Does Integrability Solve String Theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$?

Ryo Suzuki

(Trinity College Dublin)

Based on 0902.3368

March 24, 2009, at Tokyo University (Hongo)

Introduction

Branes and SCFT

Rich physics has been discovered in

String/M Theory $\leftrightarrow$ Superconformal Field Theories

Noteworthy examples are:

- ✔ Effective Theory on Dp-branes in flat space

  e.g. $\boldsymbol{p = 3} \quad \rightarrow \quad \boldsymbol{\mathcal{N} = 4,\ D = 4}$ super Yang-Mills

- ✔ Effective Theory on Membranes in flat space?

Coincident $\boldsymbol{N}$ M2-branes probing $\mathbb{C}^{4}/\mathbb{Z}_{\boldsymbol{k}}$ singularity $\leftrightarrow$

$\boldsymbol{\mathcal{N} = 6,\ D = 3}$ Chern-Simons-Matter theory

with $\boldsymbol{SU(N)_k \times SU(N)_{-k}}$ symmetry

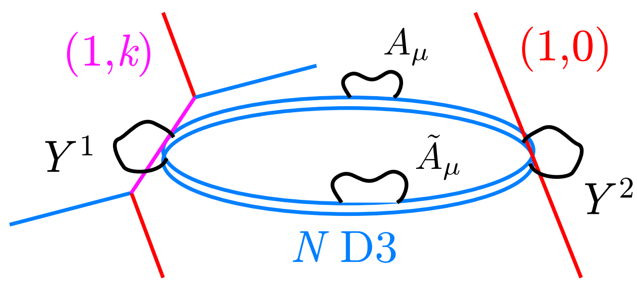
[Aharony Bergmann Jafferis Maldacena]

The ABJM Model

This is an $\mathcal{N} = 6$ Chern-Simons theory coupled to matter,
Or Extension of the BLG model with complex f^{abcd}

[Bagger Lambert, Gustavsson]

- ✔ Contents: Bifundamental scalars Y^A, $Y_A^\dagger$, Chern-Simons fields A_μ, $\tilde{A}_\mu$ (not dynamical), and spinors ψ_A, ψ_A^c $(A = 1, 2, 3, 4)$
- ✔ Coupling constant $\lambda \equiv N/k$ can be small $\lambda \ll 1$
- ✔ The $SU(2)_k \times SU(2)_{-k}$ theory is equivalent to the BLG model



Type IIB brane setup [Hanany Witten (1996), Kitao Ohta Ohta (1998)]

Gravity Duals

The ABJM model has two dual gravity backgrounds:

- ✔ M Theory on $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$ for $k \ll N^{1/5}$
- ✔ IIA Superstring on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ for $k \gg N^{1/5}$

1. Take the near-horizon limit of the blackbrane solution,

$$ds^2 = f^{-3/2} ds^2_{\mathbb{R}^{1,2}} + f^{1/3}\left(dr^2 + r^2 d\Omega_{\mathrm{S}^7}\right), \quad f = 1 + \frac{32\pi^2 N' \ell_p^6}{r^6}$$

$$\to (R^2/4) ds^2_{\mathrm{AdS}_4} + R^2 ds^2_{\mathrm{S}^7}, \quad R = \left(32\pi^2 N'\right)^{1/6} \ell_p$$

2. Take the $\mathbb{Z}_k$ quotient, $z_i \sim e^{2\pi i/k} z_i\,, \quad \int_{\mathrm{S}^7} *F_4 = kN$

3. Rewrite $\mathrm{S}^7/\mathbb{Z}_k$ as $\mathrm{S}^1/\mathbb{Z}_k$ fibration of $\mathbb{CP}^3$

$$ds^2_{\mathrm{S}^7/\mathbb{Z}_k} = 1/k^2 \left(d\varphi + k\omega\right)^2 + ds^2_{\mathbb{C}P^3}\,, \quad \varphi \sim \varphi + 2\pi$$

$\Rightarrow$ The 11th dimension is characterized by $R/(k\ell_p) \propto (kN)^{1/6}/k$

Gravity Duals

The ABJM model has two dual gravity backgrounds:

- ✔ M Theory on $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$ for $k \ll N^{1/5}$
- ✔ IIA Superstring on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ for $k \gg N^{1/5}$

Type IIA background:

$$ds^2 = R_s^2 \left(\frac{1}{4} ds^2_{\mathrm{AdS}_4} + ds^2_{\mathbb{CP}^3} \right), \quad R_s \equiv \frac{R}{k^{1/3}} = 4\pi \sqrt{\frac{\lambda}{2}},$$

$$\text{where} \quad \lambda \equiv \frac{N}{k} = \left(g_s N^2 \right)^{2/5},$$

with RR fluxes $F_4 \propto R^3$ and $F_2 \propto k$

Supergravity approximation is valid for $1 \ll \lambda \ll N^{4/5}$

Integrability of $\mathbf{AdS_4/CFT_3}$

The AdS_4/CFT_3 possesses "integrability" as in AdS_5/CFT_4

- ✔ Diagonalization of Dilatation operator Δ

  $\langle \mathcal{O}(x)^\dagger \mathcal{O}(y) \rangle \sim |x-y|^{-2\Delta}, \quad (\Delta_0 + \delta\Delta)_A{}^B |\mathcal{O}_B\rangle = \lambda_A |\mathcal{O}_A\rangle$

  $\delta\Delta$ is the Hamiltonian of an integrable **alternating** spin chain,

$$\delta\Delta = \frac{\lambda^2}{2} \sum_{\ell=1}^{2L} \left(2 - 2P_{\ell,\ell+2} + P_{\ell,\ell+2}K_{\ell,\ell+1} + K_{\ell,\ell+1}P_{\ell,\ell+2}\right)$$

[Minahan Zarembo, Bak Rey]

- ✔ Classical integrability of IIA superstring

  Green-Schwarz action on $AdS_4 \times \mathbb{CP}^3$ [Arutyunov Frolov, Stefanski]

  $\Rightarrow$ $Osp(2,2|6)/[SO(3,1)\times U(3)]$ supercoset sigma model

  Classical spectrum classified by algebraic curves [Gromov Vieira]

Bethe Ansatz Equations

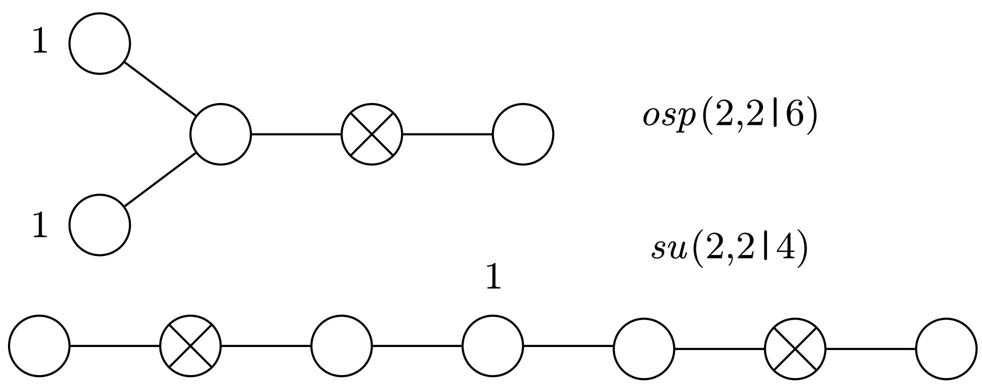
Multiparticle S-matrix factorizes in integrable theories

BAE are the periodicity conditions with factorized scatterings

$$e^{ip_j R} = \prod_{k\neq j} S(p_k, p_j)$$

Two-loop BAE of ABJM are similar to one-loop BAE of $\mathcal{N}=4$ SYM:

$$\left(\frac{u_{p,j}+\frac{i}{2}V_j}{u_{p,j}-\frac{i}{2}V_j}\right)^L = \prod_{(k,q)\neq(j,p)} \frac{u_{p,j}-u_{q,k}+\frac{i}{2}M_{jk}}{u_{p,j}-u_{q,k}-\frac{i}{2}M_{jk}}$$



M_{jk} is a super-Cartan matrix

V_j is the Dynkin label of the vector representation.

Bethe Ansatz Equations

Multiparticle S-matrix factorizes in integrable theories

BAE are the periodicity conditions with factorized scatterings

$$e^{ip_j R} = \prod_{k\neq j} S(p_k, p_j)$$

Two-loop BAE of ABJM are similar to one-loop BAE of $\mathcal{N}=4$ SYM:

$$\left(\frac{u_{p,j}+\frac{i}{2}V_j}{u_{p,j}-\frac{i}{2}V_j}\right)^L = \prod_{(k,q)\neq(j,p)} \frac{u_{p,j}-u_{q,k}+\frac{i}{2}M_{jk}}{u_{p,j}-u_{q,k}-\frac{i}{2}M_{jk}}$$

Gromov and Vieira proposed all-loop generalization of Bethe Ansatz Equations imitating BAE of AdS_5/CFT_4 [Beisert Staudacher (2005)]

- ✔ Consistent with classical string, when $\lambda \to \infty$
- ✔ Used the same dressing phase as in AdS_5/CFT_4

GV Conjecture

$$1 = \frac{u_1 - u_2 + \frac{i}{2}}{u_1 - u_2 - \frac{i}{2}} \frac{x_1 - \frac{1}{x_4^+}}{x_1 - \frac{1}{x_4^-}} \frac{x_1 - \frac{1}{x_{\bar{4}}^+}}{x_1 - \frac{1}{x_{\bar{4}}^-}}$$

$$1 = \frac{u_2 - u_2 + i}{u_2 - u_2 - i} \frac{u_2 - u_1 + \frac{i}{2}}{u_2 - u_1 - \frac{i}{2}} \frac{u_1 - u_3 + \frac{i}{2}}{u_1 - u_3 - \frac{i}{2}}$$

$$1 = \frac{u_3 - u_2 + \frac{i}{2}}{u_3 - u_2 - \frac{i}{2}} \frac{x_3 - x_4^+}{x_3 - x_4^-} \frac{x_3 - x_{\bar{4}}^+}{x_3 - x_{\bar{4}}^-}$$

$$\left(\frac{x_4^+}{x_4^-}\right)^L = \frac{u_4 - u_4 + i}{u_4 - u_4 - i} \frac{x_1 - \frac{1}{x_4^+}}{x_1 - \frac{1}{x_4^-}} \frac{x_4^- - x_3}{x_4^+ - x_3} \sigma_{4,4} \sigma_{4,\bar{4}}$$

$$\left(\frac{x_{\bar{4}}^+}{x_{\bar{4}}^-}\right)^L = \frac{u_{\bar{4}} - u_{\bar{4}} + i}{u_{\bar{4}} - u_{\bar{4}} - i} \frac{x_1 - \frac{1}{x_{\bar{4}}^+}}{x_1 - \frac{1}{x_{\bar{4}}^-}} \frac{x_{\bar{4}}^- - x_3}{x_{\bar{4}}^+ - x_3} \sigma_{\bar{4},\bar{4}} \sigma_{\bar{4},4}$$

Omitted the symbols $\prod_{j \neq k}$ and subscripts $j, k, \ldots$

GV Conjecture

Notation

$$x^{\pm}(p) = e^{\pm ip/2}\,\frac{1+\sqrt{1+16\,h(\lambda)^2\sin^2\frac{p}{2}}}{4\,h(\lambda)\sin\frac{p}{2}}$$

$$u(p) = \frac{1}{2}\cot\frac{p}{2}\sqrt{1+16\,h(\lambda)^2\sin^2\frac{p}{2}}$$

The function $h(\lambda)$ is left undetermined

The energy (anomalous dimension) and higher conserved charges

$$E = h(\lambda)\,Q_2\,, \quad Q_n = \sum_{j=1}^{K_4} q_n(p_{4,j}) + \sum_{j=1}^{K_{\bar{4}}} q_n(p_{\bar{4},j})$$

$$q_n(p) = \frac{i}{n-1}\left\{\frac{1}{(x^+)^{n-1}} - \frac{1}{(x^-)^{n-1}}\right\}$$

Plan of Talk

Asymptotic Bethe Ansatz Equations should correctly compute
the Δ of long operators and the E of spinning strings

Does it really work? Why?

Plan of Talk

Asymptotic Bethe Ansatz Equations should correctly compute
the Δ of long operators and the E of spinning strings

Does it really work? Why?

- ✔ Is quantum string on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ integrable?
- ✔ Is GV's proposal correct?
- ✔ Are there any rules to derive asymptotic BAE?

Plan of Talk

Asymptotic Bethe Ansatz Equations should correctly compute
the Δ of long operators and the E of spinning strings

Does it really work? Why?

- ✔ Is quantum string on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ integrable?
- ✔ Is GV's proposal correct?
- ✔ Are there any rules to derive asymptotic BAE?

It reminds me of the sudden fall of stock prices... (N.B.)

Plan of Talk

Asymptotic Bethe Ansatz Equations should correctly compute
the Δ of long operators and the E of spinning strings

Does it really work? Why?

- ✔ Is quantum string on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ integrable?
- ✔ Is GV's proposal correct?
- ✔ Are there any rules to derive asymptotic BAE?

It reminds me of the sudden fall of stock prices... (N.B.)

The magic of $su(2|2)$ symmetry is surely one of the reasons. Still we need more data to believe them. (M.S.)

Plan of Talk

- ✔ Introduction
- ✔ Review of Classical Integrability
- ✔ Agreements and Disagreements
- ✔ Infinite-size spectrum
- ✔ Summary and Outlook

Review of Classical Integrability

Green-Schwarz action

- ✔ The $\mathrm{AdS}_4 \times \mathbb{CP}^3$ background preserves 24 supersymmetry
- ✔ **Partially** gauge-fixed Green-Schwarz action $(24 = 32 - 8)$ becomes sigma model on supercoset

$$Osp(2,2|6)/[SO(1,3) \times U(3)]$$

The action

$$S = -2h \int d^2\sigma \ \mathrm{str} \left[\gamma^{ab} A_a^{(2)} A_b^{(2)} \pm \epsilon^{ab} A_a^{(1)} A_b^{(3)} \right]$$

where γ^{ab} is worldsheet metric, and

$$A \equiv -gdg = \sum_{j=0}^{3} A^{(j)}, \quad \Omega \cdot A^{(k)} = i^k A^{(k)}$$

[Arutyunov Frolov, Stefanski]

Algebraic curve for $\mathrm{AdS}_4 \times \mathbb{CP}^3$

Integrability of supercoset $\Rightarrow$ Classical string as an algebraic curve

Parametrize the bosonic coset

$$h = \begin{pmatrix} 1 - 2yy^\dagger & 0 \\ 0 & 1 - 2zz^\dagger \end{pmatrix}, \quad (y, z) \in \left(\frac{SO(2,3)}{SO(1,3)}, \frac{SU(4)}{U(3)} \right)$$

Bosonic part of Green-Schwarz action,

$$\mathcal{L} = -1/2 \, \mathrm{tr} \, (j_{\mathrm{AdS}})^2 + 2 \, \mathrm{tr} \, (j_{\mathbb{CP}})^2 \,, \quad j \equiv h^{-1} dh$$

From Bianchi identity $d * j = 0$ and e.o.m. $dj + j \wedge j = 0$,

$$dJ + J \wedge J = 0, \quad J = \frac{j + x * j}{1 - x^2}, \quad x \in \mathbb{C}$$

Algebraic curve for $AdS_4 \times \mathbb{CP}^3$

Eigenvalues of monodromy matrix are independent of w.s. coordinates,

$$\Omega(x) = \overline{P} \exp\left(\oint d\sigma J_\sigma \right)$$

$$\simeq \text{diag}\left(e^{i\hat{p}_1}, e^{-i\hat{p}_1}, e^{i\hat{p}_2}, e^{-i\hat{p}_2}, 1 \,;\, e^{i\tilde{p}_1}, e^{i\tilde{p}_2}, e^{i\tilde{p}_3}, e^{i\tilde{p}_4} \right)$$

Constraints on its eigenvalues (quasimomenta) $\{p(x)\}$

- ✔ $\sum_{i=1}^{4} \tilde{p}_i = 0$ from unitarity
- ✔ The identity $h = h^{-1}$ relates $p(x)$ with $p(1/x)$
- ✔ Virasoro constraints fix $p(x = \pm 1)$
- ✔ Global charges (E, S, J_1, J_2, J_3) fix $p(x \to \infty)$

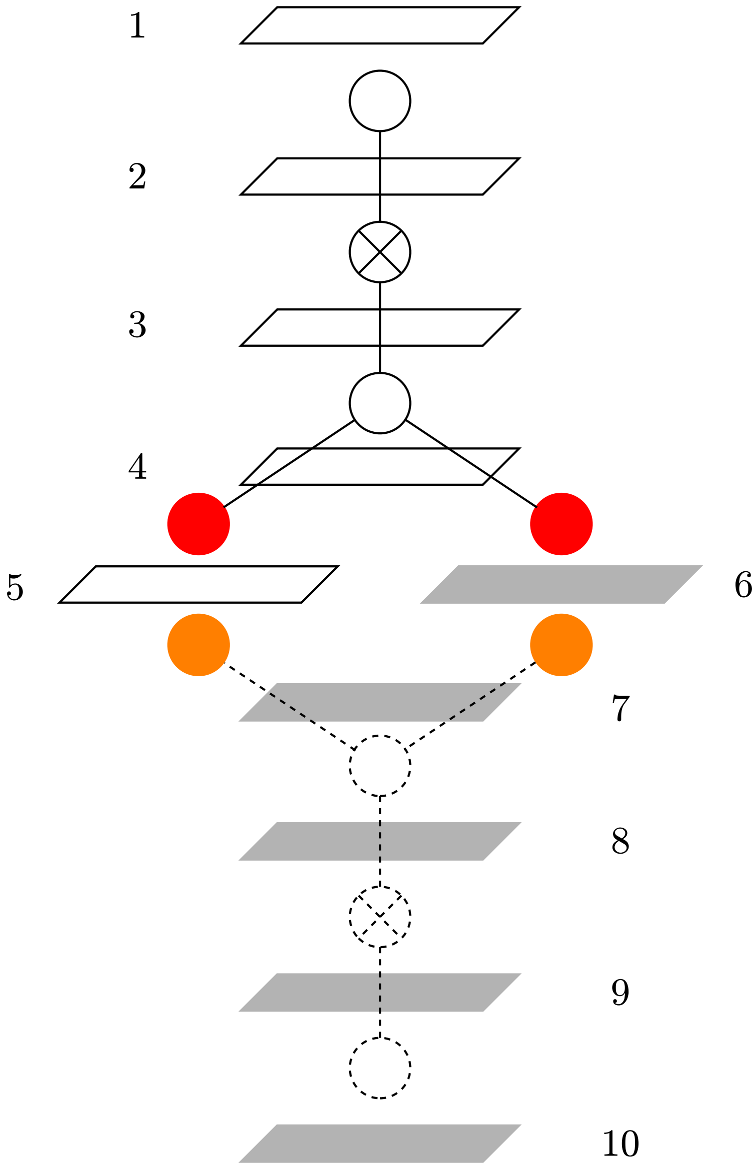
  i.e. Can compute charges from a consistent set of $\{p(x)\}$

Relation to Bethe Ansatz

As the vector representation of $Osp(4|6)$,

$$
\begin{aligned}
&(q_1\,,q_2\,,q_3\,,q_4\,,q_5)\\
&=-\,(q_6\,,q_7\,,q_8\,,q_9\,,q_{10})\\
&=\left(\frac{\hat{p}_1+\hat{p}_2}{2},\frac{\hat{p}_1-\hat{p}_2}{2},\,\tilde{p}_1+\tilde{p}_2\,,\,\tilde{p}_1+\tilde{p}_3\,,\,\tilde{p}_1+\tilde{p}_4\right)
\end{aligned}
$$

Relation to Bethe Ansatz



String modes
$\leftrightarrow$ how to connect two of ten sheets

Light Bosonic $\mathbb{CP}^3$:	$(3,5)$ $(3,6)$ $(4,5)$ $(4,6)$,
Light Fermonic :	$(1,5)$ $(1,6)$ $(2,5)$ $(2,6)$,
Heavy Bosonic $\mathbb{CP}^3$:	$(3,7)$,
Heavy Bosonic AdS_4 :	$(1,9)$ $(2,9)$ $(1,10)$,
Heavy Fermonic :	$(1,7)$ $(1,8)$ $(2,7)$ $(2,8)$.

Interpret the mode $\boldsymbol{q_i} - \boldsymbol{q_j}$ as the excitation of Bethe roots in between

Then, by construction, the scaling limit of asymptotic BAE
$=$ Integral equations for quasimomenta

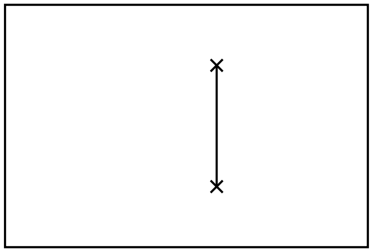
Semiclassical quantization

Semiclassical quantization in the sigma model
= Sum over *quadratic* fluctuation around classical background

Can we do the same by algebraic curve?

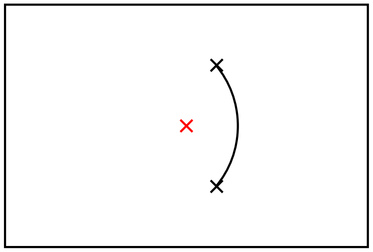
1. Regard quantum fluctuation as

   adding poles to the background curve



2. Find consistent $\{q_i(x) + \delta q_i(x)\}$,

   including backreaction



3. Quantize the mode number

$$q_i(x_n^{(ij)}) - q_j(x_n^{(ij)}) = 2\pi n, \quad n \in \mathbb{Z}$$

$\left(x_n^{(ij)} \in \text{“tiny cut” } \textit{i.e.} \text{ the pole}\right)$

4. Sum over the mode number n,

   and over the sixteen polarizations (i, j)

[Gromov Vieira]

Agreements and Disagreements

Summary Table

Whether worldsheet calculation agrees with BAE:

	One-loop spectrum
Plane wave	✓ ($i.e.$ $1/J$ corrections agree)
Folded/circular string	×
Giant magnon	△

[Gaiotto Giombi Yin, Nishioka Takayanagi, Grignani Harnark Orselli]

[McLoughlin Roiban, Alday Arutyunov Bykov, Krishnan, Gromov Mikhaylov, McLoughlin Roiban Tseytlin]

[Gromov Vieira, Shenderovich]

One-loop correction to the energy of AdS_3 folded string:

$$E - S \simeq \left(2h - \frac{5}{2\pi}\ln 2\right)\ln S, \quad 2h \simeq \sqrt{2\lambda} \qquad \text{for } AdS_4 \times \mathbb{CP}^3$$

$$E - S \simeq \left(4g - \frac{3}{2\pi}\ln 2\right)\ln S, \quad 4g = \frac{\sqrt{\lambda}}{\pi} \qquad \text{for } AdS_5 \times S^5$$

Does not agree with GV's conjecture if $h(\lambda) = \sqrt{\lambda/2} + \mathcal{O}(1/\sqrt{\lambda})$

Plane-wave spectrum

The geodesic of $\mathrm{AdS}_4 \times \mathbb{CP}^3$

$$[z_1 : z_2 : z_3 : z_4] = \left[e^{it/2} : 0 : 0 : e^{-it/2}\right], \quad E - \frac{J_1 - J_4}{2} = 0$$

Corresponds to the BPS state of ABJM model

$$\mathcal{O} = \mathrm{tr}\left[Y^1 Y_4^\dagger Y^1 Y_4^\dagger \cdots\right]$$

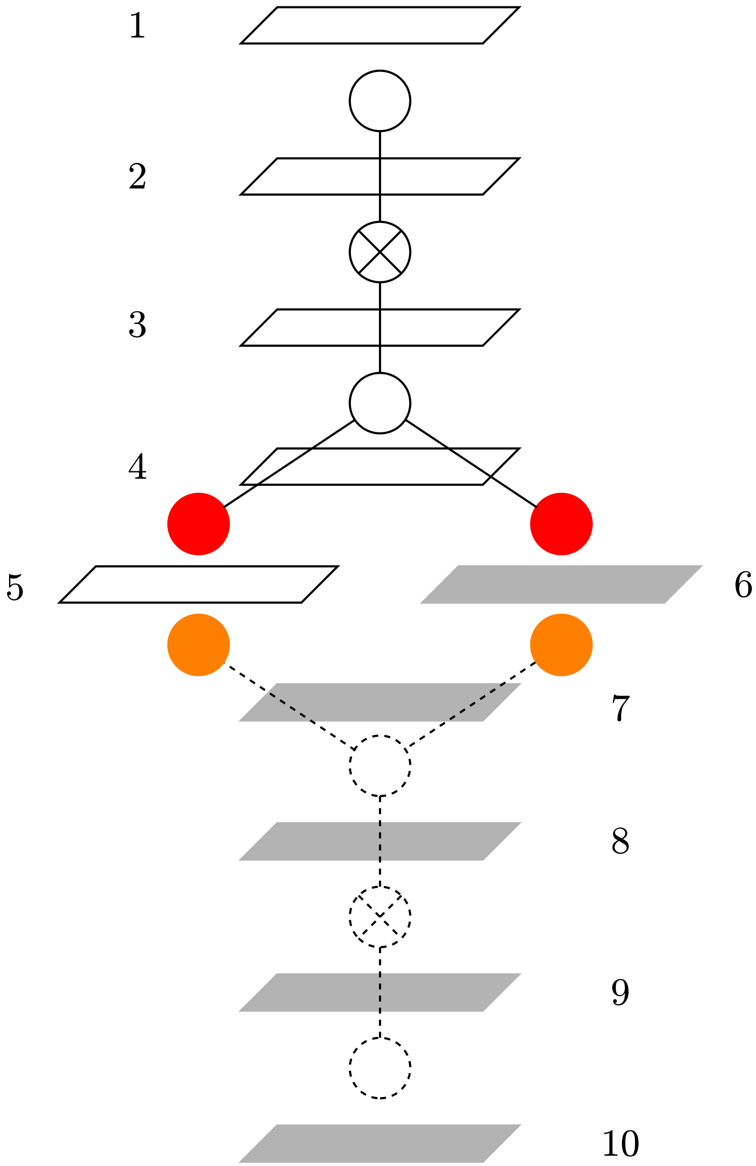
String plane-wave spectrum:

$$\epsilon_{\text{light}} = \sqrt{\tfrac{1}{4} + p^2} \quad 2 + 2\ \mathbb{CP}^3 \text{ modes}$$

$$\epsilon_{\text{heavy}} = \sqrt{1 + p^2} \quad \text{AdS triplet, } \mathbb{CP}^3 \text{ singlet}$$

with $p = n/P_+$

Plane-wave spectrum



Algebraic-curve description of plane-wave spectrum:

Light Bosonic $\mathbb{CP}^3$:	$(3,5)\ (3,6)\ (4,5)\ (4,6)$,
Light Fermonic :	$(1,5)\ (1,6)\ (2,5)\ (2,6)$,
Heavy Bosonic $\mathbb{CP}^3$:	$(3,7)$,
Heavy Bosonic AdS_4 :	$(1,9)\ (2,9)\ (1,10)$,
Heavy Fermonic :	$(1,7)\ (1,8)\ (2,7)\ (2,8)$.

Plane-wave from Spin Chain

Light modes correspond to the BMN-like operators,

$$\mathcal{O}_n = \frac{1}{\sqrt{2J}} \sum_{\ell=0}^{J-1} e^{2\pi i \ell n/J} \mathrm{tr}\left[(Y^1 Y_4^\dagger)^\ell X (Y^1 Y_4^\dagger)^{J-\ell} X\right]$$

where $X = Y^2 Y_4^\dagger$ or $X = Y^1 Y_3^\dagger$

Their anomalous dimensions consistent with GV's conjecture

$$\Delta - J = \lambda^2 \left(\frac{2\pi n}{J}\right)^2 \quad \leftarrow \sum_{i=1}^{2J} \left\{ \sqrt{\frac{1}{4} + 4h^2 \sin^2 \frac{p_i}{2}} - \frac{1}{2} \right\}$$

$$\text{if} \quad p_i \sim 2\pi n/J \,, \quad h \sim \lambda + \mathcal{O}(\lambda^2)$$

Heavy modes are not bound states of two light magnons,
But a composite of the two, having the same momentum

[Bombardelli Fioravanti, Łukowski Ohlsson-Sax, Zarembo]

Folded/Circular string

- ✔ BAE and the one-loop energy of folded string disagrees
- ✔ Try to rescue by modifying $h(\lambda)$

$$\bar{h}(\lambda) = \sqrt{\lambda/2} - \ln 2/(2\pi) + \mathcal{O}(1/\sqrt{\lambda})$$

- ✔ Still a mismatch in the one-loop energy of circular string

[McLoughlin Roiban Tseytlin]

Classical energy (BMN-like expansion)

$$E_0 = S + J + \frac{\lambda}{2J} k^2 u(1+u) + \mathcal{O}\left(\frac{\lambda^2}{J^3}\right),$$

$$u = \frac{S}{J}, \quad k = \text{winding}$$

Folded/Circular string

One-loop correction

$$E_1 = 1/(2\kappa) \sum_{n=-\infty}^{\infty} e(n), \quad (t = \kappa\tau)$$

Split the region of summation into two, small n and large n

$$E_1 = \frac{1}{\kappa} \sum_{n=1}^{\infty} e^{\text{sum}}(n) + \frac{\omega}{2\kappa} \int_{-\infty}^{\infty} dx\, e^{\text{int}}(x)$$

where $x = n/\omega,\ J = \pi\omega\sqrt{2\lambda}$

1. Integration part (odd):

$$E_1^{\text{odd}} = -\frac{\sqrt{\lambda}\, k^2 u(1+u)}{J} \ln 2 + \mathcal{O}\left(\frac{\lambda^{3/2}}{J^3}\right)$$

2. Summation part (even):

$$E_1^{\text{even}} = \frac{\lambda\, k^4 u^2 (1+u)^2}{8\, J^2} \left(6\zeta(2) - 15k^2 u(1+u)\zeta(4) + \cdots\right) + \mathcal{O}\left(\frac{\lambda^2}{J^4}\right)$$

Folded/Circular string

The Mismatch

✔ Integration (odd) part is same as AdS_5/CFT_4

$$\left\{E_0 + E_1^{\rm odd}\right\}_{\rm AdS_4\times\mathbb{CP}^3}\,(S,J,k,\sqrt{\lambda}) = \frac{1}{2}\left\{E_0 + E_1^{\rm odd}\right\}_{\rm AdS_5\times S^5}\,(2S,2J,k,2\bar{h}(\lambda))$$

where $$\bar{h}(\lambda) = \sqrt{\lambda/2} - \ln 2/(2\pi) + \mathcal{O}(1/\sqrt{\lambda})$$

✔ Summation (even) part different. They agree if we replace

$$\sum_{n=1}^{\infty} e^{\rm sum}(n) \quad \longrightarrow \quad 2\sum_{n=1}^{\infty} e^{\rm sum}(2n+1)$$

Such regularization looks unnatural in view of worldsheet

Giant magnons

✔ Giant magnons are solutions of the decompactified worldsheet

$$\sigma \sim \sigma + 2\pi\mu, \quad \mu \to \infty$$

✔ There are two types of giant magnons in $AdS_4 \times \mathbb{CP}^3$

$$\epsilon_{\mathbb{CP}^1} = \sqrt{\frac{1}{4} + 4h^2 \sin^2 \frac{\Delta\phi}{2}}, \quad \epsilon_{\mathbb{RP}^2} = \sqrt{1 + 16h^2 \sin^2 \frac{\Delta\phi}{4}}$$

under the b.c. $\left(\frac{z_1}{z_4}, \frac{z_2}{z_4}, \frac{z_3}{z_4}\right) \to \left(e^{it \pm i\Delta\phi/2}, 0, 0\right)$ as $\sigma \to \pm\infty$

✔ Limit of the decompactified spectrum $\Rightarrow$ Plane-wave spectrum

Take $h \to \infty$ **with** $p = h\Delta\phi$ **fixed**

$$\epsilon_{\mathbb{CP}^1} \to \epsilon_{\text{light}} = \sqrt{\frac{1}{4} + p^2}, \quad \epsilon_{\mathbb{RP}^2} \to \epsilon_{\text{heavy}} = \sqrt{1 + p^2}$$

Giant magnons

Quantization by algebraic curve favors another ‘unnatural’ regularization

$$x_{2n}^{\text{heavy}} = x_n^{\text{light}} \quad \leftrightarrow \quad \omega_{2n}^{\text{heavy}} = 2\omega_n^{\text{light}}$$

One-loop corrections to the energy of both GMs vanish, *e.g.*

$$\left(E_{\mathbb{CP}^1} - \frac{J_1 - J_4}{2}\right) = \sqrt{2\lambda}\sin\frac{p}{2} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)$$

Consistent with

$$\epsilon_{\mathbb{CP}^1} = \sqrt{\frac{1}{4} + 4h^2\sin^2\frac{p}{2}}, \quad h(\lambda) = \sqrt{\lambda/2} + 0 + \mathcal{O}\left(1/\sqrt{\lambda}\right)$$

(Note: the BPS relation does not fix $h(\lambda)$) [Shenderovich, Gromov Mikhaylov]

Infinite-size spectrum

Motivation

Classify giant magnons in the $AdS_4 \times \mathbb{CP}^3$ string

$\leftrightarrow$ BPS (and possibly non-BPS) states of infinite-size spin chain

✔ Elementary excitations $\leftrightarrow$ $\mathbb{CP}^1$ GM and $\mathbb{RP}^2$ GM

$$Y^1 Y_4^\dagger \to X \quad \text{and} \quad Y^1 Y_4^\dagger \to Y^2 Y_3^\dagger, \quad (X = Y^2 Y_4^\dagger \text{ or } Y^1 Y_3^\dagger)$$

✔ Boundstates of $\mathbb{CP}^1$ and $\mathbb{RP}^2$ (become $\mathbb{CP}^2$ and $\mathbb{RP}^3$)

$$(Y^1 Y_4^\dagger)^Q \to X^Q \quad \text{and} \quad (Y^1 Y_4^\dagger)^Q \to (Y^2 Y_3^\dagger)^Q$$

✔ Scattering states of the above

Motivation

Classify giant magnons in the $AdS_4 \times \mathbb{CP}^3$ string

$\leftrightarrow$ BPS (and possibly non-BPS) states of infinite-size spin chain

- ✔ Elementary excitations $\leftrightarrow$ $\mathbb{CP}^1$ GM and $\mathbb{RP}^2$ GM

$$Y^1 Y_4^\dagger \to X \quad \text{and} \quad Y^1 Y_4^\dagger \to Y^2 Y_3^\dagger, \quad (X = Y^2 Y_4^\dagger \text{ or } Y^1 Y_3^\dagger)$$

- ✔ Boundstates of $\mathbb{CP}^1$ and $\mathbb{RP}^2$ (become $\mathbb{CP}^2$ and $\mathbb{RP}^3$)

$$(Y^1 Y_4^\dagger)^Q \to X^Q \quad \text{and} \quad (Y^1 Y_4^\dagger)^Q \to (Y^2 Y_3^\dagger)^Q$$

- ✔ What sort of string solutions correspond to neutral modes?

$$(Y^1 Y_4^\dagger)^Q \to Z^Q, \quad Z = Y^1 Y_1^\dagger \text{ or } Y^4 Y_4^\dagger$$

Finding solitons

Techniques for solving 2d classically integrable field theories

Pohlmeyer's reduction on $\mathbb{CP}^3 \sim \mathbf{S}^7/U(1) \subset \mathbb{C}^4$

[Pohlmeyer, Lund Regge]

1. Choose $U(1)$-invariant orthonormal basis $\{\vec{v}^k\}$ of $\mathbb{C}^4$

2. Differentiate the basis vectors, expand them by basis vectors

$$\partial_a \vec{v}^k = M_a \cdot \vec{v}^k$$

3. Compatibility conditions give (generalized) sine-Gordon

$$(D_a D_b - D_b D_a)\, \vec{v}^k = -i F_{ab} \vec{v}^k$$

4. Solution of reduced system linearizes the string e.o.m.

$$D_a^2 z_i - [\cos(u)/4]\, z_i = 0, \quad \cos(u)/4 = -\left|D_a z_k\right|^2$$

However, the reduced sG system is too complicated for $\mathbb{R} \times \mathbb{CP}^2$

Finding solitons

Techniques for solving 2d classically integrable field theories

Dressing Method

[Zakhalov Mikhailov, Zakhalov Shabat, Harnad Saint-Aubin Shnider] [Sasaki, Antoine Piette, Piette]

Embed $\mathbb{CP}^3$ into the coset $SU(4)/U(3)$ (or $SO(6)/U(3)$)

Consider auxiliary linear problem $\quad$ ($\Rightarrow$ e.o.m. $dJ + J \wedge J = 0$)

$$\left\{\partial_\pm - \frac{\partial_\pm g \cdot g^{-1}}{1 \pm x}\right\}\psi(x) = 0, \quad g = \psi(x=0) \in \frac{SU(4)}{U(3)}$$

under the following constraints

Unitarity $\quad [\psi(\bar{x})]^\dagger \psi(x) = 1$

Inversion symmetry $\quad \psi(1/x) = g\theta\psi(x)\theta, \;\; \theta = \mathrm{diag}\;(1,-1,-1,-1)$

which follow from $g^\dagger g = 1$ and $g^{-1}\partial g = \theta\partial g\, g^{-1}\theta$

Finding solitons

Techniques for solving 2d classically integrable field theories

Dressing Method

[Zakhalov Mikhailov, Zakhalov Shabat, Harnad Saint-Aubin Shnider] [Sasaki, Antoine Piette, Piette]

Try to construct new solution from the known solution

$$\tilde{\psi}(x) = \chi(x)\,\psi(x)$$

Most (but not all) solutions follow from the following Ansatz

$$\chi(x) = 1 + \sum_{i=1}^{N} \frac{X_i F_i^\dagger}{x - x_i}\,, \quad X_i\,, F_i \;:\; \text{vectors}$$

The simplest dressing matrix for $SU(4)/U(3)$ has two poles,

at $x = x_1^+,\ 1/x_1^+$

Results

$$\tilde{z}_1 = \frac{e^{i\tau/2}}{2\sqrt{2}\,\Lambda_z}\left[\cos^2\rho_1\left(-\frac{e^{X\sin\alpha}x_1^+ + e^{-X\sin\alpha}x_1^-}{x_1^+ - x_1^-} + \frac{e^{-iT\cos\alpha}\left(e^{2iT\cos\alpha} + x_1^+ x_1^-\right)}{x_1^+ x_1^- - 1}\right) - \frac{2\sin^2\rho_1\left((x_1^+)^2 - 1\right)x_1^-}{(x_1^+ - x_1^-)(x_1^+ x_1^- - 1)}\right]$$

$$\tilde{z}_2 = -\frac{\sin 2\rho_1}{2\Lambda_z}\cosh\left(\frac{X\sin\alpha - iT\cos\alpha}{2}\right), \quad \tilde{z}_3 = 0$$

$$\tilde{z}_4 = \frac{e^{-i\tau/2}}{2\sqrt{2}\,\Lambda_z}\left[\cos^2\rho_1\left(-\frac{e^{-X\sin\alpha}x_1^+ + e^{X\sin\alpha}x_1^-}{x_1^+ - x_1^-} + \frac{e^{-iT\cos\alpha}\left(1 + e^{2iT\cos\alpha}x_1^+ x_1^-\right)}{x_1^+ x_1^- - 1}\right) - \frac{2\sin^2\rho_1\left((x_1^+)^2 - 1\right)x_1^-}{(x_1^+ - x_1^-)(x_1^+ x_1^- - 1)}\right]$$

$$\Lambda_z = \left\{x_1^+ x_1^-\left[\left(\frac{\sin^2\rho_1 - \cos^2\rho_1\cos(T\cos\alpha)}{x_1^- x_1^+ - 1}\right)^2 - \left(\frac{\sin^2\rho_1 + \cos^2\rho_1\cosh(X\sin\alpha)}{x_1^+ - x_1^-}\right)^2\right]\right\}^{1/2}$$

$(\boldsymbol{X}, \boldsymbol{T})$ are worldsheet coordinates $(\boldsymbol{\sigma}, \boldsymbol{\tau})$ after Lorentz boost. The velocity and $\boldsymbol{\alpha}$ are functions of $\boldsymbol{x}_1^{\pm}$

Results

All solutions obey the dispersion ($h = \sqrt{\lambda/2}$)

$$E - \frac{J_1 - J_4}{2} = \frac{h}{i}\left\{x_1^+ - \frac{1}{x_1^+} - x_1^- + \frac{1}{x_1^-}\right\}, \quad J_2 = J_3 = 0$$

$$= \sqrt{n^2 + 16h^2 \sin^2 \frac{p}{2}}, \quad x_1^\pm = e^{(\pm ip+q)/2}$$

which should correspond to neutral modes

Reduce to the $\mathbb{RP}^2$ giant magnons when $\rho_1 \to \pm\pi/4, \ \left|x_1^\pm\right| \to 1$

(Because classical string cannot distinguish $Y^1 Y_4^\dagger \to Y^2 Y_3^\dagger$ from $\to Y^1 Y_1^\dagger$)

Some questions remain unanswered:

- ✔ Are they BPS or non-BPS? What is $h(\lambda)$?
- ✔ How to construct $\mathbb{CP}^2$ DGMs? And their scattering states?

Summary and Outlook

Summary

We reviewed:

- ✔ Recent developments of $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence
- ✔ Proposal of Asymptotic Bethe Ansatz Equations
- ✔ Classical/One-loop tests of BAE from string worldsheet

On top of them, this work focused on also [Hollowood Miramontes, Kalousis Spradlin Volovich]

- ✔ Construction of soliton solutions

$$E - \frac{J_1 - J_4}{2} = \sqrt{n^2 + 16h^2 \sin^2 \frac{p}{2}}, \quad J_2 = J_3 = 0$$

Hoping to clarify: States of infinite-size spin chain $\leftrightarrow$
Spectrum of decompactified worldsheet

Outlook

Future directions

- ✔ Dyonic solutions still under way

  [Abbotto, Aniceto, Ohlsson Sax]

- ✔ Scattering phase agree with BAE?
- ✔ Dressing phase?
- ✔ Finite-size corrections?