Finite-size effects in $AdS_5 \times S^5$ superstring

Ryo Suzuki[†]

Based on papers 0801.0747, 0807.0643 with Yasuyuki Hatsuda[‡]

[†] Trinity College Dublin, [‡] University of Tokyo

November 10, 2008

Sigma model on $\mathrm{AdS}_5 \times \mathrm{S}^5$

The σ-model on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is interesting, because of

- The AdS/CFT correspondence: [Maldacena (1997)]

  Superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5 \quad \leftrightarrow \quad \mathcal{N} = 4\ SU(N)$ SYM

  string tension $\lambda = R^4/\alpha'^2 \quad \leftrightarrow \quad$ 't Hooft coupling $\lambda = N g_{\mathrm{YM}}^2$

  In the limit $N \to \infty$ with λ fixed.

  Large $N \Leftrightarrow$ String is free $g_s = g_{\mathrm{YM}}^2 = 0$,

- The (classical) integrability:

  The classical σ-model on

$$\mathrm{AdS}_5 \times \mathrm{S}^5 = \frac{PSU(2,2|4)}{SO(1,4) \times SO(5)} \ \text{supercoset}$$

  is integrable. [Bena, Polchinski, Roiban (2003)]

Decompactification limit

There are many classically integrable 2d field theories known.
Usually they are defined on a plane, rather than a cylinder.

Let us take the following decompactification limit:

- Rescale $\sigma \in [-\pi, \pi] \;\mapsto\; \tilde{\sigma} \in [-\infty, \infty]$
- Forget periodicity (level-matching) condition (tentatively)

[Hofman, Maldacena (2006)]

Under this limit,

- We can define **asymptotic states** for soliton-like solutions
- We can define their **scattering on worldsheet**

Sigma model in uniform light-cone gauge

Uniform light-cone gauge is useful for decompactification limit.

$$H_{\rm ws} = -P_- \equiv E - J_1, \quad p_{\rm ws} \equiv -\int_{-r}^{r} d\sigma\, \pi_i\, \partial_\sigma X^i, \quad r \equiv \frac{\pi P_+}{\sqrt{\lambda}}.$$

E and J_1 are conserved charges of $\mathrm{AdS}_5 \times \mathrm{S}^5$.

[Kruczenski, Ryzhov, Tseytlin (2004); Arutyunov, Frolov, Zamaklar (2006)]

In the limit $P_+ \to \infty$, we can find soliton-like solutions with

$$\epsilon_{\rm ws}(p_{\rm ws}) = \frac{\sqrt{\lambda}}{\pi} \left| \sin \frac{p_{\rm ws}}{2} \right| \qquad p_{\rm ws} \sim \mathcal{O}(1)$$

$$\epsilon_{\rm ws}(p_{\rm ws}) = \sqrt{1 + \left(\frac{\sqrt{\lambda}\, p_{\rm ws}}{2\pi} \right)^2} \qquad p_{\rm ws} \sim \mathcal{O}\left(\frac{1}{\sqrt{\lambda}} \right) \ll 1$$

Sigma model in uniform light-cone gauge

Uniform light-cone gauge is useful for decompactification limit.

$$H_{\rm ws} = -P_- \equiv E - J_1, \quad p_{\rm ws} \equiv -\int_{-r}^{r} d\sigma\, \pi_i\, \partial_\sigma X^i, \quad r \equiv \frac{\pi P_+}{\sqrt{\lambda}}\,.$$

E and J_1 are conserved charges of $AdS_5 \times S^5$.

[Kruczenski, Ryzhov, Tseytlin (2004); Arutyunov, Frolov, Zamaklar (2006)]

In the limit $P_+ \to \infty$, we can find soliton-like solutions with

$$\epsilon_{\rm ws}(p_{\rm ws}) = \sqrt{1 + \frac{\lambda}{\pi^2}\,\sin^2\frac{p_{\rm ws}}{2}} \qquad \text{(Giant magnon)}$$

Dispersion relation of (decompactified) $AdS_5 \times S^5$ σ-model is non-relativistic

Dispersion and S-matrix

Asymptotic states and their S-matrix are important in this limit.
The residual $su(2|2)^2$ symmetry of the action constrains them.

[Arutyunov, Frolov, Zamaklar (2006)]

- Asymptotic spectrum?
  - Classified by atypical representations of $su(2|2)$
  - There are also boundstates, $(2Q|2Q)$-representation
- Dispersion relations?
  - Follow from BPS relation of the $su(2|2)^2$
- S-matrix among them?
  - The $su(2|2)$ determines the S-matrix up to a scalar factor

[Beisert (2005)]

Dispersion and S-matrix

Asymptotic states and their S-matrix are important in this limit.
The residual $su(2|2)^2$ symmetry of the action constrains them.

[Arutyunov, Frolov, Zamaklar (2006)]

- The scalar factor?
    - The generalized crossing symmetry constrains this factor

      [Janik (2006)]
    - There is a conjecture on the exact dressing phase σ_{dress}

      [Beisert, Eden, Staudacher (2006)]

$$S_0(y^\pm, x^\pm) = \frac{y^- - x^+}{y^+ - x^-} \frac{1 - \frac{1}{y^+ x^-}}{1 - \frac{1}{y^- x^+}} \sigma^2_{\mathrm{dress}}(y^\pm, x^\pm)$$

$$x^\pm(p) = e^{\pm ip/2} \frac{1 + \sqrt{1 + 16 g^2 \sin^2 \frac{p}{2}}}{4g \sin \frac{p}{2}}, \quad g \equiv \frac{\sqrt{\lambda}}{4\pi}$$

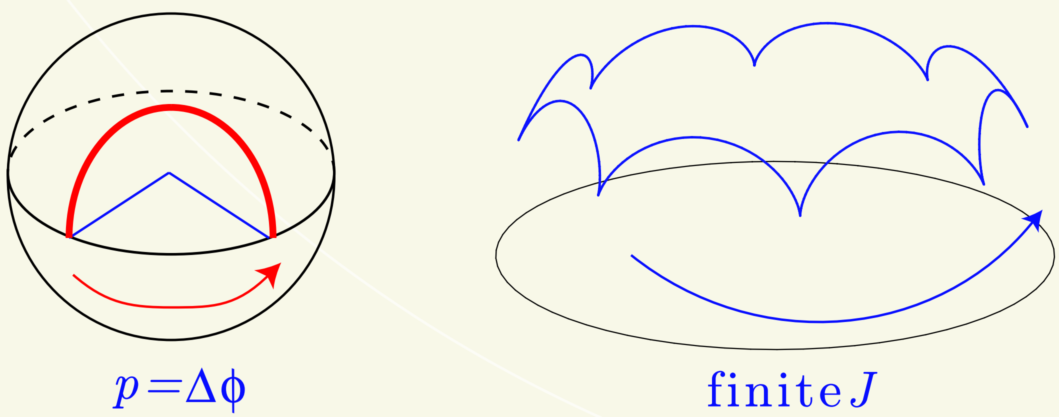
Giant magnons at finite J

$$\begin{aligned} \text{Size} &= \text{Circumference of worldsheet cylinder} \\ &= \text{Angular momentum } J_1 \end{aligned}$$

In the target-space language,

$$\epsilon_{\rm ws}(p_{\rm ws}) = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_{\rm ws}}{2}}\,, \qquad P_+ \to \infty, \quad \text{is}$$

$$E - J_1 = \sqrt{1 + 16 g^2 \sin^2 \frac{\Delta\phi_1}{2}}\,, \qquad E \ \text{ and } \ J_1 \to \infty$$



Giant magnons at finite J

$$\begin{aligned} \textbf{Size} &= \textbf{Circumference of worldsheet cylinder} \\ &= \textbf{Angular momentum } J_1 \end{aligned}$$

Dispersion for Finite-J_1

$$\epsilon(\Delta\phi_1) = \sqrt{1 + 16g^2 \sin^2 \frac{\Delta\phi_1}{2}} + \delta\epsilon(\Delta\phi_1)$$

At strong coupling $g \gg 1$,

$$\delta\epsilon(\Delta\phi_1) = -16g \sin^3 \frac{\Delta\phi_1}{2} \exp\left(-2 - \frac{J_1}{2g \sin \frac{\Delta\phi_1}{2}}\right) + \dots$$

[Arutyunov, Frolov, Zamaklar (2006); Astolfi, Forini, Grignani, Semenoff (2007)]

The Lüscher formula

Finite-size effects can be computed also from the Lüscher formula:

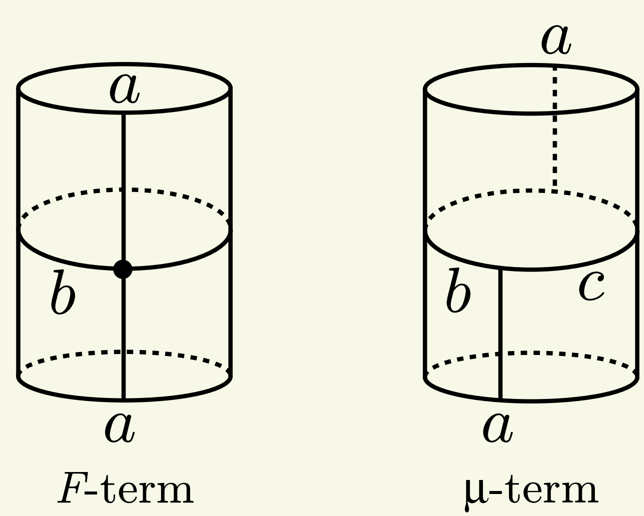
- Consider a relativistic QFT on a cylinder with size L
- Mass of a particle receives corrections of

$$\delta m = \mathcal{O}\left(e^{-cmL}\right) + \mathcal{O}\left(e^{-2cmL}\right) + \dots$$

- The **leading** correction is related to the S-matrix

for a theory with the single mass scale m.

[Lüscher (1986); Klassen, Melzer (1991)]



The Lüscher formula

Finite-size effects can be computed also from the Lüscher formula:

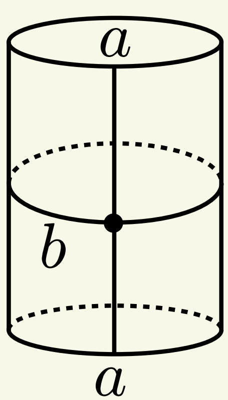
$$\delta m = \delta m^F + \delta m^\mu$$

$$\delta m^F = -m \sum_b \int \frac{d\theta}{2\pi} \cosh\theta \, e^{-mL\cosh\theta} \left[S_{ba}^{ba}\left(\theta + \frac{i\pi}{2}\right) - 1 \right]$$

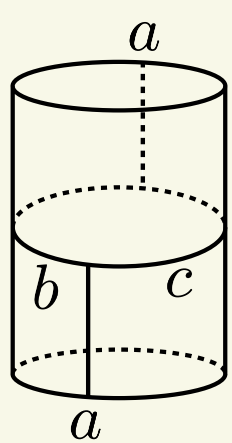
$$\delta m^\mu = -i\frac{\sqrt{3}}{2} m \sum_b e^{-\frac{\sqrt{3}}{2} mL} \operatorname*{Res}_{\theta=\theta_*} S_{ba}^{ba}(\theta)$$

for a theory with the single mass scale m.

[Lüscher (1986); Klassen, Melzer (1991)]



F-term



μ-term

Generalized Lüscher formula

The generalized formula for non-relativistic dispersion

$$\delta\epsilon_a = \delta\epsilon_a^F + \delta\epsilon_a^\mu$$

$$\delta\epsilon_a^F = -\sum_b (-1)^{F_b} \int \frac{d\tilde{q}}{2\pi} \left(1 - \frac{\epsilon_a'(p)}{\epsilon_b'(q)}\right) e^{-iqL} \left[S_{ba}^{ba}(q,p) - 1\right]$$

$$\delta\epsilon_a^\mu = -i \sum_b (-1)^{F_b} \left(1 - \frac{\epsilon_a'(p)}{\epsilon_b'(q_*)}\right) e^{-iq_*L} \operatorname*{Res}_{\tilde{q}=\tilde{q}_*} S_{ba}^{ba}(q,p)$$

[Janik, Łukowski (2007)]

- Incoming particle is a with $(p^0, p^1) = (\epsilon_a(p), p)$.
- Wrapping particle is b with $(q^0, q^1) = (\epsilon_b(q), q)$ and $\tilde{q} = iq^0$.
- The exponent $-iq_*L$ must be negative and large..

Generalized Lüscher formula

Apply the formula to the gauge-fixed σ-model on $AdS_5 \times S^5$

- μ-term $\leftrightarrow$ Correction to **classical** energy of a giant magnon

$$\delta\epsilon^{\mu}(p) = -16g\sin^3\frac{p}{2}\,\exp\left(-2-\frac{J_1}{2g\sin\frac{p}{2}}\right)$$

[Janik, Łukowski (2007)]

- F-term $\leftrightarrow$ Correction to **one-loop** energy of a giant magnon

$$\delta\epsilon^{F}(p) = -\sqrt{\frac{g}{\pi J_1}}\,\frac{16\sin^2\frac{p}{4}}{1-\sin\frac{p}{2}}\,\exp\left(-2\sin\frac{p}{2}-\frac{J_1}{2g}\right)$$

[Heller, Janik, Łukowski (2008); Gromov, Schäfer-Nameki, Vieira (2008)]

Further generalization

There is a rich variety of spectrum.

- Boundstates

  $\leftrightarrow$ Dyonic Giant Magnon with spin $J_2 = Q$

$$\epsilon_Q(p) = \sqrt{Q^2 + 16g^2 \sin^2 \frac{p}{2}}\,, \qquad Q \in \mathbb{Z}_{\geq 1}$$

[Chen, Dorey, Okamura (2006); Roiban (2006)]

- Multi-magnon states

  Incoming particles being M giant magnons,

  $(Q_1, \ldots, Q_M)$ boundstates with momenta $(p_1, \ldots, p_M)$

$$\epsilon_{\rm total} = \sum_{k=1}^{M} \epsilon_{Q_k}(p_k)$$

Further generalization

Now the questions are:

- What are the finite-J corrections?
- Is the Lüscher formula modified?

Can we still find agreement of the two results in the end?

Further generalization

Now the questions are:

- What are the finite-J corrections?
- Is the Lüscher formula modified?

Can we still find agreement of the two results in the end?

YES
WE
CAN

Let's see how...

Sigma model in conformal gauge

Conformal gauge is useful to study classical string spectrum through integrability methods.

1. Equations of motion in terms of coset current

$$\Longrightarrow \quad [\partial_\sigma - J_\sigma(x), \partial_\tau - J_\tau(x)] = 0 \qquad \forall x \in \mathbb{C}.$$

2. Given a classical string solution,

$$\overline{\mathcal{P}} \exp\left(\oint d\sigma J_\sigma(x) \right) \simeq \operatorname{diag}\left\{ e^{ip_1(x)}, \ldots, e^{ip_8(x)} \right\}$$

defines a (non-)algebraic curve on a Riemann surface.

[Kazakov, Marshakov, Minahan, Zarembo (2004); Beisert, Kazakov, Sakai, Zarembo (2005)]

Efficient method to compute classical (and one-loop) energy

[Gromov, Vieira (2007)]

Modification to the F-term formula

Now we know finite-J_1 corrections in string theory.
Look for the modified formula that agrees with these results

- For boundstates, use the dispersion for boundstates and the S-matrix between fundamental magnon and boundstates

- For multi-magnon states,

$$\delta\epsilon_a^F = -\sum_b (-1)^{F_b} \int \frac{d\tilde{q}}{2\pi} \left(1 - \frac{\epsilon_a'(p)}{\epsilon_b'(q)}\right) e^{-iqL} \left[S_{ba}^{ba}(q,p) - 1\right]$$

should become

$$\delta\epsilon_A^F = -\sum_b (-1)^{F_b} \int \frac{d\tilde{q}}{2\pi} \left(1 - \sum_{\ell=1}^{M} \alpha_\ell \frac{\epsilon_{a_k}'(p_k)}{\epsilon_b'(q)}\right) e^{-iqL} \times \left(\prod_{\ell=1}^{M} S_{ba_\ell}^{ba_\ell}(q,p_\ell) - 1\right), \quad \sum_{\ell=1}^{M} \alpha_\ell = 1$$

[Hatsuda, RS (2008)]

Relation to Thermodynamic Bethe Ansatz

This is consistent with the TBA-like result in sinh-Gordon model:

$$E(L) = \sum_{j=1}^{M} m \cosh\theta_j - \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} m \cosh\theta \ \log[1 + Y(\theta)]$$

$$\log Y(\theta) = -mL\cosh\theta - \sum_{j=1}^{M} \log S(\theta - \theta_j - i\frac{\pi}{2})$$

$$-i \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \left(\frac{d \log S}{d\theta} \right) (\theta - \theta') \log[1 + Y(\theta')]$$

If $L \gg 1$, they can be solved order by order.

[Teschner (2007)]

Relation to Thermodynamic Bethe Ansatz

This is consistent with the TBA-like result in sinh-Gordon model:

The result

$$\delta E_{\mathrm{shG}}(L) \approx -\int_{-\infty}^{\infty} \frac{d\theta}{2\pi} m \cosh\theta\, e^{-mL\cosh\theta} \prod_{k=1}^{M} S(\tilde{\theta}_k - \theta + i\frac{\pi}{2}) + \sum_{j=1}^{M} m \sinh\tilde{\theta}_j \delta\theta_j$$

can be compared with

$$\delta\epsilon_A^F = -\sum_b (-1)^{F_b} \int \frac{d\tilde{q}}{2\pi} \left(1 - \sum_{\ell=1}^{M} \alpha_\ell \frac{\epsilon'_{a_k}(p_k)}{\epsilon'_b(q)}\right) e^{-iqL} \times \left(\prod_{\ell=1}^{M} S_{ba_\ell}^{ba_\ell}(q, p_\ell) - 1\right)$$

[Bajnok, Janik (2008); Hatsuda, RS (2008)]

Modification to the μ-term formula

Corrections to classical energy are found in [Minahan, Ohlsson-Sax (2008)]

$$\delta\epsilon_a^{\mu} = -i\sum_b (-1)^{F_b}\left(1-\frac{\epsilon_a'(p)}{\epsilon_b'(q_*)}\right) e^{-iq_* L}\operatorname*{Res}_{\tilde q=\tilde q_*} S_{ba}^{ba}(q,p)$$

should become

$$\delta\epsilon_A^{\mu} = \mathrm{Re}\left\{\sum_{\ell=1}^{M}\sum_b (-1)^{F_b}\left\{\epsilon_b'(q_\ell^*) - \epsilon_{a_\ell}'(p_\ell)\right\} e^{-iq_\ell^* L}\times \right.$$
$$\left.\operatorname*{Res}_{q^1=q_\ell^*} S_{ba_\ell}^{ba_\ell}(q^1, p_\ell)\prod_{k\neq\ell}^{M} S_{ba_k}^{ba_k}(q_\ell^*, p_k)\right\}$$

Implying

μ-term $\leftrightarrow$ Residue of F-term at simple poles

[Hatsuda, RS (2008)]

Comments on μ-term

Some remarks:

- Take the real part ($\because$ originally $2\,e^{-iqL}$ was $e^{-iqL}+e^{iqL}$)
- Sum the residues of two 'physical' poles

  There are four poles in the fundamental-boundstate S-matrix

$$x_q^{\pm} = X^{+}\,, \qquad x_q^{\pm} = 1/X^{+}$$

Only the first two combinations satisfy

$$\left|x_q^{\pm}\right| > 1 \quad \text{and} \quad \left|X^{\pm}\right| > 1$$

[Arutyunov, Frolov (2007)]

Moreover, both $x_q^{\pm} = X^{+}$ satisfy the conservation law

$$E_Q(p_x) = E_{Q-1}(p_y) + E_1(p - p_y) \qquad (Q \gg 1)$$

Comments on μ-term

Some questions:

- The same poles do not satisfy the energy-momentum conservation at weak coupling.
    - Why the μ-term appears only at strong coupling?
- The residues at $x_q^- = X^+$ and $x_q^+ = X^+$ come in "wrong" sign.

$$\delta E_{\text{classical string}} = \mathrm{Re}\left\{ \delta\epsilon_\mu \Big|_{x_q^- = X^+} - \delta\epsilon_\mu \Big|_{x_q^+ = X^+} \right\}$$

For $Q \sim \mathcal{O}(1) \ll g$,

$$-16g\sin^3\frac{p}{2} = -8g\sin^3\frac{p}{2}\left\{\left(1+\frac{1}{Q}\right) + \left(1-\frac{1}{Q}\right)\right\}$$

Summary and outlook

Summary:

- Proposed Lüscher formula for multi-particle (bound)states
- That agrees with the results of σ-model on $AdS_5 \times S^5$

Some Applications:

- Remarkable agreement beyond perturbative region $\lambda \ll 1$

$$\Delta_{\text{Konishi from SYM}} = \Delta_{\text{Bethe}} + \Delta_{\text{Lüscher}} \quad \text{up to } \mathcal{O}(\lambda^4)$$

[Bajnok, Janik (2008)]

- Other models, like σ-model on $AdS_4 \times \mathbb{CP}^3$

[Bombardelli, Fioravanti (2008); Łukowski, Ohlsson-Sax (2008); Ahn, Bozhilov (2008)]