Hybrid nonlinear integral equations for $AdS_5 \times S^5$

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Introduction

Exact spectrum of string on $AdS_5 \times S^5$

In the large *N* limit, this theory becomes

1) Dual to $\mathcal{N}=4,\ D=4$ super Yang-Mills

$$\Delta(\lambda) \stackrel{?}{=} E(\lambda)$$

2) Integrable --- Thermodynamic Bethe Ansatz (TBA)

$$\Delta(\lambda) \stackrel{?}{=} E_{\mathrm{TBA}}(\lambda) \stackrel{?}{=} E(\lambda)$$

Conformal dimensions in $\mathcal{N}$=4 SYM

Konishi $\mathfrak{sl}(2)$ descendant, $\mathcal{O} = \mathrm{tr}\left[D_+^2 Z^2 - (D_+ Z)^2\right]$

$$\Delta = \Delta_0 + 3\,g^2 - 3\,g^4 + \frac{21}{4}g^6$$

[Beisert, Staudacher; hep-th/0504190]

$$+\left(-\frac{39}{4} + \frac{9\zeta(3)}{4} - \frac{45\zeta(5)}{8}\right)g^8$$

[Bajnok, Janik; 0807.0399]
[Fiamberti, Santambrogio, Sieg, Zanon; 0712.3522]

$$+\left(\frac{27\zeta(3)}{4} - \frac{81\zeta(3)^2}{16} - \frac{135\zeta(5)}{16} + \frac{945\zeta(7)}{32} + \frac{237}{16}\right)g^{10} + \ldots$$

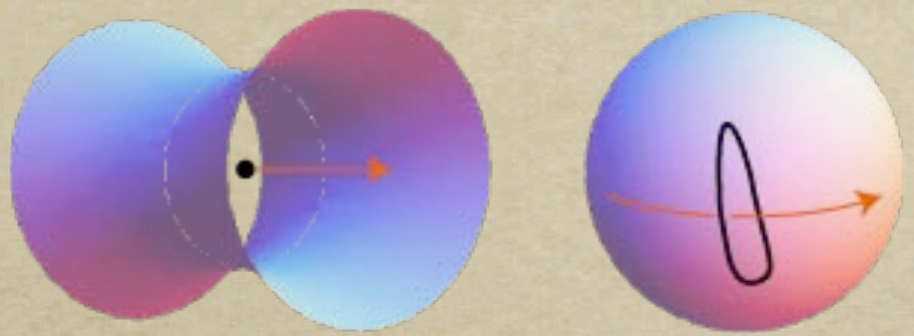
[Arutyunov, Frolov, RS; 1002.1701] [Balog Hegedus; 1002.4142]
[Bajnok, Janik, Hegedus, Lukowski; 0906.4062]

$$\left[g \equiv \frac{\sqrt{\lambda}}{2\pi}\right]$$

String energies on $AdS^5 \times S_5$

Short spinning string

with $S = J = 2$,



$(S, J) =$ angular momenta in $\mathrm{AdS}_3 \times \mathrm{S}^1$

$$E = 2\sqrt[4]{\lambda} + E_0 + \frac{2}{\sqrt[4]{\lambda}} + \dots$$

[Gromov, Serban, Shenderovich, Volin; 1102.1040]

[Roiban, Tseytlin; 1102.1209] [Mazzucato, Vallilo; 1102.1219]

Thermodynamic Bethe Ansatz

Two-particle states in the $\mathfrak{sl}(2)$ sector

Exact energy:

$$E - J = \sum_{k=1}^{2} \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_k}{2}} - \sum_{Q=1}^{\infty} \int \frac{d\widetilde{p}_Q}{2\pi} \log(1 + Y_Q)$$

Asymptotic part (large *J*) + exact finite *J* corrections

Exact Bethe equation: $Y_{1_*}(p_k) = -1$

TBA equations: $\log Y_a = \log(1 + Y_b) \star K_{ba} + \dots$

$$\left[f \star K \equiv \int_{-\infty}^{\infty} dt\, f(t) K(t,v), \quad K(t,v) \equiv \frac{1}{2\pi i} \frac{\partial}{\partial t} \log S(t,v) \right]$$

$$\log Y_{M|w} = \log(1+Y_{M-1|w})(1+Y_{M+1|w}) \star s + \delta_{M1}\ \log \frac{1-\frac{1}{Y_-}}{1-\frac{1}{Y_+}} \,\hat{\star}\, s$$

$$\log Y_{M|vw}(v) = -\delta_{M1} \sum_{j=1}^{2} \log S(u_j^- - v) - \log(1+Y_{M+1}) \star s$$

$$+\log(1+Y_{M-1|vw})(1+Y_{M+1|vw}) \star s + \delta_{M1} \log \frac{1-Y_-}{1-Y_+} \,\hat{\star}\, s$$

$$\log \frac{Y_+}{Y_-}(v) = -\sum_{j=1}^{2} \log S_{1_* y}(u_j, v) + \log(1+Y_Q) \star K_{Qy}$$

$$\log Y_+ Y_-(v) = -\sum_{j=1}^{2} \log \frac{\left(S_{xv}^{1_* 1}\right)^2}{S_2} \star s(u_j, v)$$

$$+2 \log \frac{1+Y_{1|vw}}{1+Y_{1|w}} \star s - \log\left(1+Y_Q\right) \star K_Q + 2\log(1+Y_Q) \star K_{xv}^{Q1} \star s$$

$$\log Y_Q(v) = -\sum_{j=1}^{2} \left(\log S_{\mathfrak{sl}(2)}^{1_* Q}(u_j, v) - 2 \log S \star K_{vwx}^{1Q}(u_j^-, v) \right)$$

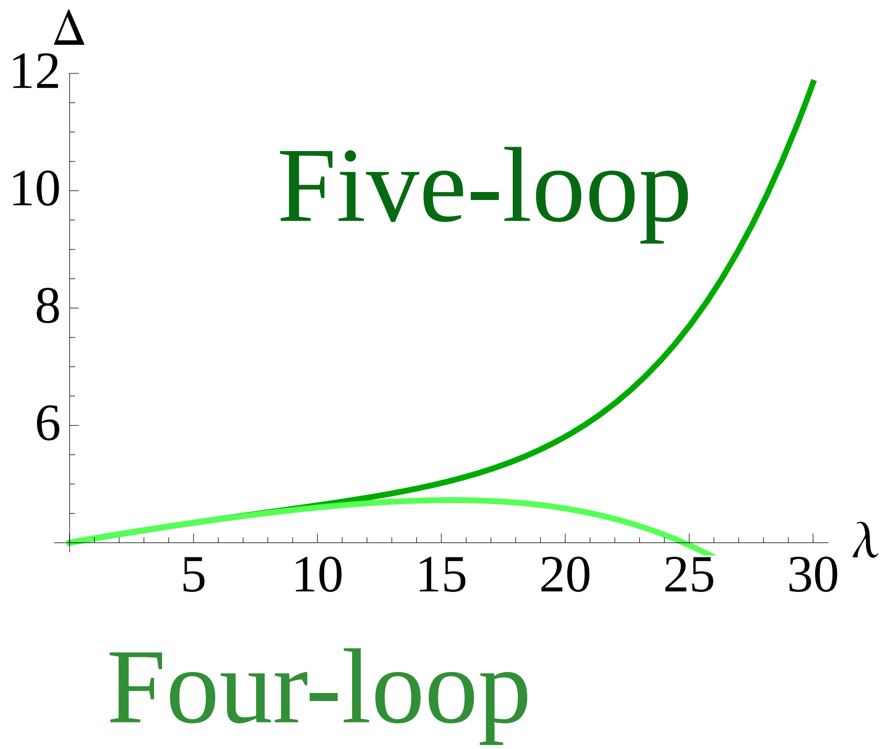
$$-L\, \widetilde{\mathcal{E}}_Q + \log\left(1+Y_{Q'}\right) \star \left(K_{\mathfrak{sl}(2)}^{Q'Q} + 2\, s \star K_{vwx}^{Q'-1,Q} \right)$$

$$+2 \log\left(1+Y_{1|vw}\right) \star s \,\hat{\star}\, K_{yQ} + 2\ \log\left(1+Y_{Q-1|vw}\right) \star s$$

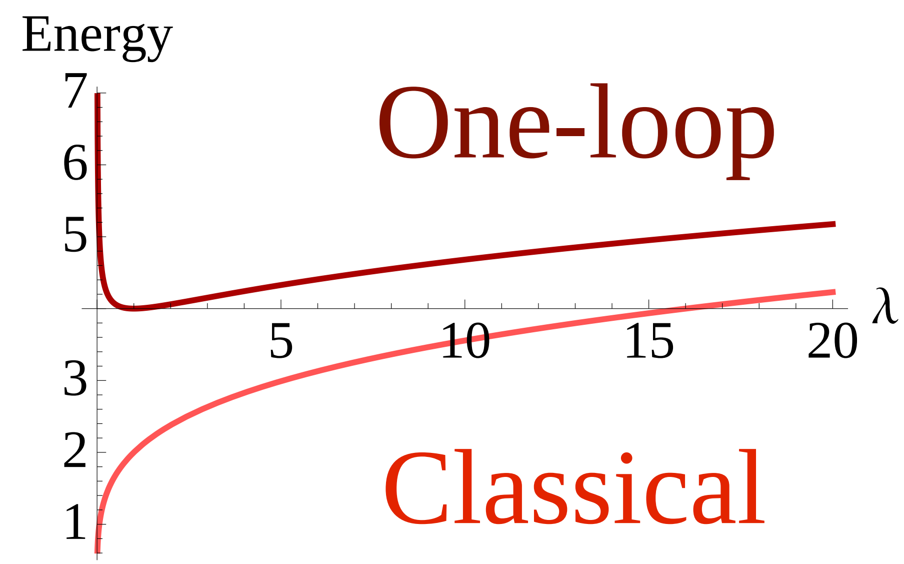
$$-2 \log \frac{1-Y_-}{1-Y_+} \,\hat{\star}\, s \star K_{vwx}^{1Q} + \log \frac{1-\frac{1}{Y_-}}{1-\frac{1}{Y_+}} \,\hat{\star}\, K_Q + \log\left(1-\frac{1}{Y_-}\right)\left(1-\frac{1}{Y_+}\right) \hat{\star}\, K_{yQ}$$

$$\Delta(\lambda) \stackrel{?}{=} E_{\mathrm{TBA}}(\lambda) \stackrel{?}{=} E(\lambda)$$

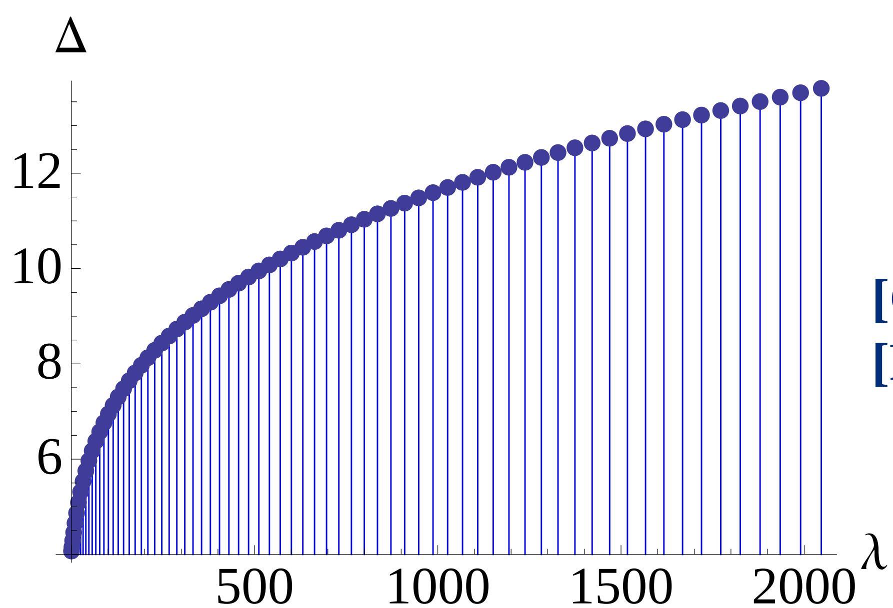
$\mathcal{N}$=4 SYM



$AdS_5 \times S^5$ string



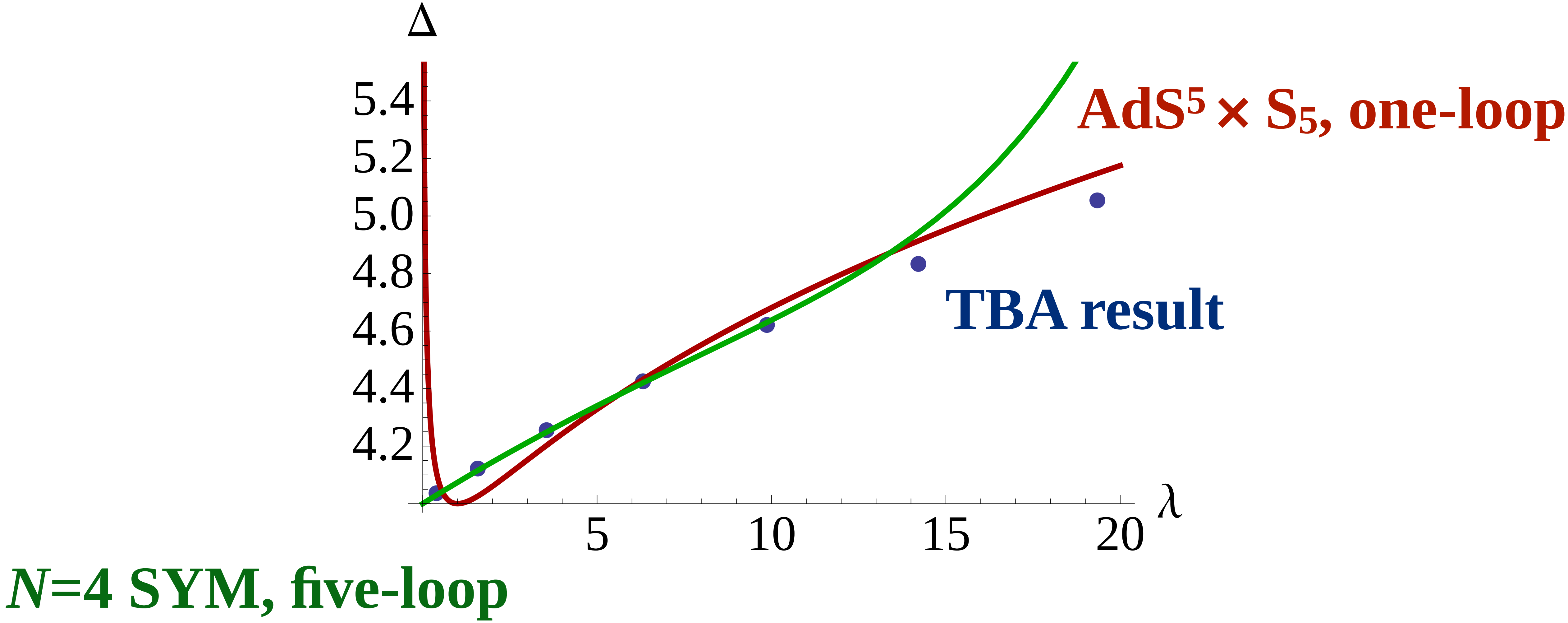
TBA result



[Gromov, Kazakov, Vieira; 0906.4240]
[Frolov; 1006.5032]

$$\Delta(\lambda) \stackrel{?}{=} E_{\text{TBA}}(\lambda) \stackrel{?}{=} E(\lambda)$$

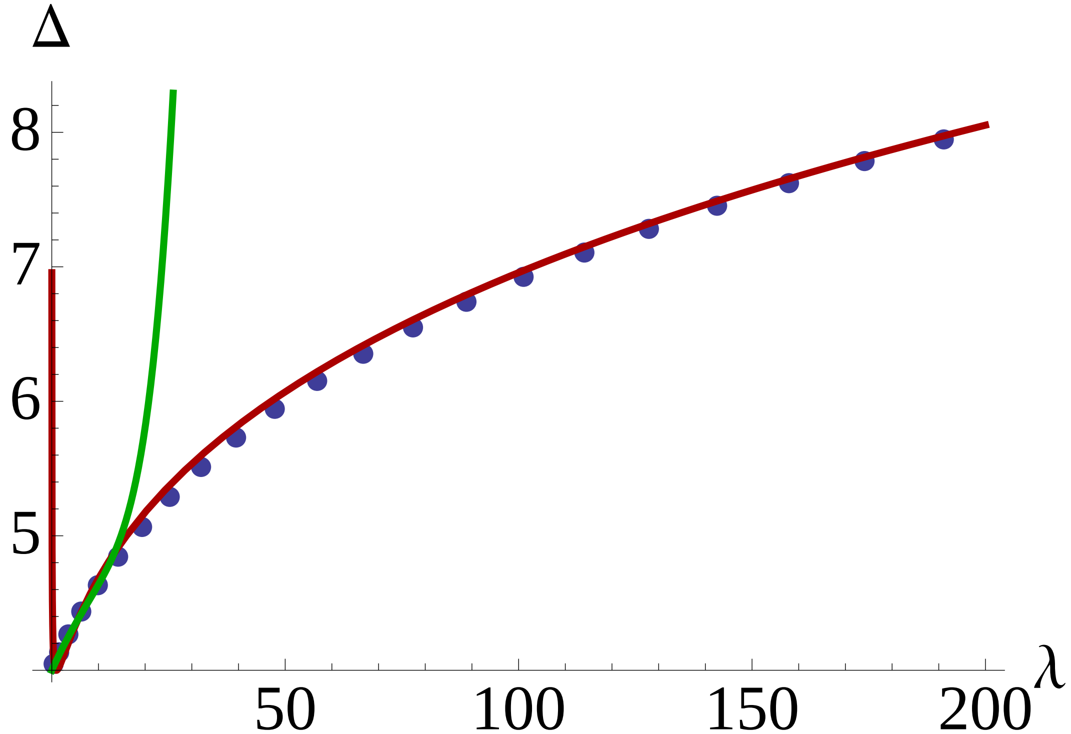
Exact Konishi dimension/short string energy

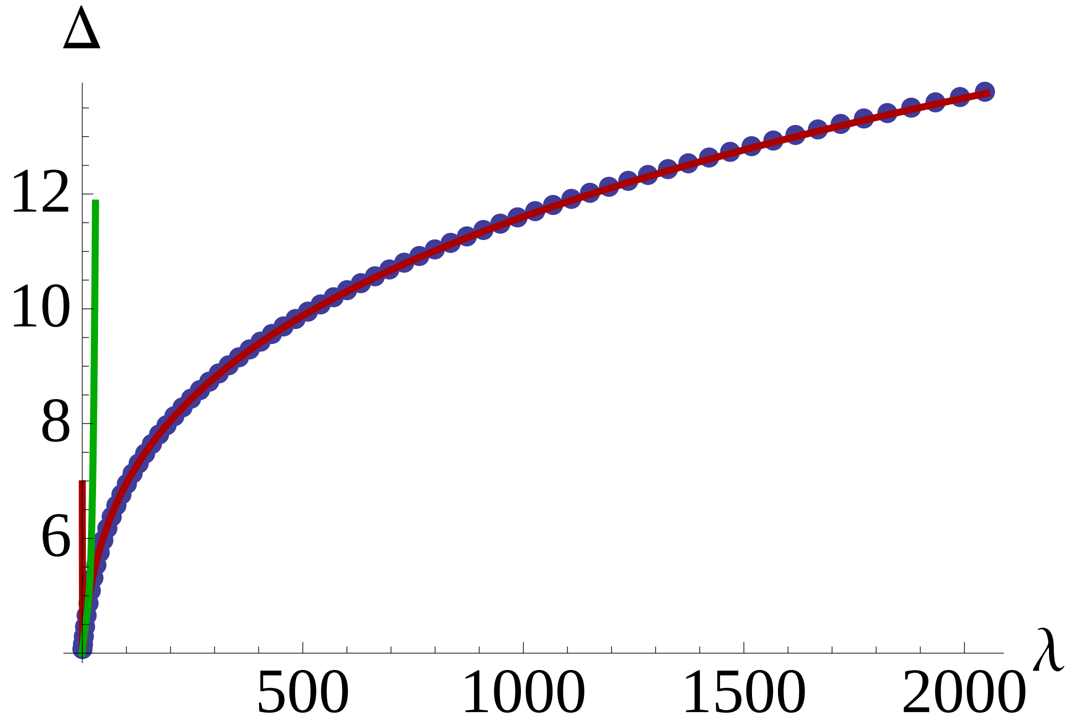


All results come close around $\lambda \approx 10$

$\Delta(\lambda) \stackrel{?}{=} E_{\mathrm{TBA}}(\lambda) \stackrel{?}{=} E(\lambda)$

Exact Konishi dimension/short string energy





***N*=4 SYM, five-loop** **TBA result** **$AdS^5 \times S_5$, one-loop**

Conclusion: $\Delta(\lambda) \approx E_{\mathrm{TBA}}(\lambda) \approx E(\lambda)$?

$\Delta(\lambda) \approx E_{\mathrm{TBA}}(\lambda) \approx E(\lambda)$

TBA computes the exact finite size corrections.

However, it turns out that the f.s. corrections are not very large in the perturbative region of Konishi state.

1) $\Delta(\lambda) \approx E_{\mathrm{TBA}}(\lambda)$

f.s. corrections are $\mathcal{O}\left(\lambda^{J}\right) \ll 1$ at weak coupling

2) $E_{\mathrm{TBA}}(\lambda) \approx E(\lambda)$

f.s. corrections are $\mathcal{O}\left(\lambda^{-1/4}\right) \ll 1$ at strong couping

What will happen if the finite size corrections are large?

Problems of Mirror $AdS_5 \times S^5$ TBA

1) TBA is too complicated

Infinite set of nonlinear integral equations

How to solve it?

Problems of Mirror $AdS_5 \times S^5$ TBA

1) TBA is too complicated

Infinite set of nonlinear integral equations

How to solve it?

2) TBA depends on coupling constant

$$\mathrm{TBA}(\lambda < \lambda_c^{(i)}) \neq \mathrm{TBA}(\lambda > \lambda_c^{(i)})$$

Numerical iteration does not converge at $\lambda = \lambda_c^{(i)}$

What is $\mathrm{TBA}(\lambda \to \infty)$? **[Arutyunov, Frolov, R.S.; 0911.2224]**

Problems of Mirror $AdS_5 \times S^5$ TBA

1) TBA is too complicated

2) TBA depends on coupling constant

Can TBA be simpler?

NLIE

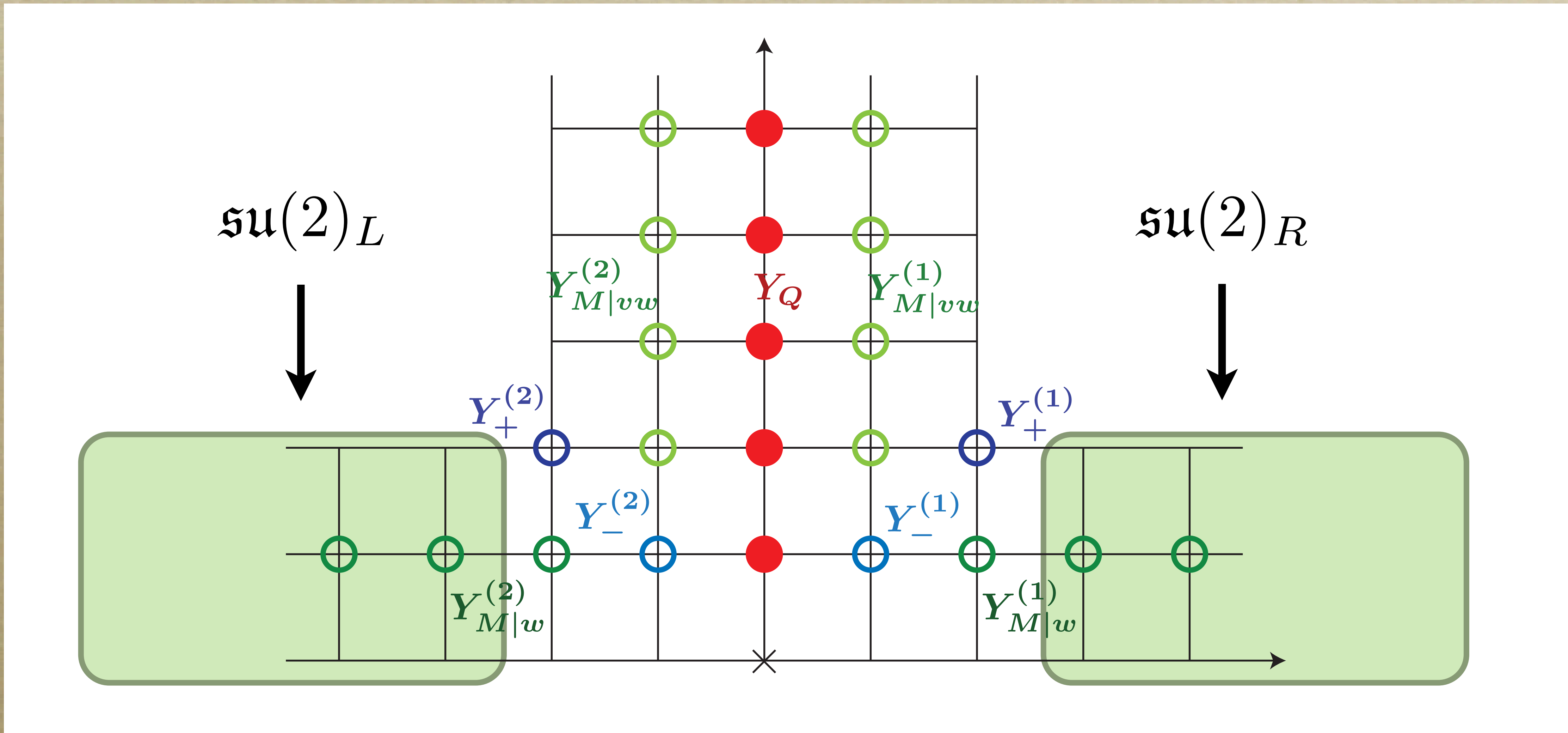
- Nonlinear integral equation (NLIE) is an efficient method for the exact finite-size or finite-temperature problem in cond-mat.
- NLIE consists of finitely many nonlinear integral equations.
- NLIE might relieve the problem of (perhaps infinite) critical coupling constants in TBA.

NLIE

- However, as of today, it is not clear if *the NLIE for the mirror* $AdS_5 \times S^5$ *exists*.
- (My opinion is hopefully yes.)
- I would like to discuss how to formulate a *hybrid NLIE*.
- Hybrid means we are halfway there.

Hybrid NLIE

A Y-function corresponds to a node of $\mathfrak{su}(2|4|2)$-hook



The horizontal wings of the hook go away in hybrid NLIE

Only Y_Q are needed to compute the exact energy

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Hybrid NLIE; key ideas

1) Use proper (or fundamental) variables

"Mesons" vs. "Quarks"

Q-functions seem more fundamental than T-functions.
The relation between T and Q is like mesons and quarks.
M. Staudacher

Hybrid NLIE; key ideas

1) Use proper (or fundamental) variables

"Mesons" vs. "Quarks"

2) Two steps of derivation

a) TQ-relations

b) Analyticity conditions

These ideas are well known in condensed matter physics,

[J. Suzuki, J Phys A32 (1999)]

But their methods were not general enough for the application to AdS/CFT

Variables Y, T

[Cavaglia, Fioravanti, Tateo; 1005.3016] [Cavaglia, Fioravanti, Mattelliano, Tateo; 1103.0499] [Balog, Hegedus; 1104.4054]

Mirror TBA = Y-system + analyticity

Difference equations

$$Y_{a,s}^{+}\,Y_{a,s}^{-}=\frac{\left(1+Y_{a,s+1}\right)\left(1+Y_{a,s-1}\right)}{\left(1+\frac{1}{Y_{a+1,s}}\right)\left(1+\frac{1}{Y_{a,s-1}}\right)}$$

Zeroes, poles and gaps

$$\log\frac{Y_{a,s}(v+i0)}{Y_{a,s}(v-i0)}=\dots$$

Change of variables

$$Y_{a,s}=\frac{T_{a,s+1}\,T_{a,s-1}}{T_{a+1,s}\,T_{a-1,s}}$$

Y-system $\Leftrightarrow$ T-system

$$T_{a,s}^{-}\,T_{a,s}^{+}=T_{a-1,s}\,T_{a+1,s}+T_{a,s-1}\,T_{a,s+1}$$

$$\left[f^{[+s]}\equiv f\left(v+\frac{is}{g}\right),\quad f^{\pm}=f^{[\pm 1]},\quad g\equiv\frac{\sqrt{\lambda}}{2\pi}\right]$$

Variables T, Q

Mirror TBA = T-system + analyticity

Difference equations

$$T_{a,s}^{-}\, T_{a,s}^{+} = T_{a-1,s}\, T_{a+1,s} + T_{a,s-1}\, T_{a,s+1}$$

Zeroes, poles and gaps

$$\log \frac{T_{a,s}(v+i0)}{T_{a,s}(v-i0)} = \dots$$

The general solution of T-system (without implementing analyticity) is given by Wronskian of 8 fundamental Q-functions

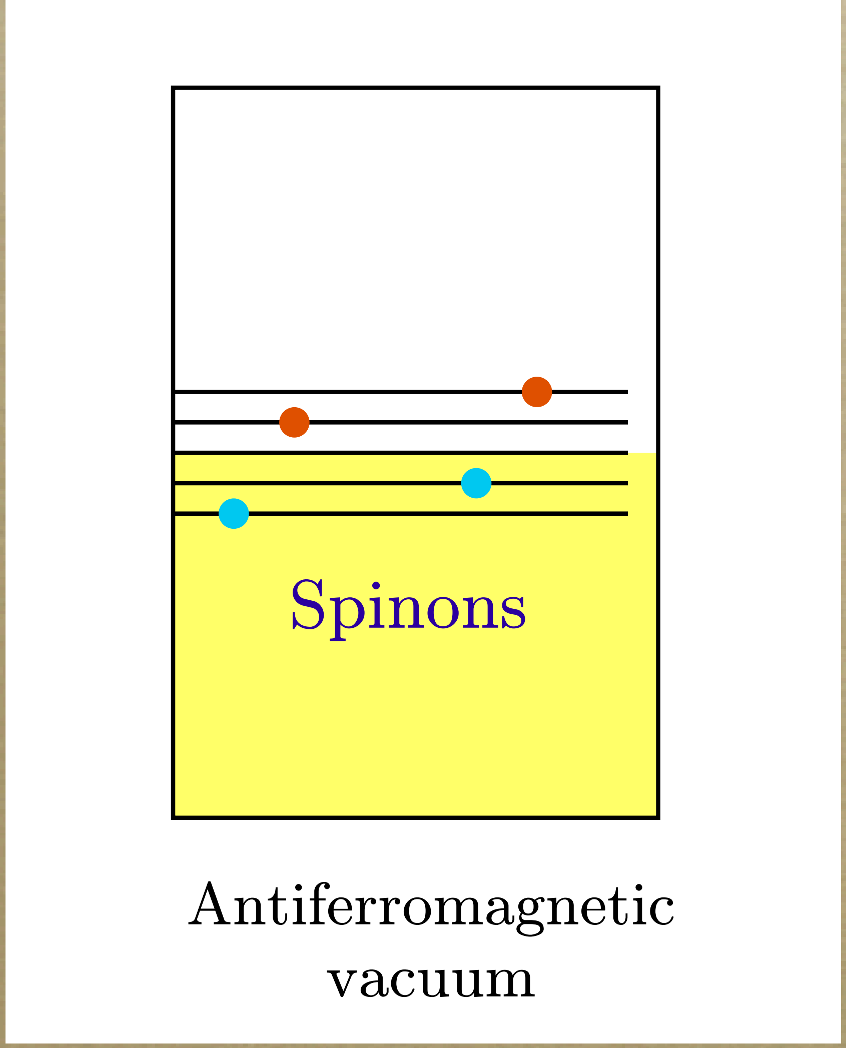
[Gromov, Kazakov, Leurent, Tsuboi; 1010.2720]

Y, T, Q; which variables should we use?

$$\left[f^{[+s]} \equiv f\left(v + \frac{is}{g}\right), \quad f^{\pm} = f^{[\pm 1]}, \quad g \equiv \frac{\sqrt{\lambda}}{2\pi} \right]$$

Spinon variables

Proper choice of variables is crucial in (hybrid) NLIE



$$1 + Y_{1,s} = (1 + \mathfrak{b}_s)\,(1 + \bar{\mathfrak{b}}_s)$$

"mesons" or magnons

"quarks" or spinons

Spinons are elementary excitations over the antiferromagnetic vacuum

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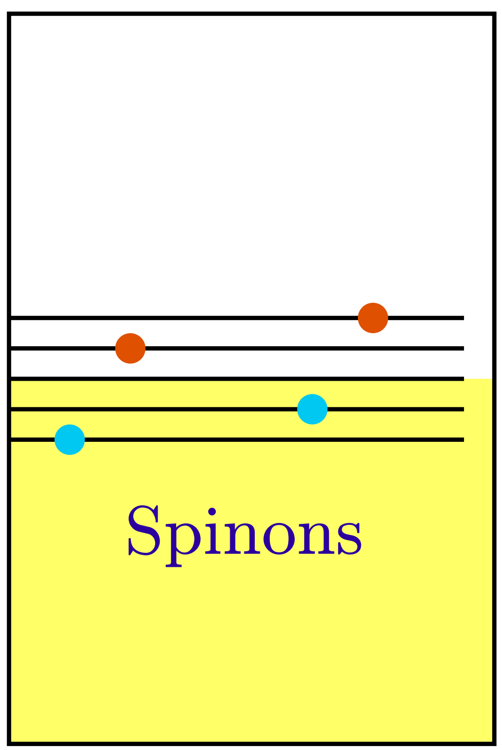
Spinon variables

Proper choice of variables is crucial in (hybrid) NLIE

$$1 + Y_{1,s} = (1 + \mathfrak{b}_s)\,(1 + \bar{\mathfrak{b}}_s)$$

"mesons" or magnons — "quarks" or spinons



Antiferromagnetic vacuum

Spinons are elementary excitations over the antiferromagnetic vacuum

$$\mathfrak{b}_s \sim e^{iZ},\ Z \equiv \text{Counting function}$$

In spin-chain models of cond-mat, Z counts **Bethe roots** and **holes** on an equal footing

Derivation of hybrid NLIE
a) TQ-relations

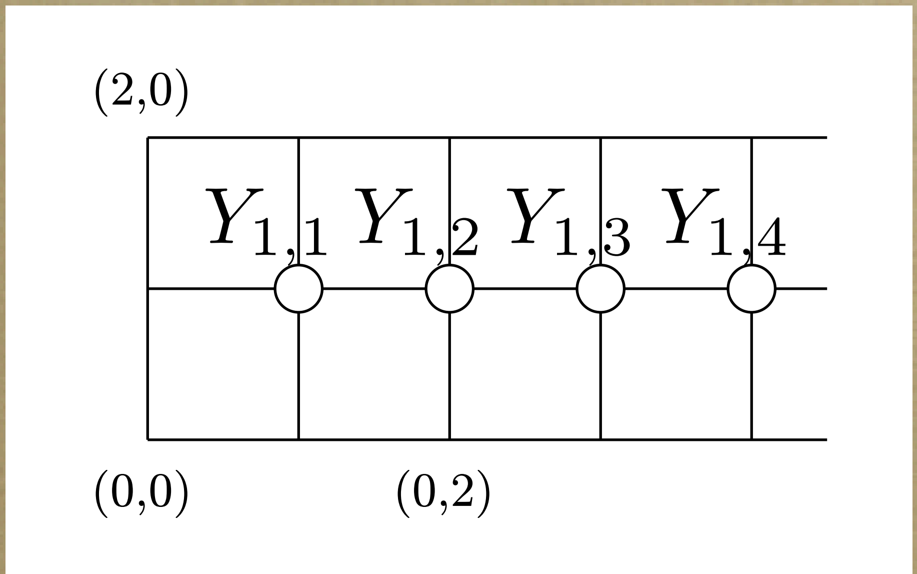
T-functions in the $\mathfrak{su}(2)$ wing of the $\mathfrak{su}(2|4|2)$-hook satisfy the linear difference equations (TQ-relation)

[Krichever, Lipan, Wiegmann, Zabrodin; hep-th/9604080]

$$Q^{-}_{1,s-1}\,T_{1,s} - Q_{1,s}\,T^{-}_{1,s-1} = \overline{Q}^{-}_{1,s-1}\,L_{1,s}$$

$$\overline{Q}^{+}_{1,s-1}\,T_{1,s} - \overline{Q}_{1,s}\,T^{+}_{1,s-1} = Q^{+}_{1,s-1}\,\overline{L}_{1,s}$$

$$T_{0,s}\,T_{2,s} = L^{+}_{1,s}\,\overline{L}^{-}_{1,s}$$



Q and L are translationally invariant

$$Q_{1,s} = Q\left(v + \frac{is}{g}\right) \equiv Q^{[+s]}\,,$$

$$L_{1,s} = L^{[+s]}\,,\ \overline{Q}_{1,s} = \overline{Q}^{[-s]}\,,\ \overline{L}_{1,s} = \overline{L}^{[-s]}$$

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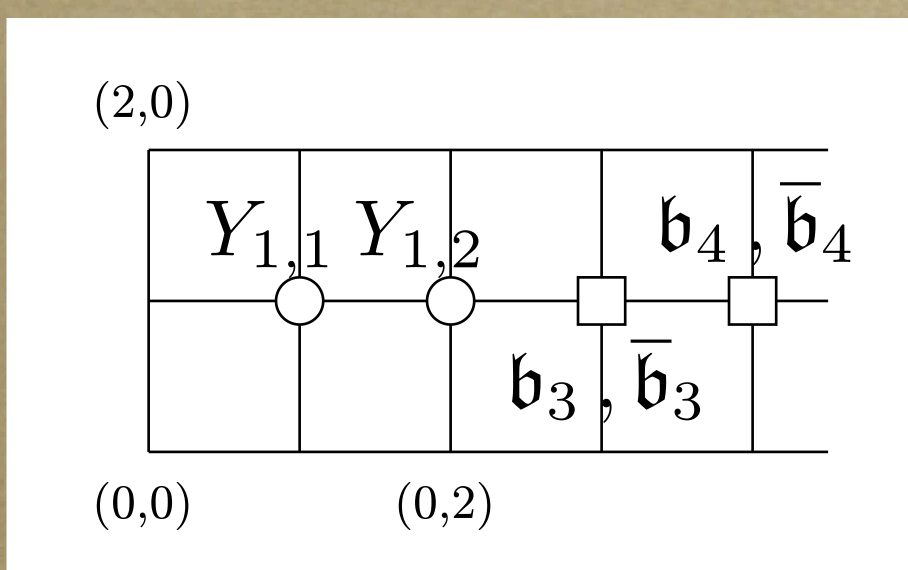
Derivation of hybrid NLIE
a) TQ-relations

Change of variables:

$$\boldsymbol{A}_{1,s} = \frac{\overline{\boldsymbol{Q}}^{-}_{1,s-1}}{\boldsymbol{Q}^{-}_{1,s-1}}\,\boldsymbol{L}_{1,s}\,, \quad \overline{\boldsymbol{A}}_{1,s} = \frac{\boldsymbol{Q}^{+}_{1,s-1}}{\overline{\boldsymbol{Q}}^{+}_{1,s-1}}\,\overline{\boldsymbol{L}}_{1,s}\,, \quad (s \geq 2)$$

Define spinon variables:

$$1 + \mathfrak{b}_s \equiv \frac{\boldsymbol{T}^{+}_{1,s}}{\boldsymbol{A}^{+}_{1,s}} = \frac{\boldsymbol{Q}_{1,s-1}\,\boldsymbol{T}^{+}_{1,s}}{\overline{\boldsymbol{Q}}_{1,s-1}\,\boldsymbol{L}^{+}_{1,s}}\,, \quad 1 + \overline{\mathfrak{b}}_s \equiv \frac{\boldsymbol{T}^{-}_{1,s}}{\overline{\boldsymbol{A}}^{-}_{1,s}} = \frac{\overline{\boldsymbol{Q}}_{1,s-1}\,\boldsymbol{T}^{-}_{1,s}}{\boldsymbol{Q}_{1,s-1}\,\overline{\boldsymbol{L}}^{-}_{1,s}}$$



One can check that:

$$(1 + \mathfrak{b}_s)(1 + \overline{\mathfrak{b}}_s) = \frac{\boldsymbol{T}^{+}_{1,s}\,\boldsymbol{T}^{-}_{1,s}}{\boldsymbol{A}^{+}_{1,s}\,\overline{\boldsymbol{A}}^{-}_{1,s}} = \frac{\boldsymbol{T}^{+}_{1,s}\,\boldsymbol{T}^{-}_{1,s}}{\boldsymbol{T}_{0,s}\,\boldsymbol{T}_{2,s}} = 1 + \boldsymbol{Y}_{1,s}$$

Derivation of hybrid NLIE
b) Analyticity conditions

By using various relations and Fourier transform, we get

$$\widehat{dl}\,\mathfrak{b}_s = e^{+2q/g}\left[\widehat{dl}\,Q_{1,s-1} - \widehat{dl}\,L^{+}_{1,s-1}\right] - \left[\widehat{dl}\,\overline{Q}_{1,s-1} - \widehat{dl}\,\overline{L}^{-}_{1,s-1}\right] + \widehat{dl}\,(1+Y_{1,s-1})\,\hat{s}_K(q)$$

From analyticity of T-functions in the physical strip,

$$\left[\widehat{dl}\,Q_{1,s-1} - \widehat{dl}\,L^{+}_{1,s-1}\right] \sim \widehat{dl}\,(1+\mathfrak{b}_s)\,\hat{s}_K(q)$$

$$\left[\widehat{dl}\,F(q) \equiv \int_{-\infty}^{+\infty} dv\, e^{iqv}\,\frac{\partial}{\partial v}\log F(v), \quad \hat{s}_K(q) = \frac{1}{2\cosh(q/g)}\right]$$

Derivation of hybrid NLIE
b) Analyticity conditions

By using various relations and Fourier transform, we get

$$\widehat{dl}\,\mathfrak{b}_s = e^{+2q/g}\left[\widehat{dl}\,Q_{1,s-1} - \widehat{dl}\,L^{+}_{1,s-1}\right]$$
$$-\left[\widehat{dl}\,\overline{Q}_{1,s-1} - \widehat{dl}\,\overline{L}^{-}_{1,s-1}\right] + \widehat{dl}\,(1+Y_{1,s-1})\,\hat{s}_K(q)$$

$$\widehat{dl}\,\mathfrak{b}_s \sim \frac{e^{+2q/g}}{2\cosh(q/g)}\,\widehat{dl}\,(1+\mathfrak{b}_s) + \dots$$

This factor diverges exponentially as $\mathrm{Re}\,q \to +\infty$

Rescued if $\left[\widehat{dl}\,Q_{1,s-1} - \widehat{dl}\,L^{+}_{1,s-1}\right] = \mathbf{0}$ for $\mathrm{Re}\,q > 0$

$\Leftrightarrow\ Q_{1,s-1}/L^{+}_{1,s-1}$ is analytic for $\mathrm{Im}\,v > 0$

Derivation of hybrid NLIE
c) Summary of results

We insert $\theta(\pm q)$, collect other terms and apply inverse Fourier transform

Regularization (useful for numerical computation)

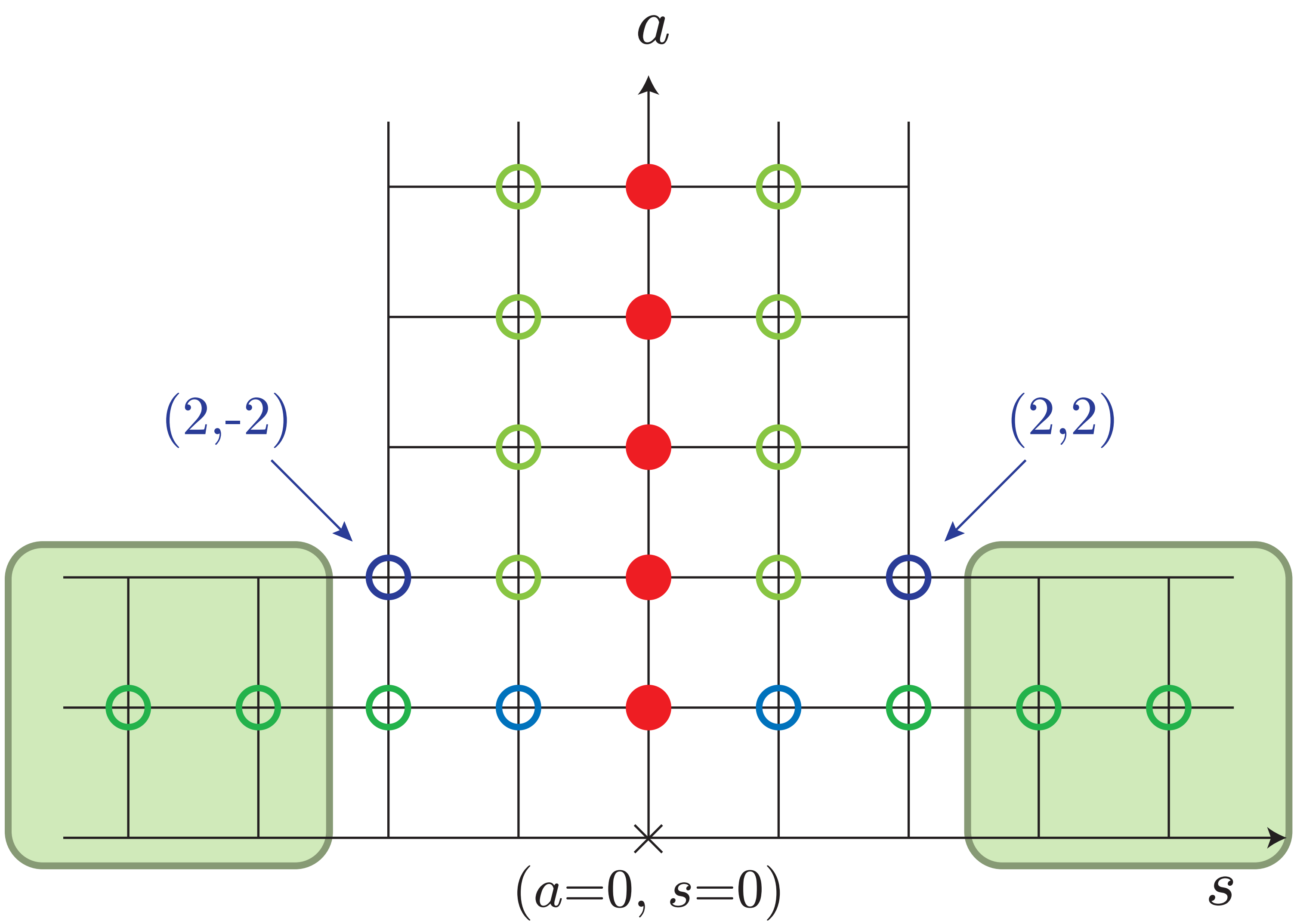
$$\mathfrak{a}_s(v) = \mathfrak{b}_s\Big(v - \frac{i\gamma}{g}\Big), \quad \bar{\mathfrak{a}}_s(v) = \bar{\mathfrak{b}}_s\Big(v + \frac{i\gamma}{g}\Big), \quad (0 < \gamma < 1)$$

Hybrid NLIE

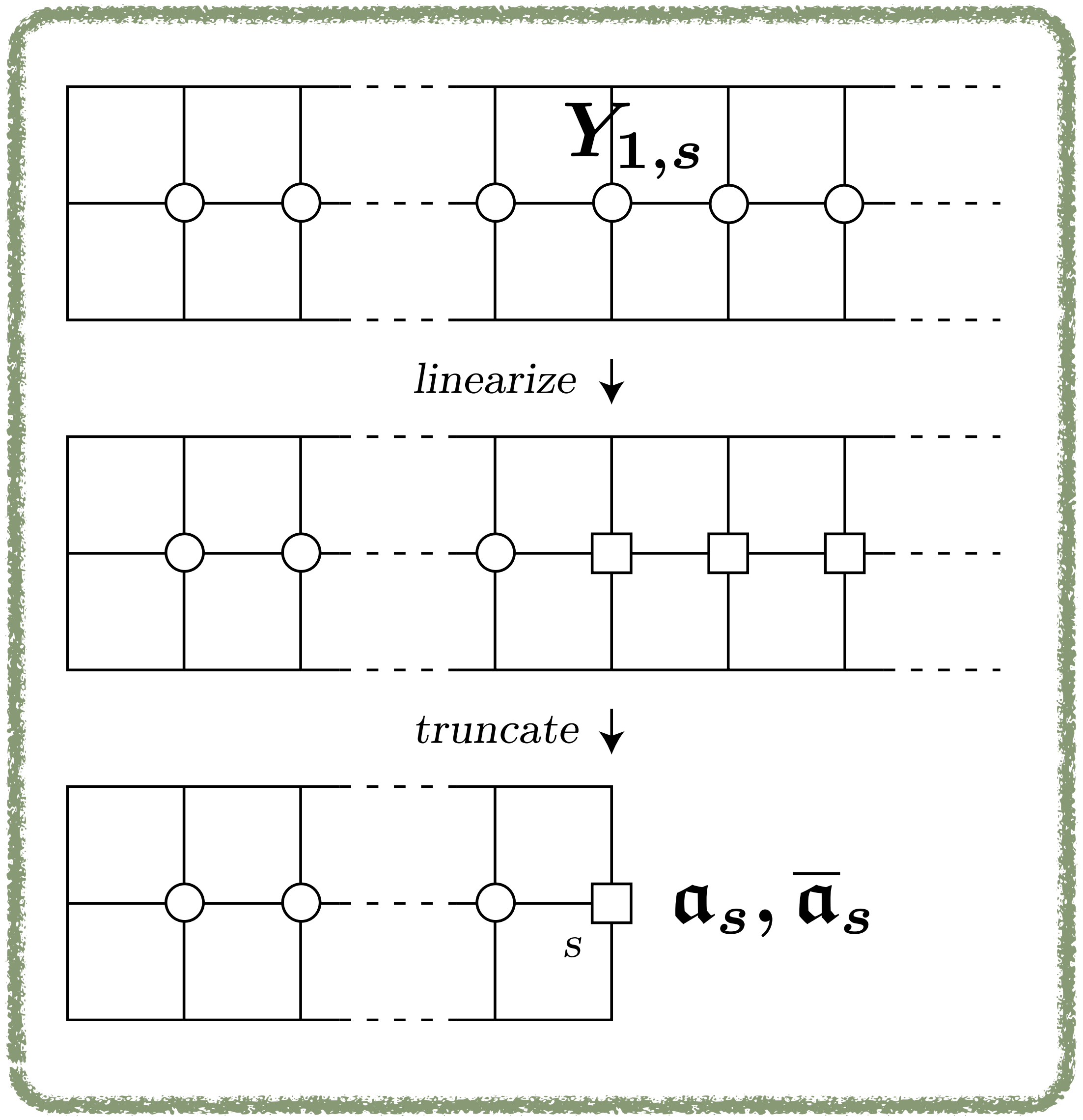
$$\log \mathfrak{a}_s = \log(1+\mathfrak{a}_s) \star K_f - \log(1+\bar{\mathfrak{a}}_s) \star K_f^{[+2-2\gamma]} + \log(1+Y_{1,s-1}^{[-\gamma]}) \star s_K + (\text{source})$$

No $Y_{1,s+1}$ nor $\mathfrak{a}_{s+1}$, $\bar{\mathfrak{a}}_{s+1}$ on the RHS!

$$\left[K_f(v) = \frac{1}{2\pi i}\frac{\partial}{\partial v}\log S_f(v), \quad S_f(v) = \frac{\Gamma\Big(\frac{g}{4i}(v+\frac{2i}{g})\Big)\Gamma\Big(-\frac{gv}{4i}\Big)}{\Gamma\Big(\frac{gv}{4i}\Big)\Gamma\Big(-\frac{g}{4i}(v-\frac{2i}{g})\Big)}\right]$$



a
(2,-2)
(2,2)
(a=0, s=0)
s



$Y_{1,s}$
linearize
truncate
s
$\mathfrak{a}_s, \bar{\mathfrak{a}}_s$

Derivation of hybrid NLIE
c) Summary of results

Hybrid NLIE

$$\log \mathfrak{a}_s = \log(1+\mathfrak{a}_s)\star K_f - \log(1+\bar{\mathfrak{a}}_s)\star K_f^{[+2-2\gamma]} + \log(1+Y_{1,s-1}^{[-\gamma]})\star s_K + (\text{source})$$

& Similar equation for $\bar{\mathfrak{a}}_s$ (We set $s=3$)

Coupling between TBA and NLIE

$$\log(1+Y_{1,s-1}) = \log(1+Y_{1,s-2})(1+\mathfrak{a}_s^{[+\gamma]})(1+\bar{\mathfrak{a}}_s^{[-\gamma]})\star s_K + (\text{source})$$

Other Y-functions obey the mirror TBA

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c.f. Old TBA equations

$$\log Y_{M|w} = \log(1+Y_{M-1|w})(1+Y_{M+1|w})\star s + \delta_{M1}\,\log\frac{1-\frac{1}{Y_-}}{1-\frac{1}{Y_+}}\,\hat{\star}\, s$$

Discussions, future works

- How to truncate the vertical direction?
- How many critical coupling constants are there in NLIE?
- What if finite size corrections are large?

- Physical roles of new variables?
- Can AdS/CFT (or strong/weak duality) be simpler?

Thank you for attention