapping corrections from $\mathcal{N}$=4 SYM agree with the Lüscher formula
- Proposed and solved BTBA equations for Y=0 & Ybar=0
- Found the lower-bound for the (B)TBA energy

Future works

- Beyond the critical coupling? Compare with string theory?
- How to compute the dimension of the L=1 state?
- AdS/CFT for unstable systems?

Thank you for attention

Infinite-dimensional symmetry

The centrally-extended $\mathfrak{su}(2|2)$ determines
the asymptotic dispersion and S-matrix
of fundamental representations almost uniquely

$$\Delta - J = \sum_{j=1}^{N} \sqrt{1 + 4 f(g)^2 \sin^2 \frac{p_j}{2}}, \quad f(g) = g \equiv \frac{\sqrt{\lambda}}{2\pi} \text{ in } \mathcal{N} = 4\,\text{SYM}$$

$$A_a^\dagger(p_1) A_b^\dagger(p_2) = \mathbb{S}_{ab}^{cd}(p_1, p_2) A_c^\dagger(p_2) A_d^\dagger(p_1), \quad \mathbb{S} = S_0[\hat{S}_{\mathfrak{su}(2|2)} \otimes \hat{S}_{\mathfrak{su}(2|2)}]$$

The (fundamental) S-matrix of AdS/CFT satisfies Yang-Baxter relation

$$\mathbb{S}_{12}\,\mathbb{S}_{13}\,\mathbb{S}_{23} = \mathbb{S}_{23}\,\mathbb{S}_{13}\,\mathbb{S}_{12} \equiv \mathbb{S}_{123}$$

NB. boundstate S-matrices are obtained by fusion while *imposing* the YBR

[Beisert (2007)] and others

Infinite-dimensional symmetry

An N-particle state and its dimension/energy is

$$|p_1, \dots p_N\rangle = A_1^\dagger(p_1) \dots A_N^\dagger(p_N)|0\rangle, \quad \Delta - J = \sum_{j=1}^{N} \sqrt{1 + 4g^2 \sin^2 \boxed{\frac{p_j}{2}}}$$

?

The creation-annihilation operators have a free-field-like representation (Zamolodchikov-Faddeev algebra)

$$A_1^\dagger A_2^\dagger = A_2^\dagger A_1^\dagger \mathbb{S}_{12}, \quad A_1 A_2 = \mathbb{S}_{12} A_2 A_1, \quad A_1 A_2^\dagger = A_2^\dagger A_1 \mathbb{S}_{12} + \delta_{12}$$

The centrally-extended su(2|2) extends further to the Hopf-algebra with a non-trivial co-product

$$\Delta \mathfrak{J}^A = \mathfrak{J}^A \otimes 1 + e^{ip[A]} \otimes \mathfrak{J}^A, \quad \mathfrak{J}^A \ : \ \mathfrak{su}(2|2) \text{ generators}$$

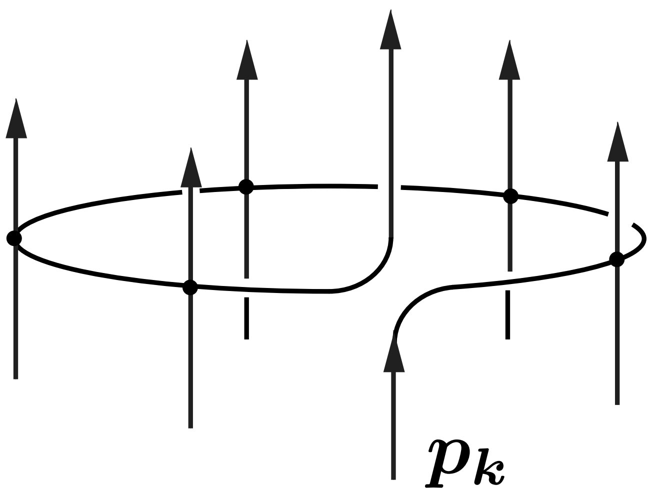
$$[\Delta \mathfrak{J}^A, \mathbb{S}] = 0$$

eventually to the Yangian of su(2|2)

[Beisert (2005)] and others

Bethe-Yang equation (BYE)

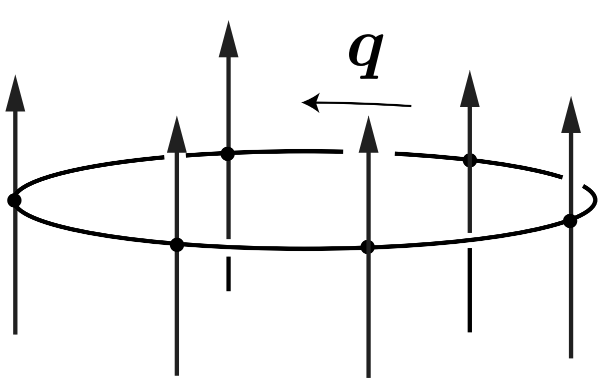
For a large and finite J, momenta of the particles are determined by the Bethe-Yang (or Bethe Ansatz) equation



$$-1 = e^{-iJp_k} \prod_{j=1}^{N} S(p_j, p_k)$$

$$S(p,p) = -1$$

BYE in terms of transfer matrix



$$T_a(q|\vec{p}) \equiv (\mathrm{s})\mathrm{tr}_{V_a}\left[\mathbb{S}_{a1}(q,p_1)\cdots\mathbb{S}_{aN}(q,p_N)\right]$$

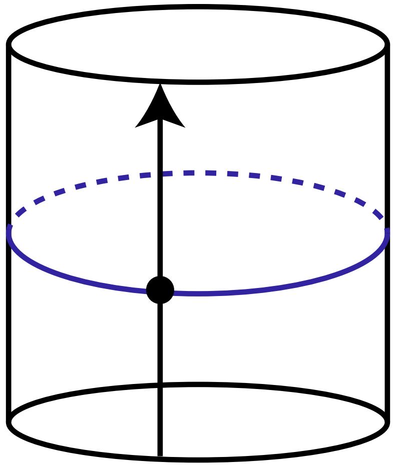
Yang-Baxter relation for integrable S-matrices $\Rightarrow \quad [T_a(q_a|\vec{p}), T_b(q_b|\vec{p})] = 0$

BYE $\quad \Leftrightarrow \quad -1 = e^{-iJq}\, T(q|\vec{p})\Big|_{q=p_k}$

Wrapping corrections

- The dimension Δ of SYM operator with a finite R-charge J receives exponentially small "wrapping" corrections
- The leading wrapping correction is related to the transfer matrix via the Lüscher formula

$$\Delta_{\text{Lüscher}} \sim \sum_{Q} \int_{-\infty}^{\infty} d\widetilde{p}_Q \, e^{-\widetilde{\mathcal{E}}_Q(\widetilde{p}_Q) J}$$



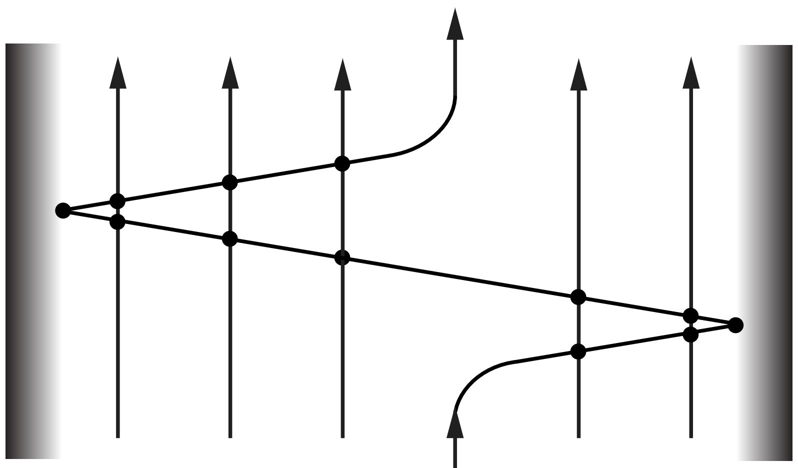
$$(\mathcal{E}_Q, p_Q) = (-i\widetilde{p}_Q, -i\widetilde{\mathcal{E}}_Q), \quad \widetilde{\mathcal{E}}_Q = 2\text{arcsinh}\left(\sqrt{Q^2 + \widetilde{p}_Q^2}/(2g)\right)$$

$$\Delta_{\text{Lüscher}} = -\sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{d\widetilde{p}_Q}{2\pi} Y_Q^{\bullet}(\widetilde{p}_Q), \quad Y_Q^{\bullet}(\widetilde{p}_Q) = e^{-\widetilde{\mathcal{E}}_Q J} \underline{T_Q(\widetilde{p}_Q|\vec{p})}$$

[Lüscher (1986)] [Janik Łukowski (2007)] and others

Boundary Bethe-Yang equation

Integrable open spin chains obey boundary BYE

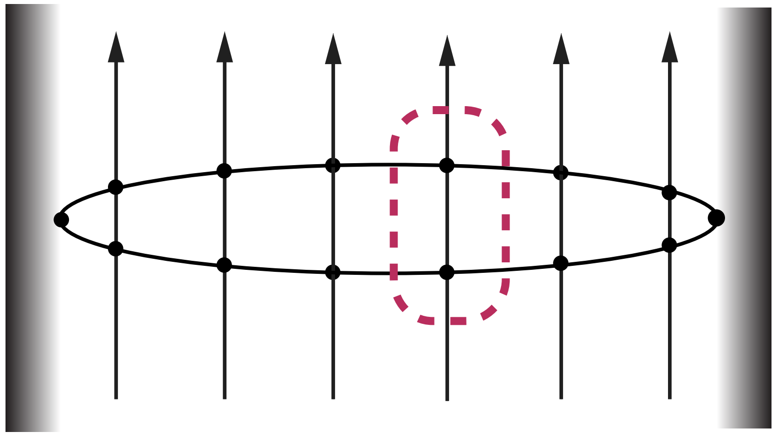


$$1 = e^{-i2Jp_K} \prod_{j \neq k}^{N} S(p_k, p_j) R^{-}(p_k) \times \prod_{j \neq k}^{N} S(p_j, -p_k) R^{+}(-p_k)$$

BBYE from double-row transfer matrix

$$D_a = \mathrm{tr}_a \Big[\mathbb{S}_{aN} \cdots \mathbb{S}_{a1} \, \mathbb{R}^{-} \mathbb{S}_{1a} \cdots \mathbb{S}_{Na} \, \tilde{\mathbb{R}}^{+} \Big]$$

$\mathbb{R}^{\pm}$: **reflection matrix**



Boundary Yang-Baxter for $\mathbb{R}^{\pm} \quad \Rightarrow \quad [D_a, D_b] = 0$

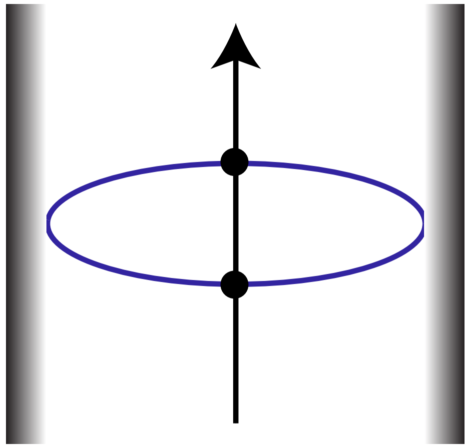
BBYE $\quad \Leftrightarrow \quad -1 = e^{-2iqJ} \, D_a(q|\vec{p}) \Big|_{q=p_k}$

[Sklyanin (1988)]

Boundary wrapping corrections

- Boundary Lüscher formula has been conjectured and tested

$$\Delta_{\text{Lüscher}} \sim \sum_{Q} \int_0^\infty d\widetilde{p}_Q \, e^{-\widetilde{\mathcal{E}}_Q(\widetilde{p}_Q)2J}$$



- In terms of the double-row transfer matrix

$$\Delta_{\text{Lüscher}} = -\sum_{Q=1}^{\infty} \int_0^\infty \frac{d\widetilde{p}_Q}{2\pi} Y_Q^\bullet, \quad Y_Q^\bullet = e^{-\widetilde{\mathcal{E}}_Q 2J} D_Q$$

Agree with $\mathcal{N}$=4 SYM perturbation at weak coupling for simple states

[Correa, Young (2009)] [Bajnok, Palla (2010)]

Error bars

We put Qmax=6 to draw the solid line

$$E^{(\mathrm{num})}_{\mathrm{BTBA}}(J,g) = -\sum_{Q=1}^{Q_{\max}} \int_0^\infty \frac{d\tilde{p}_Q}{2\pi} \log(1+Y_Q) - \sum_{Q=Q_{\max}+1}^{100} \int_0^\infty \frac{d\tilde{p}_Q}{2\pi} \log(1+Y_Q^{\bullet})$$

The error from the truncation of YQ is huge around the critical value

$$E_{\mathrm{BTBA}} = \sum_Q \mathrm{E}(Q), \quad \mathrm{E}(Q) = -\int \frac{d\tilde{p}_Q}{2\pi} \log(1+Y_Q) \sim Q^{-4J-4E_{\mathrm{BTBA}}}$$

We extrapolate the BTBA energy from Qmax=6 to Qmax=100

using the large Q asymptotics of $\mathrm{E}(\mathrm{Q})$

$$\tilde{E}_{\mathrm{BTBA}} = \sum_{Q=1}^{6} \mathrm{E}^{(\mathrm{original})}(Q) + \sum_{Q=7}^{100} \mathrm{E}^{(\mathrm{fit})}(Q) \ \left(< E^{(\mathrm{num})}_{\mathrm{BTBA}}\right)$$

Estimate of truncation error: $\delta E_{\mathrm{BTBA}} \equiv E^{(\mathrm{num})}_{\mathrm{BTBA}} - \tilde{E}_{\mathrm{BTBA}}$