Lectures on AdS/CFT Correspondence and Integrability

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Abstract: This is a lecture note from a summer school in 2021.[2] The main topic is the Hagedorn transitions in $\mathcal{N} = 4$ super Yang-Mills and in superstring theory on $AdS_5 \times S^5$, as well as the prediction from integrability. Appendices include an introduction to Mathematica.

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[2] The second frontier summer school for string theory, quantum field theory and information theory, held online in August 2021, hosted by the Shing-Tung Yau Center of Southeast University.

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1 Introduction

1.1 AdS/CFT Correspondence

The AdS/CFT correspondence is a conjecture about the equivalence of two theories. On one side, we have a theory of gravity or strings on thee Anti-de Sitter (AdS) spacetime in $D + 1$ dimensions. On the other side, we have a conformal field theory (CFT) in D dimensions. The AdS/CFT correspondence states that the two descriptions are dual to each other,

$$\text{Physics of gravity on AdS}_{D+1} \text{ spacetime} = \text{ Physics of CFT in } D \text{ dimensions}$$

which was originally proposed by Juan Maldacena [1].

This correspondence has many interesting features. Firstly it is a bulk-boundary correspondence, namely a CFT in D dimensions is somehow realized on the boundary of AdS spacetime in $D+1$ dimensions (the bulk). If there is a phenomenon happening on the AdS boundary, you should also be able to see the equivalent phenomenon in the bulk, and vice versa.[1]

Secondly, AdS/CFT is a concrete realization of large-N (or holographic) dualities proposed earlier by Gerard 't Hooft. He argued that the planar limit of an $SU(N)$ gauge

[1] Some people distinguish between *duality* and *correspondence*. The word 'duality' means that anything in one description has a counterpart in the other description. The word 'correspondence' means that a moderately complicated mathematical structure shows up in two different situations. We argue that AdS/CFT is an example of the duality.

theory should behave as the string worldsheet on a sphere, and the $1/N$ expansion of the gauge theory matches the genus expansion of string theory [2]. Furthermore he argued that gravity in $d+1$ dimensions should have a description in terms of a non-gravity theory in d dimensions, based on the fact that the entropy of a black hole is proportional to the area rather than the volume of the black hole [3].

Thirdly, Maldacena's idea is based on string theory in the presence of D-branes. He placed large-N coincident D3-branes on flat space. In the closed string description, D3-branes make the spacetime curved, which is interpreted as a black brane solution. The black brane solution becomes $\mathrm{AdS}_5 \times \mathrm{S}^5$ in the near-horizon limit, with N units of RR flux. In the open string description, we probe N coincident D-branes by attaching an open string ending on the branes, giving $SU(N)$ degrees of freedom. This brane setup preserves 16 supercharges and 16 superconformal charges. Thus, we discover the following correspondence

$$\begin{aligned}&\text{Type IIB superstring on } \mathrm{AdS}_5 \times \mathrm{S}^5 \text{ spacetime}\\ &\qquad \Leftrightarrow D=4\ \mathcal{N}{=}4 \text{ super Yang-Mills (SYM) theory with gauge group } SU(N). \end{aligned} \tag{1.1}$$

This is a strong-weak duality in the sense that the $\mathrm{AdS}_5 \times \mathrm{S}^5$ description is perturbative when $\lambda = \frac{R^2}{\alpha'} \gg 1$, and the SYM description is perturbative when $\lambda = N g_{\mathrm{YM}}^2 \ll 1$.[2] For this reason, it is hard to 'prove' this correspondence in a rigorous way.

The setup (1.1) is one of the most studied examples of AdS/CFT correspondence. We make comments on some of the interesting features:

- It is notoriously hard to quantize the superstring theories covariantly on the spacetime with RR-fluxes, which are sourced by D-branes.
- $\mathcal{N}{=}4$ SYM is an interacting superconformal theory in four dimensions, with the central charge equal to $\frac{\dim SU(N)}{4} = \frac{N^2-1}{4}$.
- Numerous examples of AdS/CFT have already been proposed in the literature. There exist other versions of AdS/CFT which may or may not be related to Maldacena's proposal.[3] In this regard, the setup (1.1) is special because we can study both sides precisely, as we discuss shortly.

1.2 Integrability

One way to introduce integrability is the so-called Liouville (or classical) integrability. Suppose that we have a classical system with n dynamical degrees of freedom $(q_i\,, p_i)$ with $i = 1, 2, \ldots, n$. If the system has as many conserved charges as the degrees of freedom, the system is Liouville integrable. In such a system, q_i as a function of some parameter s satisfies

$$q_i(t) - q_i(0) = \int_0^t ds\ \dot{q}_i \tag{1.2}$$

[2]Here R is the radius of AdS_5 and S^5, $1/\alpha'$ is the string tension, g_{YM} is the Yang-Mills coupling, $\mathcal{N}$ is the number of supercharges and N is the rank of the gauge group.

[3]You can propose your own version of AdS/CFT.

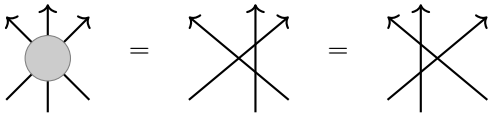
where $\dot{q}_i$ is a constant determined by the conserved charges like energy, angular momentum, etc.

Quantum integrability is more difficult to introduce, especially from a rigorous mathematical point of view. One working definition is that quantum integrable systems, typically with a mass gap, have infinite-dimensional symmetry related to Yang-Baxter relations.

Integrable systems are often solved by an Ansatz (a working hypothesis like Bethe Ansatz). An integrable QFT is mostly found in $1+1$ dimensions. Consider the S-matrix in a QFT in 1+1 dimensions, defined as the operator which evolves the asymptotic states with n particles,

$$S_{12\ldots n}\,|a_1 a_2 \cdots a_n\rangle_{in} = |a'_1 a'_2 \cdots a'_n\rangle_{out}\ . \tag{1.3}$$

When this n-body S-matrix factorizes into products of two-body S-matrices, and is insensitive to the order of the product, then this QFT is integrable.[4] This factorization condition at $n = 3$ is called the Yang-Baxter (YB) relation, which is diagrammatically presented as:

$S_{123} =$  (1.4)

By using the YB relations, we can compute physical observables, such as the energy spectrum and correlation functions. This is because behind the YB relations we find an infinite-dimensional algebra called Yangian, which guarantees the existence of an infinite number of conserved quantities. The process of how to make use of quantum integrability is well-understood, known as the quantum inverse scattering method.

One may want to know if a given system is integrable or not. This is a hard question, but there are some ways to check integrability. One way is to study whether the classical phase space trajectory is periodic or chaotic. If the trajectory is chaotic, it is unlikely that this system is integrable in the sense of Liouville. Another way is to study the level-spacing distributions. If the level-spacing pattern follows the Poisson distribution, then this system is likely to be classical or quantum integrable [5].

1.3 AdS/CFT and Integrability

The (maximally supersymmetric) AdS/CFT pair like (1.1) is believed to be integrable in the planar large N limit. The AdS/CFT dictates that the conformal dimensions Δ of a single-trace operator $\mathcal{O}_\alpha$ should correspond to the energy E of a closed string state Ψ_α on $\mathrm{AdS}_5 \times \mathrm{S}^5$ as a function of the 't Hooft coupling

$$\Delta_\alpha(\lambda) \ \leftrightarrow\ E_\alpha(\lambda). \tag{1.5}$$

A priori, we do not know which operator $\mathcal{O}_\alpha$ corresponds to which string state Ψ_α due to the strong/weak nature of the AdS/CFT duality. Remarkably, the integrability method enables us to compute yet another energy $E_\alpha^{(\mathrm{int})}(\lambda)$ at any value of λ. By comparing $E_\alpha^{(\mathrm{int})}(\lambda)$ with

[4] The opposite is not true, particularly when a QFT contains massless excitations [4].

$\Delta_\alpha(\lambda)$ at $\lambda \ll 1$, and with $E_\alpha(\lambda)$ at $\lambda \gg 1$, we can construct the dictionary between CFT operators and string states.

In this lecture I discuss partition functions, not the spectral problem. Let me finish the introduction by making a few more comments:

- The integrability method can be applied to QCD, in the sense that $\mathcal{N} = 4$ SYM can predict part of the QCD data [6, 7].
- We do not fully understand why the integrability shows up in the maximally supersymmetric setups. Whether other AdS/CFT pairs can be integrable remains a challenging question [8].
- Integrable systems can also be called exactly solvable models. However, this does not always mean that problems can be exactly solved. The computations are often quite involved, and we need to work hard to find a good Ansatz based on explicit computations using e.g. Mathematica. For this reason, we explain the basic usage of Mathematica in appendices.

1.4 Collection of reviews

There is a large amount of literature on this subject. We pick up review articles closely related to the topics in this lecture.

- Collection of reviews on integrability in AdS/CFT correspondence up to 2010 [9]
- Lecture note on integrability at ETH [10]
- Collection of lecture notes on integrability at YRIS summer school [11].
- A classic introduction to the XXX spin chain [12]
- Thermodynamic Bethe Ansatz for one-dimensional Hubbard model [13].[5]
- More classics are found in [14].

More review materials will be introduced as we discuss each topic.

1.5 Structure of this lecture note

In sections 2-5 and in section 9, we introduce $\mathcal{N} = 4$ SYM, superstring and gravity on AdS_5 with an emphasis on the partition functions and the Hagedorn behavior. Most of the materials are elementary, so students should be able to follow part of the calculations.

In sections 6-8, we summarize the minimal material for explaining TBA equations for the Hagedorn temperature in $\mathcal{N} = 4$ SYM. Some topics are advanced, and students should be able to understand basic ideas.

In appendices, we explain the basic usage of Mathematica.

2 $\mathcal{N} = 4$ Super Yang-Mills Theory

The simplest way to obtain the Lagrangian of $D = 4$, $\mathcal{N} = 4$ SYM is to start from $D = 10$, $\mathcal{N}$=1 SYM and apply trivial dimensional reduction. To be more concrete, we start from

[5] The one-dimensional Hubbard model is closely related to the asymptotic spin chain of $\mathcal{N} = 4$ SYM.

the Lagrangian:

$$\mathcal{L}_{D=10} = -\frac{1}{4g_{YM}^2}\,\mathrm{tr}\left(F_{MN}F^{MN}\right) + \text{fermions} \quad (\text{like } \lambda_{a\alpha}) \tag{2.1}$$

$$F_{KM} = [\nabla_K, \nabla_M]\,, \qquad K, M = 0, 1, \ldots, 9 \tag{2.2}$$

where F_{KM} is the field strength associated with the $SU(N)$ gauge field A_M.[6] We split A_M into two sets $A_M = (A_\mu, \phi_I)$, where $\mu = 0, 1, 2, 3$ and $I = 4, \ldots, 9$. The trivial dimensional reduction throws away ∂_I for all I, and thus

$$F_{\mu\nu} = [\nabla_\mu, \nabla_\nu]\,, \qquad F_{\mu I} = -F_{I\mu} = \nabla_\mu \phi_I\,, \qquad F_{IJ} = -g_{\mathrm{YM}}^2\,[\phi_I, \phi_J]\,. \tag{2.3}$$

By substituting this into (2.1), we obtain the (bosonic part of) $D = 4$, $\mathcal{N} = 4$ SYM Lagrangian,[7]

$$\mathcal{L}_{D=4} \to -\frac{1}{4g_{\mathrm{YM}}^2}\,\mathrm{tr}\left(F_{\mu\nu}^2 + 2\,(\nabla_\mu \phi_I)^2 - g_{\mathrm{YM}}^2\,[\phi_I, \phi_J]^2\right) + \ldots. \tag{2.4}$$

Here we list important properties of $\mathcal{N} = 4$ SYM:

- All fields in $\mathcal{N} = 4$ SYM are adjoint under the gauge group $SU(N)$. There are no dynamical quarks (i.e. the elementary fields in the fundamental representation) as in QCD.[8]
- We can count the (off-shell) degrees of freedom of each elementary field as

$$\begin{aligned} &(A_\mu)_{ij} : 4(N^2-1), && (\phi_I)_{ij} : 6(N^2-1), \\ &(\lambda_{a\alpha})_{ij} : 4 \times 2(N^2-1), && \left(\bar{\lambda}_{a\dot{\alpha}}\right)_{ij} : 4 \times 2(N^2-1) \end{aligned} \tag{2.5}$$

  We need to add auxiliary fields to match the degrees of freedom between bosons and fermions, as usual.
- This theory is maximally supersymmetric in $D = 4$; maximal in the sense that the helicity h of all elementary fields must satisfy $|h| \leqslant 2$.
- This is an example of an interacting 4d (S)CFT, already at UV. To show the conformal invariance perturbatively, we need to check that the β function of the Yang-Mills coupling is zero, and the trace of the energy-momentum tensor vanishes,[9]

$$\beta(g_{YM}) = 0 \quad \Rightarrow \quad \text{scale invariant}, \qquad \langle T_\mu{}^\mu \rangle = 0 \quad \Rightarrow \quad \text{conformal invariant}. \tag{2.6}$$

From the maximal supersymmetry and the conformal invariance, we expect that $\mathcal{N} = 4$ SYM has the global superconformal symmetry $PSU(2,2|4)$.[10] The bosonic subgroup of

[6] The commutator of the covariant derivatives is well-defined, as one can see from $\nabla_K \phi = \partial_K \phi + i g_{\mathrm{YM}}\,[A_K, \phi]$ for any ϕ.

[7] Strictly speaking, we should redefine the Yang-Mills coupling $g_{\mathrm{YM}}(D = 10)$ to $g_{\mathrm{YM}}(D = 4)$ after integrating $\mathcal{L}_{D=10}$ over the internal 6-dimensional space.

[8] One can construct baryons in $\mathcal{N} = 4$ SYM by using a D5-brane [15].

[9] Proving the conformal invariance of $\mathcal{N} = 4$ SYM non-perturbatively is a challenging problem [16].

[10] The superconformal symmetry has been classified in [17]; see also [18, 19].

$PSU(2,2|4)$ is $SU(2,2) \times SU(4)_R$, where $SU(2,2)$ represents 4d conformal symmetry and $SU(4)_R$ is the R-symmetry of the $\mathcal{N} = 4$ supersymmetry. The same symmetry shows up on the superstring theory side, since the spacetime $\mathrm{AdS}_5 \times \mathrm{S}^5$ is realized as a supercoset

$$\frac{PSU(2,2|4)}{USp(2,2) \times USp(4)} \supset \frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)} = \mathrm{AdS}_5 \times \mathrm{S}^5. \tag{2.7}$$

More details about $\mathcal{N} = 4$ SYM can be found e.g. in [20–22].

Consider the Lie superalgebra $\mathfrak{psu}(2,2|4)$. Its Cartan subalgebra consists of six elements; the conformal dimensions Δ, Lorentz spins (S_1, S_2), and R-charges (J_1, J_2, J_3). The Lorentz spins and R-charges correspond to the angular momenta along the compact directions of $\mathrm{AdS}_5 \times \mathrm{S}^5$, and thus they are quantized. The conformal dimensions can be an arbitrary real number that may depend on the coupling constant.

3 Hagedorn Transition in Gauge Theory

3.1 Partition Function

Let us compute the thermal partition function of $\mathcal{N} = 4$ SYM on $S^1 \times S^3$ at zero coupling. The spacetime $S^1 \times S^3$ is constructed by taking the polar coordinates in the flat Euclidean space, $\mathbb{R}^4 \simeq \mathbb{R} \times S^3$, and by periodically identifying the radial direction with the period of $\beta = 1/T$. The radial direction is regarded as time, and T is the temperature of the system. This also implies that the Hamiltonian (the generator of time translation) is the dilatation operator D (the generator of overall scaling). Since $\mathcal{N} = 4$ SYM is superconformal, only the ratio of the radii between S^1 and S^3 is meaningful. We set the radius of S^3 to 1.[11]

The partition function is given by,

$$Z_{S^1 \times S^3} = \int DA_\mu D\phi_I D\lambda_A D\bar{\lambda}_A \; e^{-S} = \mathrm{tr}\left(e^{-\beta D}\right). \tag{3.1}$$

We should integrate over gauge-invariant states in the path-integral formalism, and should take the trace over gauge-invariant states in the Hamiltonian formalism.

We argue that all states should satisfy the Gauss law constraint on S^3. The charge density of $SU(N)$ gauge fields at weak coupling is given by

$$j^0 = \nabla_\mu F^{\mu 0} = \partial_k \, \partial^k A^0 - \partial_k \, \partial^0 A^k + O(g_{\mathrm{YM}}) \tag{3.2}$$

where $k = 1, 2, 3$ and $\mu = 0, 1, 2, 3$. The second term disappears if we choose the gauge $\partial_k A^k = 0$. Now the conserved charge is given by

$$Q = \int_{S^3} d^3x j^0 = \int_{S^3} d^3x \, \partial_k \partial^k A^0 = 0 \tag{3.3}$$

where we used the fact that the integral of a total derivative is equal to boundary terms, and $\partial S^3 = \varnothing$. Physically, this means that the total $SU(N)$ charge over S^3 is zero, which

[11] CFT has the unique vacuum owing to the state-operator correspondence. The radii break the conformal symmetry, which makes it possible that the theory undergoes a phase transition.

is called Gauss law constraint on S^3. This argument fails for $\mathbb{R}^3$ because there can be an extra charge at infinity. A non-trivial distribution $j^0(x)$ is allowed as long as its integral over S^3 vanishes. Note that the Gauss law constraint is a result of the topology of S^3, and not related to the QCD color confinement that comes from the strong coupling dynamics.

What kind of local degrees of freedom contribute to $j^0(x)$ is a non-trivial question. If all local degrees of freedom are $SU(N)$ singlets, then the Gauss law constraint is satisfied trivially. By $SU(N)$ singlets, we mean composite operators whose color indices are contracted, e.g. $\mathrm{tr}\,(\phi_1\phi_2\dots)=\sum_{a,b,\dots}(\phi_1)_{ab}(\phi_2)_{bc}\cdots$. We may also imagine the situation that the local degrees of freedom are $SU(N)$ adjoints, because all elementary fields in the $\mathcal{N}=4$ SYM Lagrangian are in the adjoint representation. The $SU(N)$ adjoints are a collection of N^2-1 fields like $\{(\phi_I)_{ab}\}$ before taking a trace. The two situations correspond to different vacua of the theory. We should compare the free energy

$$Z_{S^1\times S^3}=e^{-\beta F} \tag{3.4}$$

to determine the true vacuum.

A physical expectation of the phase space of $\mathcal{N}=4$ SYM is shown in Table 1. Around $T=0$, the color degrees of freedom are confined due to the Gauss law constraints. The free energy is of order 1, and typical states are multi-trace operators. When $T\gg 1$, a large number of virtual particles with color degrees of freedom is created and annihilated. The Gauss law constraints become unimportant, and the free energy becomes of order N^2. The phase transition occurs at the critical temperature T_H, which is called the Hagedorn temperature.

Temperature	Below T_H	Above T_H
Color degrees of freedom	Confined	Deconfined
Free energy	$O(N^0)$	$O(N^2)$
Typical states	Multi-trace operators	Particles along color flux tubes

Table 1. Expected phase space of $\mathcal{N}=4$ SYM.

3.2 Counting single letters

In order to check our expectation in Table 1, we compute $Z_{S^1\times S^3}=\mathrm{tr}\left(e^{-\beta D}\right)$ at tree-level, by counting gauge-invariant local operators of $\mathcal{N}=4$ SYM at a low-temperature phase [23–25]. The gauge-invariant operators are equivalent to multi-trace operators, which are products of single-trace operators. A general single-trace operator can be written as

$$O=\mathrm{tr}\left(\mathcal{W}^{A_1}\mathcal{W}^{A_2}\cdots\mathcal{W}^{A_L}\right) \tag{3.5}$$

where $\mathcal{W}^A$ is an adjoint field of $SU(N)$, also called a single letter of $\mathcal{N}=4$ SYM. Written explicitly, a single letter should be chosen from

$$\mathcal{W}^A \in \Big\{ \nabla_{(\mu_1}\nabla_{\mu_2}\cdots\nabla_{\mu_s)}\phi^I,\ \bar{\not\nabla}\bar{\not\nabla}\ldots\bar{\not\nabla}\lambda_{a\alpha},\ \bar{\not\nabla}\bar{\not\nabla}\ldots\bar{\not\nabla}\bar{\lambda}_{a\dot\alpha},\ \nabla_{(\mu_1}\nabla_{\mu_2}\cdots\nabla_{\mu_s)}F_{\mu\nu} \Big\} \qquad (s\in\mathbb{Z}_{\geqslant 0}). \tag{3.6}$$

Since $[\nabla_\mu\,,\nabla_\nu]=F_{\mu\nu}$, we can always symmetrize $\nabla_{\mu_1}\nabla_{\mu_2}\cdots\nabla_{\mu_s}$ with respect to Lorentz indices.

Let us define $x\equiv e^{-\beta}$ and count independent on-shell degrees of freedom. There are six scalars ϕ^I with $I=1,\ldots,6$ of dimensions $\Delta_0=1$. We subtract the degrees of freedom killed by the equations of motion, $\phi^I-\nabla^2\phi^I=0$. In total, they contribute to the partition function Z by $6x-6x^3$. For each ϕ^I, its conformal descendants exist as shown in (3.6),

$$\phi \ \mapsto\ \nabla_{(\mu_1}\nabla_{\mu_2}\cdots\nabla_{\mu_s)}\phi, \qquad (\mu_k=0,1,2,3). \tag{3.7}$$

Therefore, the contribution to Z from scalars is

$$Z_{\text{scalars}}=\frac{6x-6x^3}{(1-x)^4}\,. \tag{3.8}$$

The denominator can be expanded as $(1-x)^{-4}=1+4x+10x^2+\cdots$, and the coefficients agree with the counting of (3.7). For example, $4x$ corresponds to $\nabla_\mu\phi$ and $10x^2$ corresponds to $\nabla_{(\mu}\nabla_{\nu)}\phi$.

The same argument can be repeated for fermions $\lambda_{a\alpha}$ and $\bar\lambda_{a\dot\alpha}$, where $a=1,2,3,4$ is the $SU(4)_R$ index and $\alpha,\dot\alpha=1,2$ are Lorentz spinor indices. They have the canonical dimensions $\Delta_0=3/2$, and their equations of motion are $\lambda_{a\alpha}-(\bar{\not\nabla}\lambda_a)_\alpha=0$ and $\bar\lambda_{a\dot\alpha}-(\bar{\not\nabla}\bar\lambda_a)_{\dot\alpha}=0$. Thus the contribution from fermions is

$$Z_{\text{fermions}}=\frac{16\,x^{3/2}-16\,x^{5/2}}{(1-x)^4}\,. \tag{3.9}$$

Note that fermions are anti-periodic when going around the thermal circle S^1_β. If we want to impose anti-periodicity, we should multiply x by $e^{2\pi i}$, so that $(e^{2\pi i}x)^{1/2}=-x^{1/2}$.

There are six gauge fields $F_{\mu\nu}=[\nabla_\mu,\nabla_\nu]$ with $\mu,\nu=0,1,2,3$ at $\Delta_0=2$, and this contribution is $Z_{\text{gauge}}=6x^2/(1-x)^4+\ldots$. Then we subtract the degrees of freedom corresponding to the equations of motion and Bianchi identities,

$$0=\nabla^\mu F_{\mu\nu} \tag{3.10}$$

$$0=\nabla^\mu\tilde F_{\mu\nu}=\varepsilon_{\mu\nu\rho\sigma}\nabla^\mu F^{\rho\sigma}. \tag{3.11}$$

Since both equations contain one derivative ∇_μ, they have fewer descendants compared to (3.7). We count the additional contribution as $-8x^3/(1-x)^3$.

This is not the end of the story about counting the descendants. Consider trivial

identities

$$X_{\mu\nu} \equiv (\nabla^\mu \nabla^\nu + \nabla^\nu \nabla^\mu)\, F_{\mu\nu} = 0 \tag{3.12}$$

$$\tilde{X}_{\mu\nu} \equiv (\nabla^\mu \nabla^\nu + \nabla^\nu \nabla^\mu)\, \tilde{F}_{\mu\nu} = 0 \tag{3.13}$$

where we do not sum over μ, ν. They cancel part of the contribution from the conformal descendants of $F_{\mu\nu}$ to Z_{gauge}.

We can create $X_{\mu\nu}$ for each $F_{\mu\nu}$, so there exist 6 constraint equations $X_{\mu\nu} = 0$. However, when we sum over μ in $X_{\mu\nu}$, we encounter the equations of motion which we have already subtracted,

$$\sum_\mu X_{\mu\nu} = \sum_\mu (\nabla^\mu \nabla^\nu + \nabla^\nu \nabla^\mu)\, F_{\mu\nu} = 0. \tag{3.14}$$

This implies that 4 equations are doubly counted. Furthermore, when we sum over μ, ν we have a trivial relation $\sum_{\mu,\nu} X_{\mu\nu} = 0$. Thus we doubly counted 1 equation in (3.14). In total $6 - 4 + 1 = 3$. We apply the same argument for $\tilde{X}_{\mu\nu}$, and they contribute to Z by $-6x^4/(1-x)^4$.

The total contribution from gauge fields to the partition function is

$$Z_{\text{gauge}} = \frac{6x^2}{(1-x)^4} - \frac{8x^3}{(1-x)^3} - \frac{6x^4}{(1-x)^4}\,. \tag{3.15}$$

We have counted all single letters in (3.6). Let us define the single-letter function by collecting Z_{scalars}, Z_{fermions}, Z_{gauge} in (3.8), (3.9), (3.15) as

$$\zeta(x) = \zeta_B(x) + \zeta_F(x), \qquad \zeta_B(x) = Z_{\text{scalars}} + Z_{\text{gauge}}\,, \qquad \zeta_F(x) = Z_{\text{fermions}}\,. \tag{3.16}$$

Explicitly, this function is

$$\zeta(x) = \frac{2x(3 - x^{1/2})}{(1-x^{1/2})^3} = 6x + 16\, x^{3/2} + (24+6)x^2 + \cdots \tag{3.17}$$

where $6x$ comes from ϕ^I, $16\, x^{3/2}$ from $\lambda_{a\alpha}$ and $\bar{\lambda}_{a\dot{\alpha}}$, and $(24+6)x^2$ from $\nabla_\mu \phi^I$, $F_{\mu\nu}$.

The single-letter function $\zeta(x)$ is related to the partition function of single-trace operators, $Z_{\text{single-tr}}(x)$. The original partition function $Z_{S^1 \times S^3} = \text{tr}\left(e^{-\beta D}\right)$ contains the contribution from all multi-trace operators, and the two quantities are related by the plethystic exponential [26],

$$\text{tr}\left(e^{-\beta D}\right) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\, Z_{\text{single-tr}}(e^{-n\beta})\right). \tag{3.18}$$

Our argument so far is purely classical. However, the partition functions receive quantum corrections because the spectrum of the dilatation operator D depends on the 't Hooft coupling λ. This property is different from the computation of superconformal indices, which are SUSY protected.

3.3 Pólya Enumeration Theorem

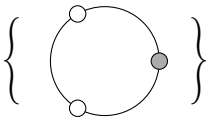
We compute the single-trace partition function $Z_{\text{single-tr}}(x)$ again by an alternative combinatorial argument, called Pólya enumeration theorem (or power enumeration theorem). This method works well if we do not introduce more than one chemical potentials.

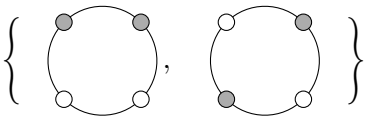
Consider the problem of counting the number of maps from a given domain to a given range. For simplicity, we choose the domain as $\{1, 2, \ldots, L\}$ and the range as $\{Z, Y\}$.[12] We want to count the number of single-trace operators of length L made out of Z or Y,

$$\text{tr}(\underbrace{ZZ\ldots YY\ldots}_{L}) \quad \leftrightarrow \quad \text{Map}\,(\mathbb{Z}_L\backslash\text{Domain} \to \text{Range}) \tag{3.19}$$

where $\mathbb{Z}_L$ comes from the cyclic translation symmetry of single-traces. Define the weight function $c(x, y) = x + y$ such that $x^m y^n$ corresponds to $Z^m Y^n$ with $m + n = L$. If we want to count the number of all possible single-trace operators of $\mathcal{N} = 4$ SYM, we should replace $c(x, y)$ by $\zeta(x)$ with $x = e^{-\beta}$.

Some examples of counting are illustrated as

$Z^2Y =$ 

$Z^2Y^2 =$  (3.20)

where the white and gray circles represent Z and Y, respectively. This suggests that there is one single-trace operator of weight x^2y, and two single-trace operators of weight x^2y^2. It will turn out that the single-trace partition function has the following series expansion,

$$\begin{aligned} Z_{\text{single-tr}} = 1 + (x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) \\ + (x^4 + x^3y + 2x^2y^2 + xy^3 + y^4) + \cdots . \end{aligned} \tag{3.21}$$

The coefficients in front of x^2y and x^2y^2 are consistent with the examples (3.20).

Pólya enumeration theorem gives the generating function of the number of inequivalent maps (3.19). The (weighted version of) theorem relates this generating function to the cycle index of the cyclic group $\mathbb{Z}_L$. The cycle index of $\mathbb{Z}_L$ is given by

$$\text{CI}_{\mathbb{Z}_L}(t_1, t_2, \ldots, t_L) = \frac{1}{L} \sum_{h|L} \text{Tot}(h)\, t_h^{L/h} \tag{3.22}$$

where $h|L$ (h divides L) means we sum over h such that L/h is an integer, and $\text{Tot}(h)$ is

[12] Here (Z, Y) are the complex scalars of $\mathcal{N} = 4$ SYM, defined by $Z = \phi_5 + i\phi_6$ and $Y = \phi_3 + i\phi_4$.

Euler's totient function,[13]

$$\mathrm{Tot}(h) = \sum_{d=1}^{h} \delta(\gcd(h,d),1) \qquad (\mathrm{Tot}(1),\mathrm{Tot}(2),\mathrm{Tot}(3),\ldots) = (1,1,2,2,4,\ldots) \tag{3.23}$$

which counts the number of positive integers no more than h that are relatively prime to h. Skipping the details, the generating function for the number of single-trace operators made of (Z,Y) is given by[14]

$$Z^{\text{Pólya}}(x,y) = \sum_{L=1} \frac{1}{L} \sum_{h|L} \mathrm{Tot}(h) c(x^h, y^h)^{L/h} \tag{3.24}$$

By performing the sum over L, we find

$$Z^{\text{Pólya}}(x,y) = -\sum_{h=1}^{\infty} \frac{\mathrm{Tot}(h)}{h} \log\Big(1 - c(x^h, y^h)\Big)\,. \tag{3.25}$$

The generating function of all possible single-traces of $\mathcal{N}=4$ SYM is

$$\begin{aligned} Z_{\text{single-tr}}(x) &= -\zeta(x) - \sum_{h=1}^{\infty} \frac{\mathrm{Tot}(h)}{h} \log\left[1 - \zeta\left((-1)^{n+1} x^n\right)\right] \\ &= 21x^2 + 96x^{5/2} + 376x^3 + \cdots \end{aligned}$$

where we subtracted $\zeta(x)$ to impose the traceless condition $\mathrm{tr}\,\mathcal{W}^A = 0$ for the $SU(N)$ gauge group. The factor $(-1)^{n+1}$ is added to ensure that fermions are anti-periodic when going around S^1_β.

The Euler's totient function satisfies

$$\sum_{h|L} \mathrm{Tot}(h) = L, \qquad \sum_{k=1} \frac{\mathrm{Tot}(k)}{k} \log(1-x^k) = -\frac{x}{1-x}\,. \tag{3.26}$$

By applying the plethystic exponential (3.18), we obtain the multi-trace partition function as

$$\mathrm{tr}\left(e^{-\beta D}\right) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} Z_{\text{single-tr}}(x^m)\right) = \exp\left(-\sum_m \frac{\zeta(x^m)}{m}\right) \cdot \prod_{m=1}^{\infty} \frac{1}{1-\zeta(x^m)} \tag{3.27}$$

We see that both $Z_{\text{single-tr}}$ and $Z_{S^1\times S^3} = \mathrm{tr}\left(e^{-\beta D}\right)$ diverge if $\zeta(x_H) = 1$ at some x_H. The possible values of x_H are

$$x_H = e^{-\beta_H} = 7 \pm 4\sqrt{3} \quad \sim \quad \{0.07, 13.9\}. \tag{3.28}$$

[13] gcd is the greatest common divisor.

[14] We added the sum over L because the terms with different L do not mix. The statement and proof of the weighted version of Pólya enumeration theorem can be found in [27].

We find $x_H = 7-4\sqrt{3}$ is the correct value because $e^{-\beta} < 1$ for $\beta > 0$. This is the Hagedorn temperature. Further developments about $Z_{S^1\times S^3}$ can be found in [28–31].

4 Hagedorn Temperature in String Theory

We discuss the Hagedorn transition of superstring theory on the flat spacetime following [32].

4.1 Hagedorn behavior

Consider the low-temperature expansion of the partition function

$$\mathrm{tr}\left(e^{-\beta D}\right) = \mathrm{tr}\left(x^D\right) \equiv \sum_{m=0}^{\infty} C_{m/2}\, x^{m/2}. \tag{4.1}$$

Naively, large m terms are small because $x = e^{-\beta} < 1$. This expectation is not true if the number of states $C_{m/2}$ grows exponentially with respect to m,

$$C_{m/2} \sim e^{\alpha m/2} \qquad \Rightarrow \qquad \mathrm{tr}\left(x^D\right) \sim \sum_m e^{\frac{m}{2}(\alpha-\beta)}\,. \tag{4.2}$$

The partition function diverges at $\beta_* = \alpha$. The exponential growth of the number of states is typical in string theory, and is called Hagedorn behavior.

4.2 Density of states

Consider the spacetime (not worldsheet) partition function of IIB superstring theory on $S^1_\beta \times \mathbb{R}^9$ at zero string coupling (i.e. $g_s = 0$), which is equivalent to the partition function of supergravity on $S^1_\beta \times \mathbb{R}^9$ with infinite types of particles with mass $\alpha' m^2 = 0, 4, 8, \ldots$. We write this partition function as

$$Z = \int \prod_n [d\phi_n d\psi_n]\, e^{-\sum_n S[\phi_n,\psi_n]}, \tag{4.3}$$

where $S[\phi_n, \psi_n]$ is the action for the free bosons and fermions in ten dimensions. Since all fields are free, we can compute its logarithm as [33]

$$\log Z = \int dm\ \rho(m) \int \frac{d^9k}{\sqrt{k^2+m^2}} \log\left(\frac{1+e^{-\beta\sqrt{k^2+m^2}}}{1-e^{-\beta\sqrt{k^2+m^2}}}\right), \tag{4.4}$$

where $\rho(m)$ is the density of states with mass m. The superstring partition function takes the form

$$Z = \Pi_{NS,NS} + \Pi_{NS,R} + \Pi_{R,NS} + \Pi_{R,R}\,. \tag{4.5}$$

We compute only the first term, because other sectors are not important for computing the Hagedorn temperature at least for $g_s = 0$.

Let us calculate $\rho(m)$ in the NS-NS sector. We apply the usual CFT techniques in the flat spacetime, which is to count the number of states of the form

$$|0\rangle\,, \quad \alpha^\mu_{-1}\,|0\rangle\,, \quad \psi^\mu_{-1}\,|0\rangle\,, \quad \alpha^\mu_{-1}\alpha^\nu_{-1}\,|0\rangle\,, \quad \psi^\mu_{-2}\,|0\rangle\,, \quad \ldots\,, \tag{4.6}$$

where $\{\alpha^\mu_{-m}, \psi^\nu_{-n}\}$ are the (left-moving) bosonic and fermionic oscillators. The commutation relations are given by

$$[\alpha^\mu_m, \alpha^\nu_n] = m\,\eta^{\mu\nu}\delta_{m+n,0}\,, \qquad \{\psi^\mu_m, \psi^\nu_n\} = \eta^{\mu\nu}\delta_{m+n,0}\,. \tag{4.7}$$

There are also right movers $\{\bar\alpha^\mu_{-m}, \bar\psi^\nu_{-n}\}$. The mass is given by

$$\alpha' m^2 = 4N_{\rm tot} = 4\,(N_B + N_{\bar B} + N_F + N_{\bar F}) \tag{4.8}$$

where $N_{\rm tot}$ is the sum of the number operators for left- and right-moving bosons and fermions.

We compute the quantity

$$\mathrm{tr}\,\left(x^{N_B+N_F}\,\bar x^{N_{\bar B}+N_{\bar F}}\right) \tag{4.9}$$

and impose the level-matching conditions later. We are interested in the case $N_{\rm tot} \gg 1$. First we count the states created by bosons. Such states are generally given by $(\alpha_{-m})^p(\bar\alpha_{-n})^q\,|0\rangle$ for all $p, q \in \mathbb{Z}_{\geqslant 0}$, and they contribute to (4.9) as

$$\sum_{p\geqslant 0}\sum_{q\geqslant 0} x^{mp}\,\bar x^{nq} = \frac{1}{1-x^m}\frac{1}{1-\bar x^n}, \qquad m, n \in \mathbb{Z}_{\geqslant 1}\,. \tag{4.10}$$

Second, we count the states created by fermions. They are given by $(\psi_{-m})^p(\bar\psi_{-n})^q\,|0\rangle$ for $p, q = 0, 1$, and contribute to (4.9) as

$$\sum_{p=0,1}\sum_{q=0,1} x^{mp}\,\bar x^{nq} = (1+x^m)(1+\bar x^n). \tag{4.11}$$

Recall that both bosonic and fermionic oscillators carry the spacetime index μ. Since there are 8 physical directions, we sum over all $8_B + 8_F$ directions to get

$$\mathrm{tr}\, x^{N_B+N_F}\,\bar x^{N_{\bar B}+N_{\bar F}} = \prod_{m=1}^{\infty}\left(\frac{1+x^m}{1-x^m}\right)^8\left(\frac{1+\bar x^m}{1-\bar x^m}\right)^8 \equiv \sum_{n,\bar n} d_{n,\bar n}\,x^n\,\bar x^{\bar n}. \tag{4.12}$$

Now we impose the level-matching conditions $n = \bar n$, and redefine $z \equiv \bar x = x$. It follows that

$$\Pi(z) \equiv \sum_N d_{N,N}\,z^{2N} = \frac{1}{\theta_4(0|\tau)^{16}}\,, \tag{4.13}$$

where θ_4 is the elliptic theta function with the modulus τ defined by $z = e^{\pi i\tau}$. This is the partition function for the NS-NS sector (4.5).

From (4.13), we can determine $d_{N,N}$ as the residue,

$$d_{N,N} = \oint_{z=0} \frac{dz}{2\pi i} \frac{\Pi(z)}{z^{2N+1}}. \tag{4.14}$$

The integrand $\Pi(z)$ is singular at $z = 1$ or $\tau = 0$. Mathematica expands $\Pi(z)$ as

$$\Pi(z) \sim \left(\frac{1-z}{4\pi}\right)^8 \exp\left(2\pi^2 \frac{1+z}{1-z}\right), \qquad (z \sim 1). \tag{4.15}$$

We evaluate (4.14) by the saddle point approximation at large N. Thanks to the factor z^{-2N-1} the saddle point is shifted to $z_* \sim 1 - \pi\sqrt{\frac{2}{N}}$. The integral becomes

$$(d_{N,N})_* \sim \frac{e^{4\pi\sqrt{2N}}}{4096N^4} \qquad \Rightarrow \qquad \rho(m) \sim \frac{e^{m\sqrt{8\pi^2\alpha'}}}{16\,(\alpha' m^2)^4} \quad (m \gg 1), \tag{4.16}$$

where we used $\alpha' m^2 = 4N_{\rm tot}$ in (4.8).

Let us simplify the kinematic factor in the spacetime partition function (4.4). By taking the polar coordinates in $\vec{k}$, we get

$$\begin{aligned} \int \frac{d^9k}{\sqrt{k^2+m^2}} \log\left(\frac{1+e^{-\beta\sqrt{k^2+m^2}}}{1-e^{-\beta\sqrt{k^2+m^2}}}\right) &= \mathrm{Vol}(S^9) \sum_{\text{odd } n} \frac{2}{n} \int \frac{dk\, k^8}{\sqrt{k^2+m^2}} e^{-n\beta\sqrt{k^2+m^2}} \\ &= \mathrm{Vol}(S^9) \sum_{\text{odd } n} \frac{2}{n} \left(\frac{m}{n}\right)^4 K_4(n\beta m), \end{aligned} \tag{4.17}$$

where K_4 is the modified Bessel function. Substituting the density of states (4.16), the partition function becomes

$$\ln Z \sim \sum_n \int dm\ \rho(m) K_4(n\beta m) \sim \sum_n \int_0^\infty dm\ \exp\left(m\sqrt{8\pi^2\alpha'} - mn\beta\right). \tag{4.18}$$

This integral diverges at $n = 1$ if

$$\beta \leqslant \beta_H = \sqrt{8\pi^2\alpha'}\,. \tag{4.19}$$

The critical value β_H is the Hagedorn temperature in string theory.

Note that this Z in (4.18) is the partition function of single strings. For the comparison with (3.27) we should compute the partition function of multiple strings.[15]

4.3 Closed string tachyons

The Hagedorn temperature in string theory can be understood as the special radius of the thermal circle such that the closed string winding mode around S^1_β becomes tachyonic. Above this temperature closed string tachyons will condensate, deforming the background

[15] We thank Troels Harmark for this comment.

spacetime [34]. In the AdS background, this interpretation is consistent with the Hawking-Page transition, where AdS becomes AdS-BH [35].

5 Thermodynamics of AdS Gravity

We discuss the thermodynamical properties of gravity in AdS_5 . If we start from $AdS_5 \times S^5$ superstring, we will obtain $AdS_5 \times S^5$ IIB supergravity by taking the $\alpha' \to 0$ limit, i.e. by neglecting all massive string states. The $AdS_5 \times S^5$ supergravity is equivalent to supergravity on AdS_5 with an infinite tower of KK modes coming from S^5. We may neglect fermions in the classical approximation, which gives gravity in AdS_5 .

The contents in this section are based on [36, 37] as well as reviews [38, 39].

5.1 Gravity Action

The gravity action for AdS_5 is

$$S = -\frac{1}{16\pi G_N}\int d^5x\sqrt{-g}\,(R - 2\Lambda)\,, \qquad \Lambda = -\frac{6}{L^2}\,, \tag{5.1}$$

where L is the AdS radius. The equations of motion are

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{5.2}$$

Since AdS is non-compact, we need to regularize (5.1) by adding various boundary (and boundary counter) terms.

5.2 Black hole temperature (conical defect trick)

Let us write a general black hole metric as

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + \dots\,. \tag{5.3}$$

The event horizon is defined by $g_{rr} = \infty$, namely $V(r_0) = 0$. We introduce the imaginary time $t = i\tau$ and expand the metric around $r = r_0$ as

$$ds^2 \sim \epsilon V'(r_0)d\tau^2 + \frac{d\epsilon^2}{\epsilon V'(r_0)} + \cdots\,, \qquad (\epsilon = r - r_0 \ll 1). \tag{5.4}$$

We want to impose a periodic boundary condition on the τ direction, and interpret the period as the inverse temperature of the black hole. The period of τ is determined by demanding the absence of conical singularity on the event horizon $\epsilon = 0$.

In order to check possible singularity, we rewrite the metric as a locally-flat form,

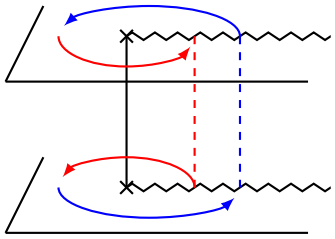
$$ds^2 = dR^2 + R^2d\theta^2 + \cdots = dzd\bar{z} + \cdots\,, \qquad (z = Re^{i\theta}). \tag{5.5}$$

If θ is periodic with the 2π period, then the metric does not have a conical singularity at $R = 0$. If $\theta \simeq \theta + 2\pi/\alpha$, we define $\Theta \equiv \alpha\theta$ and rewrite the metric as

$$ds^2 = dR^2 + (R\alpha)^2 d\Theta^2, \qquad (\Theta \simeq \Theta + 2\pi). \tag{5.6}$$

An example of the conical singularity is illustrated below:

$$z = \sqrt{w}, \tag{5.7}$$



If $\theta \sim \theta + \pi$, the variable $w = z^2$ has the 2π period. Now a point $w = w_0$ is inversely mapped to two points $z_0 = \pm\sqrt{w_0}$. The origin $z = 0$ has a non-zero curvature in general relativity, which is a defect operator insertion in the CFT language.

By comparing the two equations (5.4) and (5.5), we obtain

$$\frac{d\epsilon}{dR} = \sqrt{\epsilon V'}\,. \tag{5.8}$$

With the boundary condition $\epsilon = 0$ at $R = 0$, we get

$$\epsilon(R) = \frac{R^2 V'(r_0)}{4}\,. \tag{5.9}$$

The metric becomes

$$ds^2 = dR^2 + \left(\frac{RV'(r_0)}{2}\right)^2 d\tau^2 \tag{5.10}$$

When τ has the period $\beta = 1/T$, θ in (5.5) should have the period 2π. This means that the black hole (or Bekenstein-Hawking) temperature $T_{\rm BH}$ is given by

$$T_{\rm BH} = \frac{V'(r_0)}{4\pi}\,. \tag{5.11}$$

5.3 Black holes in AdS

In order to write down the full metric (5.3) of an AdS_5 black hole, we need to specify the topology of the event horizon. The AdS black hole with a planar horizon is

$$d\tilde{s}^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + \frac{r^2}{L^2}\sum_{i=1}^{3} dx_i^2\,, \qquad f(r) = \frac{r^2}{L^2}\left(1 - \frac{r_0^4}{r^4}\right), \tag{5.12}$$

and the black hole with a spherical horizon is

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_3\,, \qquad V(r) = 1 + \frac{r^2}{L^2} - \frac{\mu}{r^2}\,, \tag{5.13}$$

where μ is the black hole mass.

The metric $d\tilde{s}^2$ in (5.12) is invariant under the rescaling

$$(x_i, r, r_0) \to (a x_i, r/a, r_0/a), \tag{5.14}$$

which corresponds to the scale invariance of CFT_4 at the boundary. The metric ds^2 in (5.13) is *not* invariant under the rescaling, indicating a possible phase transition at a finite β. The spacetime (5.13) should be dual to CFT_4 on $S^1_\beta \times S^3$.

By applying the conical defect trick to (5.13), we obtain the horizon temperature

$$\beta_{\mathrm{BH}} = \frac{4\pi}{V'} = \frac{2\pi r_+ L^2}{L^2 + 2r_+^2}\,, \tag{5.15}$$

where r_+ is the horizon radius, namely the outermost solution of $V(r) = 0$. By plotting β_{BH} as a function of r_+ as in Figure 1, we find that there exists a minimum black hole temperature

$$\beta_m = \frac{\pi L}{\sqrt{2}}\,. \tag{5.16}$$

When $\beta < \beta_m$, there exist two values of r_+ giving the same temperature, r_+^{large} and

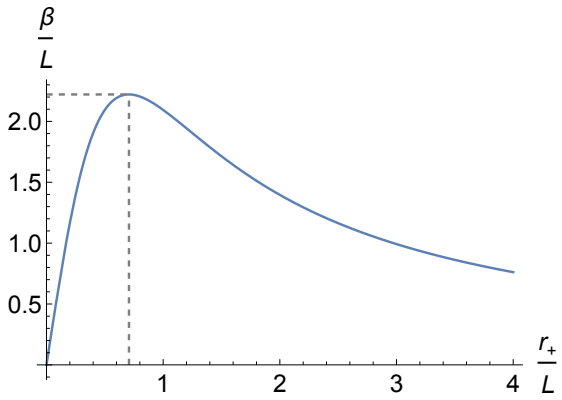


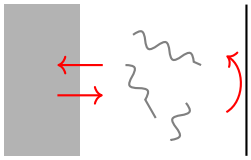
Figure 1. The temperature of AdS_5 black hole with a spherical horizon, which shows $\beta \leqslant \beta_m$.

r_+^{small} . The thermodynamical properties of a small AdS black hole is similar to those in the flat space-time, having a negative specific heat.[16] The large AdS black hole has a positive specific heat, and a new object in AdS. The large black hole gives the dominant contribution to the partition function $Z = e^{-S}$.

Physically, we can think of AdS as a finite-size box in the sense that a particle reaches the horizon in a finite proper time [38]. The black hole and thermal radiation reach thermal equilibrium, because the radiation reflects back from the AdS boundary. The free energy of a large AdS black hole *increases* by absorbing radiation, while it decreases for a small

[16] One has to calculate the free energy of a black hole to check this property; see Section 5.4.

AdS black hole.



(5.17)

5.4 On-Shell Euclidean Action

We compute the classical partition function of the on-shell Euclidean action

$$Z = e^{-S} \equiv e^{-\beta F}. \tag{5.18}$$

Various classical solutions contribute to this partition function, such as a small AdS black hole, a large AdS black hole and thermal AdS solutions.[17] Due to the non-compactness of the AdS spacetime, we need to regularize the action. We use I for the regularized on-shell action on $M = \text{AdS}_5$, which is given by

$$\begin{aligned} I = &- \frac{1}{16\pi G_N} \int d^5x\sqrt{g}\,(R(g) - 2\Lambda) \\ &+ \frac{1}{8\pi G_N} \int_{r=1/\epsilon} d^4x\sqrt{h}\,(K_{AdS} - K_{\text{flat}}) \\ &+ \frac{1}{8\pi G_N} \int_{r=1/\epsilon} d^4x\sqrt{h}\left(\frac{3}{L} + \frac{L}{4}R(h)\right). \end{aligned} \tag{5.19}$$

The first line is the Einstein-Hilbert term, the second line is the Gibbons-Hawking-York term, and the third line is the boundary counter-terms. The symbol $g_{\mu\nu}$ is the bulk metric, and h_{ab} is the metric on the boundary ∂M living on $r = 1/\epsilon$. K is the extrinsic curvature on ∂M, given by

$$2K = h^{\mu\nu}\nabla_\mu n_\nu = h^{\mu\nu}\partial_r h_\nu = \frac{\partial_r\sqrt{h}}{\sqrt{h}}\,, \tag{5.20}$$

where n_ν is normal to ∂M.

By substituting the AdS-Schwarzschild metric (5.13) we obtain

$$I_{\text{AdS-Sch}} = \frac{\pi\beta}{32G_N L^2}\left(3L^4 + 4L^2 r_+^2 - 4r_+^4\right), \qquad \mu = \left(\frac{r_+}{L}\right)^2\left(L^2 + r_+^2\right). \tag{5.21}$$

One finds that $Z = e^{-I_{\text{AdS-Sch}}}$ becomes large if r_+ is large. The thermal AdS spacetime can be obtained by taking the limit $r_+ = 0$ since the black hole mass μ approaches zero. The difference between $I_{\text{AdS-Sch}}$ and I_{th} (for thermal AdS_5) is given by

$$\Delta I \equiv I_{\text{AdS-Sch}} - I_{\text{th}} = \frac{\pi\beta}{8G_N}\left(\frac{r_+}{L}\right)^2\left(L^2 - r_+^2\right). \tag{5.22}$$

Recalling that r_+ is a function of β_{BH} as in (5.15), we can express ΔI as a function of $T = 1/\beta_{\text{BH}}$. The resulting function is plotted in Figure 2.

[17] The thermal AdS is important when $\mu \sim r_+^2 \to 0$.

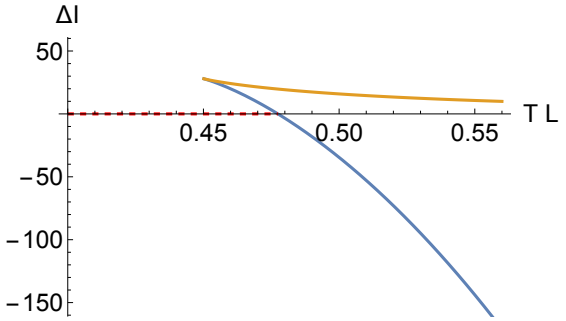


Figure 2. The upper (orange) branch corresponds to the small AdS black hole, the lower (blue) branch corresponds to the large AdS black hole, and the dashed (red) line corresponds to the thermal AdS.

The free energy of the thermal AdS agrees with the free energy of the large AdS black hole at the temperature

$$T_{\rm HP} = \frac{3}{2\pi L} \qquad \Leftrightarrow \qquad \Delta I(r_+ = L) = 0, \tag{5.23}$$

which is called the Hawking-Page temperature. The thermal AdS is stable for $T < T_{\rm HP}$ and the large black hole is stable for $T > T_{\rm HP}$.

For curiosity, let us compare the Hagedorn temperature of flat superstring, $T_H = \frac{1}{2\sqrt{2}\alpha'\pi}$ with the Hawking-Page temperature, $T_{\rm HP} = \frac{3}{2\pi L}$. The ratio between the two temperatures is

$$\left(\frac{T_H}{T_{\rm HP}}\right)^2 = \frac{L^2}{18\alpha'} = \frac{\sqrt{\lambda}}{18}\,, \tag{5.24}$$

showing that $T_H < T_{\rm HP}$ if $\sqrt{\lambda} < 18$.

Physically speaking, only the gravitons contribute to the Hawking-Page transition, whereas all massive string modes contribute to the Hagedorn temperature of the superstring. The inequality (5.24) suggests that massive string modes contribute to the thermal-black hole transition more than gravitons (massless string modes) when the string tension ($\sim \sqrt{\lambda}$) is small.

6 Integrability in $\mathcal{N} = 4$ SYM

6.1 Half BPS states

$\mathcal{N} = 4$ SYM has six real scalars, ϕ^I with $I = 1, 2, \ldots, 6$. We introduce the $SU(3)$ notation

$$Z = \phi^5 + i\phi^6, \quad Y = \phi^3 + i\phi^4, \quad X = \phi^1 + i\phi^2, \tag{6.1}$$

$$\bar{Z} = \phi^5 - i\phi^6, \quad \bar{Y} = \phi^3 - i\phi^4, \quad \bar{X} = \phi^1 - i\phi^2 \tag{6.2}$$

and the $SU(N)$ Wick rule,

$$\overbrace{(\phi^I)_i{}^j}(\phi^J)_k{}^l = \delta^{IJ}\left(\delta_i{}^l\delta_k{}^j - \frac{\delta_i{}^j\delta_k{}^l}{N}\right), \qquad (i,j,k,l = 1,2,\ldots,N). \tag{6.3}$$

This implies $\overbrace{XX} = \overbrace{YY} = \overbrace{ZZ} = 0$ but $\overbrace{Z\bar{Z}} \neq 0$.

The half-BPS states belong to the irreducible representation $[0,L,0]$ of $SU(4)_R$, which are symmetric traceless combinations of the scalars. An example with $L=2$ is

$$O^{IJ} = \mathrm{tr}\left(\phi^I\phi^J\right) - \frac{\delta^{IJ}}{6}\mathrm{tr}\left(\phi^k\phi^k\right). \tag{6.4}$$

The highest weight states (HWS) of the irreducible representation $[0,L,0]$ are

$$O^{zz} = \mathrm{tr}\, Z^2, \quad O^{zzz} = \mathrm{tr}\, Z^3, \quad \ldots \quad , O^{z\ldots z} = \mathrm{tr}\, Z^L. \tag{6.5}$$

To see that these operators are HWS, we take the generators of $\mathfrak{su}(4)_R$ as

$$\begin{pmatrix} X\partial_X & Y\partial_X & Z\partial_X \\ X\partial_Y & Y\partial_Y & Z\partial_Y \\ X\partial_Z & Y\partial_Z & Z\partial_Z \end{pmatrix} \in \mathfrak{su}(4)_R\,, \tag{6.6}$$

which shows that the upper triangular parts annihilate $\mathrm{tr}\, Z^L$.

Let us introduce γ-matrices for SUSY transformations. We need the 4D part for the spacetime, and the 6D part for the internal space. The 4D part is

$$\gamma_\mu = \begin{pmatrix} & (\sigma_\mu)_{\alpha\dot\beta} \\ (\sigma_\mu)_{\dot\alpha\beta} & \end{pmatrix}, \qquad \alpha,\beta,\dot\alpha,\dot\beta = 1,2 \ \text{ and } \ \mu = 0,1,2,3 \tag{6.7}$$

$$\gamma_{\mu\nu} \equiv [\gamma_\mu,\gamma_\nu] = \begin{pmatrix} (\sigma_{\mu\nu})_{\alpha\beta} & \\ & (\sigma_{\mu\nu})_{\dot\alpha\dot\beta} \end{pmatrix} \tag{6.8}$$

and the 6D part is

$$(\gamma_I)_{ab} = -(\gamma_I)_{ba}\,, \qquad a,b = 1,2,3,4 \ \text{ and } \ I = 1,2,\ldots. \tag{6.9}$$

By using these γ matrices, we can introduce the spinor notation for the fundamental fields,[18]

$$F_{\mu\nu} \to F^+_{\alpha\beta} = F^+_{\beta\alpha} \ \text{ and } \ F^-_{\dot\alpha\dot\beta} = F^-_{\dot\beta\dot\alpha}\,, \qquad \mathbf{6} \to \mathbf{3}\oplus\mathbf{3} \quad \in SO(1,3) \tag{6.10}$$

$$\phi_I \to \phi_{ab} = -\phi_{ba}\,, \qquad \mathbf{6} \quad \in SO(6)_R\,. \tag{6.11}$$

The $\mathcal{N}=4$ supercharges are denoted by $Q_{a\alpha}$, $\tilde{Q}^b_{\dot\alpha}$. In the spinor notation, the operator

[18] Recall that $\mathfrak{su}(4) \simeq \mathfrak{so}(6)$.

in the $[0,2,0]$ representation (6.4) becomes

$$O^{abcd} = \mathrm{tr}\left(\phi^{ab}\phi^{cd}\right) - \frac{\epsilon^{abcd}\,\epsilon_{efgh}}{24}\,\mathrm{tr}\left(\phi^{ef}\phi^{gh}\right) \tag{6.12}$$

The supercharge acts on the scalars in a simple manner,

$$Q_{a\alpha}\,\phi_{bc} = \epsilon_{abcd}\,\lambda^d_\alpha \tag{6.13}$$

If we regard $(\phi_{12}\,,\phi_{34})$ as $(\bar{Z}, Z)$, then Z is annihilated by half of the 4 supercharges,

$$Q_{1\alpha}\bar{Z} = Q_{2\alpha}\bar{Z} = 0 \qquad \text{(Half-BPS)}. \tag{6.14}$$

We also find other SUSY descendants,

$$Q_{a\alpha}\,Q_{b\beta}\,Q_{c\gamma}\,Q_{d\delta}\,O^{abcd} \sim F^+_{\alpha\beta}F^+_{\gamma\delta}\,, \qquad \bar{Q}_{a\dot{\alpha}}\,\bar{Q}_{b\dot{\beta}}\,\bar{Q}_{c\dot{\gamma}}\,\bar{Q}_{d\dot{\delta}}\,O^{abcd} \sim F^-_{\dot{\alpha}\dot{\beta}}F^-_{\dot{\gamma}\dot{\delta}} \tag{6.15}$$

Thus, the $[0,2,0]$ half-BPS multiplet contains $\mathrm{tr}\,F^2_{\mu\nu} \sim \mathrm{tr}(F^+)^2 + \mathrm{tr}(F^-)^2$, which is part of the $\mathcal{N}=4$ SYM Lagrangian. We conclude that the coefficient $1/g^2_{\mathrm{YM}}$ in (2.4) is protected by $\mathcal{N}=4$ SUSY.

6.2 Spin Chain

To introduce the spin chain of $\mathcal{N}=4$ SYM, we consider operators close to $\mathrm{tr}\,Z^L$,

$$\mathrm{tr}\left(Z^{L-2}\chi^2\right) \sim \mathrm{tr}\left(ZZ\ldots Z\chi Z\ldots Z\chi Z\ldots ZZ\right) \tag{6.16}$$

where χ can be any of the words in (3.6) in the adjoint representation of $SU(N)$.

These operators are generally non-BPS and mix under renormalization. In order to compute the conformal dimensions at a loop level, we need to diagonalize the quantum dilatation operator (or anomalous dimension matrix),

$$\Delta O_a = M_{ab}\,O_b \quad \rightarrow \quad \Delta O_\alpha = \gamma_\alpha O_\alpha\,. \tag{6.17}$$

When $\chi = Y$, another complex scalar, the one-loop dilatation takes the form

$$\Delta_1 = \frac{\lambda}{8\pi^2}\,\mathrm{tr}\left([Y,Z][\partial_Y,\partial_Z]\right), \tag{6.18}$$

which interchanges the position of Y and Z. This operator is identical to the Hamiltonian of 1D spin chain (XXX spin chain) through the map

$$(Z,Y) \quad \longleftrightarrow \quad (\uparrow,\downarrow) \tag{6.19}$$

$$\mathrm{tr}\left(ZZ\ldots YY\ldots\right) \quad \longleftrightarrow \quad |\uparrow\uparrow\ldots\downarrow\downarrow\ldots\rangle \quad \text{with PBC} \tag{6.20}$$

$$\Delta_1 \quad \longleftrightarrow \quad \left(\lambda/8\pi^2\right)\,\mathcal{H}_{XXX} \tag{6.21}$$

where the XXX Hamiltonian is

$$\mathcal{H}_{XXX} = \sum_{k=1}^{L} (I_{k,k+1} - P_{k,k+1}) \tag{6.22}$$

$$I_{k,k+1} \,|\ldots a_k\, b_{k+1}\rangle = |\ldots a_k\, b_{k+1} \ldots\rangle \tag{6.23}$$

$$P_{k,k+1} \,|\ldots a_k\, b_{k+1}\rangle = |\ldots b_k\, a_{k+1} \ldots\rangle \tag{6.24}$$

with $(a_k, b_k) \in (\uparrow, \downarrow)$. If we introduce 2×2 Pauli matrices $\{\sigma_k^x, \sigma_k^y, \sigma_k^z\}$, we get $P_{k,k+1} = (I_{k,k+1} + \vec{\sigma}_k \cdot \vec{\sigma}_{k+1})/2$, and thus

$$\mathcal{H}_{XXX} = \frac{1}{2} \sum_{k=1}^{L} (I_{k,k+1} - \vec{\sigma}_k \cdot \vec{\sigma}_{k+1})\,. \tag{6.25}$$

It is known that the XXX model is integrable [12, 40]. Its energy spectrum is given by

$$\mathcal{H}_{XXX}\, \psi_\alpha = E_\alpha\, \psi_\alpha\,, \qquad E_\alpha = \sum_{j=1}^{M} \epsilon(u_j^{(\alpha)}), \qquad \epsilon(u) = \frac{1}{4u^2+1}\,, \tag{6.26}$$

where the rapidities $u_j^{(\alpha)}$ of the state ψ_α are determined by solving the Bethe Ansatz Equations for the XXX model

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k\neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}\,, \qquad (j = 1, 2, \ldots, M). \tag{6.27}$$

The XXX Hamiltonian (6.25) has the global symmetry $\mathfrak{su}(2)$. Each regular solution of XXX Bethe Ansatz Equations is in one-to-one correspondence with the highest weight state (HWS) of $\mathfrak{su}(2)$. The $\mathfrak{su}(2)$ HWS is characterized by a Young diagram with two rows,

$$[L-M, M] \;=\; \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline & & & & & & & & & \\ \hline & & & & & & & & & \\ \hline \end{array}\,, \qquad (M \leqslant L/2). \tag{6.28}$$

It is also known that the number of consistent solutions of XXX Bethe Ansatz Equations is equal to the number of *Young tableaux* of the shape $[L-M, M]$ [41, 42]. In this lecture we just explain simple examples of energy eigenstates by using Mathematica in Appendix B.

7 All-loop Asymptotic Spin Chain

It is believed that the planar $\mathcal{N} = 4$ SYM has an all-loop planar integrable structure. The all-loop integrability typically predicts physical observables in the form

$$(\text{Observables}) = (\text{Asymptotic part}) + (\text{Wrapping part}). \tag{7.1}$$

The observables include correlation functions and scattering amplitudes. The asymptotic part is mostly determined by symmetry, and the wrapping part is given by a sum over

infinite virtual particles. By 'asymptotic', we mean an infinitely long background (e.g. $L \to \infty$ in $\mathrm{tr}\, Z^L$). Similarly, 'wrapping' correspond to the finite size (finite L) corrections. The above structure is valid at any λ at large N.

7.1 Symmetry of Asymptotic Spin Chain

The asymptotic spin chain was first introduced by Beisert [43, 44] (see [45] for refinement). As we mentioned earlier, $\mathcal{N} = 4$ SYM exhibits perturbative integrability.[19] However, all-loop planar integrability comes from a different idea. We consider $\mathrm{tr}\, Z^L$ with $L \gg 1$, and ignore the periodicity condition coming from the cyclicity of the trace. The extra freedoms enlarge the global symmetry through a central extension, which turns out to be part of the Yangian algebra.

The global symmetry algebra of $\mathcal{N} = 4$ SYM is $\mathfrak{psu}(2,2|4)$ whose Cartan algebra consists of $\{\Delta, S_1, S_2, J_1, J_2, J_3\}$, as explained in Section 2. The symmetry corresponding to Δ, J_3 are broken when we specify the operator $\mathrm{tr}\, Z^{J_3}$ as the vacuum of an effective theory. The residual symmetry is $\mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}$.

The single-letter χ entering $\mathrm{tr}(ZZ\cdots\chi\cdots ZZ)$ behaves as the fundamental representation $(\mathbf{2}|\mathbf{2})^2$ of $\mathfrak{psu}(2|2)^2$, which has 8 bosons and 8 fermions. We can relate χ with the fundamental fields of $\mathcal{N} = 4$ SYM by

$$\chi \;\in\; \begin{pmatrix} D_{\alpha\dot\alpha} Z & \lambda_{b\alpha} \\ \bar\lambda_{a\dot\alpha} & \Phi_{ab} \end{pmatrix}, \qquad a, b, \alpha, \dot\alpha = 1, 2, \tag{7.2}$$

where Φ_{ab} are $(X, Y, \bar X, \bar Y)$. This χ satisfies $\Delta_0 - J_3 = 1$. Other letters such as $\bar Z$, $D\Phi$, $D\lambda$ have $\Delta_0 - J_3 = 2$. They are the bound states of (7.2) in the asymptotic spin chain. We can also label χ in terms of $(\mathbf{2}|\mathbf{2})_L \oplus (\mathbf{2}|\mathbf{2})_R$ as

$$\chi \sim (\phi_a|\psi_\alpha)_L \otimes (\bar\phi_b \otimes \bar\psi_{\dot\alpha})_R\,, \tag{7.3}$$

which will be used in the next subsection.

Let us discuss the central extension of $\mathfrak{psu}(2|2) \ltimes \mathbb{R}$. For simplicity we consider the left moving part only. The generators consist of

$$J \;\in\; \{\mathcal{R}^a{}_b, \mathcal{L}^\alpha{}_\beta, Q^\alpha{}_a, S^b{}_\beta, \hat C\}, \tag{7.4}$$

where $\mathcal{R}^a{}_b, \mathcal{L}^\alpha{}_\beta$ are the $\mathfrak{su}(2)^2$ rotations, $Q^\alpha{}_a$ is the supercharge, $S^b{}_\beta$ is the superconformal charge, and $\hat C$ is the center related to $\Delta - J_3$. The supercharges and the superconformal charges satisfy the commutation relations

$$\{Q^\alpha{}_a, S^b{}_\beta\} = \delta^\alpha{}_\beta\, \mathcal{R}^b{}_a + \delta^b{}_a\, \mathcal{L}^\alpha{}_\beta + \delta^b{}_a \delta^\alpha{}_\beta\, \hat C\,. \tag{7.5}$$

After the central extension, the algebra becomes $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$. The extra centers $\hat K, \hat P$

[19] Also string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is classically integrable.

show up as

$$\{Q^{\alpha}{}_{a}, Q^{\beta}{}_{b}\} = \epsilon^{\alpha\beta}\,\epsilon_{ab}\,\hat{P} \tag{7.6}$$

$$\{S^{a}{}_{\alpha}, S^{b}{}_{\beta}\} = \epsilon^{ab}\,\epsilon_{\alpha\beta}\,\hat{K}. \tag{7.7}$$

7.2 One-Particle States

We make an ansatz for one-particle states as the superposition of plane-waves,

$$|\chi(p)\rangle = \sum_{n} e^{ipn} \underbrace{\left| ZZ\ldots\overset{n}{\overset{\vee}{\chi}}\ldots ZZ \right\rangle}_{L \text{ fields}}, \tag{7.8}$$

where p is the momentum and n is the position. $|ABC\ldots\rangle$ means $\mathrm{tr}(ABC\ldots)$ without trace cyclicity. In the limit $L \to \infty$, this state is invariant (up to a constant) under the addition or removal of extra Z's. The additional centers $\hat{P}, \hat{K}$ can act non-trivially on $|\chi(p)\rangle$.[20]

We write $|\chi(p)\rangle = |\chi_L\rangle \otimes |\chi_R\rangle \in (\mathbf{2}|\mathbf{2})_L \otimes (\mathbf{2}|\mathbf{2})_R$. The charges $\{Q, S\}$ act on $|\chi_L\rangle$ as[21]

$$Q^{\alpha}{}_{a} \left|\phi^{b}\right\rangle = A\,\delta^{b}{}_{a}\,|\psi^{\alpha}\rangle \tag{7.9}$$

$$Q^{\alpha}{}_{a} \left|\psi^{\beta}\right\rangle = B\,\epsilon^{\alpha\beta}\epsilon_{ab} \left|Z^{+}\phi^{b}\right\rangle \tag{7.10}$$

$$S^{a}{}_{\alpha} \left|\phi^{b}\right\rangle = C\,\epsilon^{ab}\epsilon_{\alpha\beta} \left|Z^{-}\psi^{\beta}\right\rangle \tag{7.11}$$

$$S^{a}{}_{\alpha} \left|\psi^{\beta}\right\rangle = D\,\delta^{\beta}{}_{\alpha}\,|\phi^{a}\rangle \tag{7.12}$$

where $Z^{\pm}$ is the addition or removal of extra Z's. The commutation relations of $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ give

$$\hat{C}\,|\chi\rangle = \frac{AD + BC}{2}\,|\chi\rangle, \tag{7.13}$$

$$\{Q, Q\}\,|\chi\rangle \sim \hat{P}\,|\chi\rangle = AB\,|\chi\rangle, \qquad \{S, S\}\,|\chi\rangle \sim \hat{K}\,|\chi\rangle = CD\,|\chi\rangle \tag{7.14}$$

$$\{Q, S\}\,|\chi\rangle \sim (\mathcal{R} + \mathcal{L} + \hat{C})\,|\chi\rangle \quad \Rightarrow \quad 1 = AD - BC. \tag{7.15}$$

Motivated by the perturbative data of $\mathcal{N} = 4$ SYM, we define new variables,

$$\begin{aligned} A &= \sqrt{f}\gamma, \quad B = \frac{\sqrt{f}}{\gamma}\left(1 - \frac{x^{+}}{x^{-}}\right), \\ C &= \frac{i\sqrt{f}\gamma}{x^{+}}, \quad D = \frac{\sqrt{f}x^{+}}{i\gamma}\left(1 - \frac{x^{-}}{x^{+}}\right), \end{aligned} \tag{7.16}$$

[20] If we keep the trace cyclicity, $\mathrm{tr}(ZZ\ldots\chi\ldots ZZ)$ becomes a descendant of the half-BPS state whose centers are all trivial. One should think of $|\chi(p)\rangle$ as a mathematical building block.

[21] The action on $|\chi_R\rangle$ is the same.

then $AD - BC = 1$ implies

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{f}\,. \tag{7.17}$$

This equation can be solved by Zhukovski map $x = x(u)$,

$$x(u) + \frac{1}{x(u)} = u, \qquad x^{\pm}(u) = x\left(u \pm \frac{i}{2f}\right). \tag{7.18}$$

Define the momentum p by

$$\frac{x^+}{x^-} \equiv e^{ip}. \tag{7.19}$$

The center $\hat{C}$ in (7.13) becomes

$$2\,\hat{C} = \frac{1 + \frac{1}{x^+x^-}}{1 - \frac{1}{x^+x^-}} = \sqrt{1 + 16f^2 \sin^2\frac{p}{2}}\,, \tag{7.20}$$

which is identified as $\Delta - J_3 = E(p)$ (the magnon energy). In this argument, f is an arbitrary function of the 't Hooft coupling λ. It is known that $16f^2 = \frac{\lambda}{\pi^2}$ in $\mathcal{N} = 4$ SYM. In ABJM, the comparison of the BPS Wilson lines with the localization method fixes this function $f(\lambda)$ [46].

7.3 Two-Particle States

We make the following ansatz for two-particle states,

$$\begin{aligned} |\chi_a(p_1)\chi_b(p_2)\rangle = \sum_{n_1 \ll n_2} \Big(& e^{ip_1n_1 + ip_2n_2}\, |Z \dots Z\, \chi_a\, Z \dots Z\, \chi_b\, Z \dots \rangle \\ & + e^{ip_1n_2 + ip_2n_1} S^{cd}_{ab}(p_1, p_2)\, |Z \dots Z\, \chi_c\, Z \dots Z\, \chi_d\, Z \dots \rangle \Big) \\ & + (n_1 \sim n_2), \end{aligned} \tag{7.21}$$

where $S^{cd}_{ab}(p_1, p_2)$ is a two-particle S-matrix. The last line denotes the case when n_1 is close to n_2, which is not important in our discussion. The two-particle states form a 16^2-dimensional (reducible) representation of $\mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}^3$.

We impose the condition that the S-matrix commutes with all generators of $\mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}^3$. This condition determines $S^{cd}_{ab}(p_1, p_2)$ up to an overall factor called the dressing factor. Moreover, this $\mathfrak{psu}(2|2)^2$-invariant S-matrix satisfies Yang-Baxter relations, unitarity and crossing. The entire algebra is part of Yangian of $\mathfrak{psu}(2|2)^2$.[22] [23] See [47, 48] for more details.

[22] Yangian can be defined in several ways, and one definition is more or less equivalent to the Yang-Baxter relation.

[23] The Killing form of $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ is degenerate. Some people prefer the exceptional superalgebra $\mathfrak{d}(2, 1; \varepsilon)$, which is a deformation of $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$.

7.4 Asymptotic Bethe Ansatz Equations

We have seen that the energy of one-particle states is given by (7.20). In integrable systems, the energy of a multi-particle state is a sum of the single-particle energies. Thus, the energy of an M-particle state is

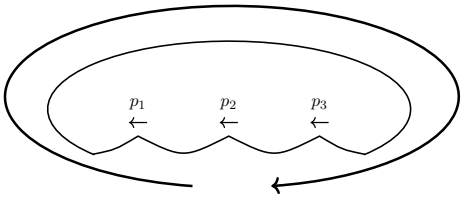
$$\Delta - J_3 = \sum_{k=1}^{M} \sqrt{1 + 16 f(\lambda)^2 \sin^2 \frac{p_k}{2}} \,. \tag{7.22}$$

The momenta $\{p_k\}$ are unconstrained in the strict $L\ (= J_3) \to \infty$ limit. When L is large but finite, we should impose the periodic boundary conditions, or equivalently asymptotic Bethe Ansatz Equations (BAE). The momenta $\{p_k\}$ should be a solution of the asymptotic BAE.

Suppose the system is periodic. We can take one excitation, say j-th particle, and let it go around the spin chain. Two factors contribute to the wave function of the system under this operation. The first factor is $e^{ip_j L}$ where p_j is the momentum of the j-th particle. The second factor is the scattering phase between the j-th and other magnons, which are given by the product of two-body S-matrices (7.21). The periodic boundary condition requires that the product of the two factors should be equal to one,

$$\mathbf{1} = e^{-ip_j L} \prod_{k \neq j} \mathbb{S}(p_j, p_k) \qquad (j = 1, 2, \ldots, M) \tag{7.23}$$

which are the asymptotic Bethe Ansatz Equations. The situation for $M = 3$ is schematically shown as:



(7.24)

Strictly speaking, both sides of the equation (7.23) are still operators. To derive algebraic equations for $\{p_k\}$ we need to diagonalize the S-matrix. After the diagonalization, we obtain the asymptotic BAE with $\mathfrak{psu}(2|2)^2$ symmetry.

Before writing down the asymptotic BAE, we fix the branch choice of the Zhukovski map by

$$x_s(u) \equiv \frac{u}{2}\left(1 + \sqrt{1 - \frac{4}{u^2}}\right) \tag{7.25}$$

The branch cut is running in between $u = \pm 2$. We require $|x_s(u)| \geqslant 1$ on the top sheet of $u \in \mathbb{C}$, from the comparison with the perturbative data. The asymptotic BAE for the momentum-carrying node are given by

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{\ell \neq k}^{K^I} S_0(u_k, u_\ell) \, \frac{x_k^+ - x_\ell^-}{x_k^- - x_\ell^+} \sqrt{\frac{x_\ell^+ x_k^-}{x_\ell^- x_k^+}} \prod_{\alpha = L,R} \prod_{m=1}^{K_\alpha^{II}} \frac{x_k^- - y_m^{(\alpha)}}{x_k^+ - y_m^{(\alpha)}} \sqrt{\frac{x_k^+}{x_k^-}} \,. \tag{7.26}$$

There are also extra four types of asymptotic BAE which determine the auxiliary Bethe roots $(y^{(\alpha)}, w^{(\alpha)})$ with $\alpha = L, R$.

7.5 Dressing Factor

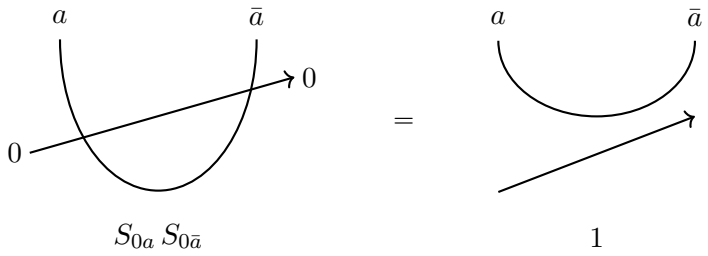
We have argued that the centrally-extended algebra $\mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}^3$ determines the two-particle S-matrix up to an overall scalar 'dressing' factor. To be more concrete, let us write the S-matrix as

$$S\left(x_1^{\pm}, x_2^{\pm}\right) = S_0\left(x_1^{\pm}, x_2^{\pm}\right)\left[\hat{S}_{\mathfrak{su}(2|2)_L} \otimes \hat{S}_{\mathfrak{su}(2|2)_R}\right] \tag{7.27}$$

where $S_0\left(x_1^{\pm}, x_2^{\pm}\right)$ is the dressing factor, and $\hat{S}_{\mathfrak{su}(2|2)_L} \otimes \hat{S}_{\mathfrak{su}(2|2)_R}$ is the matrix component determined by symmetry. We also introduces variables $x_j^{\pm} = x(u_j \pm \frac{i}{2g})$.

We know that $S_0 \neq 1$ from the perturbative data, both the dilatation spectrum of $\mathcal{N} = 4$ SYM and the energy spectrum of $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring. For example, higher-loop computation shows that the anomalous dimensions contain *transcendental* numbers such as $\zeta(3), \zeta(5), \ldots$ whereas the asymptotic BAE without dressing phase can produce only *algebraic* numbers (i.e. polynomial roots).

One way to obtain S_0 is to solve the crossing relations, which can be illustrated as



(7.28)

If a test particle (0) scatters against a pair of particle and anti-particle $(a, \bar{a})$, then the total S-matrix should be trivial owing to the Yang-Baxter relation. Here the anti-particle $(E < 0)$ is mathematically given by the crossing transformation $x^{\pm} \to 1/x^{\pm}$. See [49] for more details.

Note that 'crossing symmetry' is natural only from the viewpoint of worldsheet theory (2d QFT). From the viewpoint of $\mathcal{N} = 4$ SYM spin chain, the existence of the crossing symmetry is mysterious.

The solutions of the crossing equations are not unique due to the ambiguity of periodic (or crossing-invariant) functions. This is also called the CDD factor [50]. The correct dressing phase for $\mathcal{N} = 4$ SYM and $\mathrm{AdS}_5 \times \mathrm{S}^5$ string is determined in [51].

Finally, we do not need an explicit form of S_0 if we use the Quantum Spectral Curve method, which is equivalent to TBA [52].

8 TBA for Hagedorn Temperature

We review Thermodynamic Bethe Ansatz (TBA) equations for Hagedorn temperature, following on [53, 54]. Recall that the $\mathcal{N} = 4$ SYM partition function on $S^1_\beta \times S^3$ is given

by (3.18),

$$\mathrm{tr}\left(e^{-\beta D}\right) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\,Z_{\text{single-tr}}(e^{-n\beta})\right). \tag{8.1}$$

We look for the largest $\beta = \beta_*(\lambda)$ (= smallest $T_* \in \mathbb{R}_{\geqslant 0}$) such that the partition function diverges. This time β_* is a function of λ. Perturbative computation shows that the $n = 1$ term is dominant, and we assume that this is true at any λ.[24]

We rewrite the dilatation operator as $D = D_{\lambda=0} + \delta D$, and expand the single-trace partition as

$$Z_{\text{single-tr}}\left(e^{-n\beta}\right) = \sum_{m=2}\mathrm{tr}\, e^{-\beta\left[\frac{m}{2}+\delta D\left(\frac{m}{2}\right)\right]} \equiv \sum_{m} e^{-\beta F_m}\,. \tag{8.2}$$

Here F_m is a sum of anomalous dimensions of all single-trace operators with the canonical dimensions $D_0 = m/2$. According to Cauchy's root test, $\sum_n a_n$ converges (or diverges) if $\lim_{n\to\infty}(a_n)^{1/n} = r$ satisfies $r < 1$ (or $r > 1$).[25]

In our case, we want to test

$$\lim_{m\to\infty}\exp\left(-\frac{\beta}{m}F_m\right) \quad\longrightarrow\quad r \stackrel{?}{=} 1 \tag{8.3}$$

or equivalently,

$$\log r \quad\sim\quad -\frac{1}{m}\left\{\frac{\beta m}{2} + \underbrace{\log\ \mathrm{tr}\left[e^{\beta\delta D\left(\frac{m}{2}\right)}\right]}_{\equiv\frac{\beta m}{2}\mathcal{F}(T)}\right\} \stackrel{?}{=} 0. \tag{8.4}$$

The Hagedorn temperature T_H is now defined by

$$\mathcal{F}(T_H) = -1. \tag{8.5}$$

Here $\mathcal{F}$ can be thought of as a sum of anomalous dimensions of very long operators, which can be computed by the thermodynamic limit of asymptotic BAE.

8.1 TBA Equations

We want to sum free energy over the states which solve BAE in a thermodynamic limit. This limit is defined by letting the spin chain length $L \to \infty$, the number of particles $M \to \infty$, the ratio M/L kept fixed.[26] After performing the sum we obtain a set of nonlinear integral equations called TBA equations [56].

The free energy of a grand canonical ensemble is given by

$$\mathcal{F} = E - TS - T\cdot i\gamma N \tag{8.6}$$

[24] Strictly speaking, the $n = 2$ term (or higher) can be dominant in some range of the chemical potentials [55] where T_H for $n = 1$ is not lying on the positive real axis.

[25] Roughly speaking, we expect $\sum_m a_m \sim \sum_n n^c\, r^n$ for some fixed c.

[26] Usually, the spin chain length is the same as the parameter L in (7.26). However, in (8.2) we should take $D_0 = m/2 \to \infty$. The two limiting procedures should agree with each other because $D_0 \to \infty$ implies infinite $L \to \infty$ and $M \to \infty$.

where the third term represents a pure-imaginary chemical potential for fermions.[27] The first and the third term can be rewritten as

$$E - Ti\gamma N = \sum_a \int du\, (e_a(u) - Ti\gamma_a)\, \rho_a(u) \tag{8.7}$$

where u is rapidity (or momentum), and $\rho_a(u)$ is the density of type-a particles. The second term, the entropy, can be computed by counting the number of all possible states.

Let us pause for a moment and examine the meaning of *all possible states*. We sum over all possible states in the grand canonical ensemble. Each state is a solution of asymptotic BAE. In the thermodynamic limit, we may use the saddle-point approximation; namely only the saddle-point configuration contributes to the free energy.[28] The saddle-point configuration is also a solution of the BAE, characterized by the density of the occupied state $\rho_a(u)$ and the unoccupied state $\bar\rho_a(u)$.

We define $N_a(u) = \rho_a(u)du$ and $\bar N_a(u) = \bar\rho_a(u)du$ as the number of occupied and unoccupied states of the saddle-point configuration inside the rapidity range $[u, u+du]$. The sum $N_a(u) + \bar N_a(u)$ is the number of all possible states in the same range. Thus, the entropy S in (8.6) is given by

$$\begin{aligned} S &= \sum_a \int du\, \log \frac{(N_a(u) + \bar N_a(u))!}{N_a(u)!\bar N_a(u)!} \\ &\to \sum_a \int du \Big\{ (\rho_a + \bar\rho_a)\log(\rho_a + \bar\rho_a) - \rho_a \log\rho_a - \bar\rho_a \log\bar\rho_a \Big\}, \end{aligned} \tag{8.8}$$

where we used Stirling's formula.[29]

Let us extremize $\mathcal{F}[\rho_a, \bar\rho_a]$ as

$$\delta\mathcal{F} = \frac{\partial\mathcal{F}}{\partial\rho_a}\delta\rho_a + \frac{\partial\mathcal{F}}{\partial\bar\rho_a}\delta\bar\rho_a = 0. \tag{8.9}$$

We use BAE to relate $\delta\rho_a$ and $\delta\bar\rho_a$:

$$\log(\text{BAE}) : \quad 2\pi i\, n_{a,k} = ip(u_k)L_a + \sum_b \sum_{j\neq k} \log S_{ab}(u_k, u_j). \tag{8.10}$$

Here $n_{a,k}$ is not just an integer, but belongs to a finite interval with upper and lower bounds. This is because when we choose the branch of the logarithm as $\mathrm{Im}\log z \in (-\pi, \pi]$, and thus the imaginary part of RHS is finite. It is expected that for each appropriate choice of $\{n_{a,k}\}$,

[27] The particles in many integrable systems obey Fermion statistics, because the scattering factor becomes -1 when the rapidities coincide, $S(u,u) = -1$. See RHS of (6.27).

[28] In a free fermion system, one can compute $\mathcal{F}$ by rigorously enumerating all possible states in the ensemble, and derive a TBA-like equation.

[29] One can compute subleading corrections to Stirling's formula as in [57].

there is a consistent solution $\{u_{a,k}\}$ of BAE. Then, the continuum limit of (8.10) becomes

$$\rho_a + \bar{\rho}_a = \frac{p(u)L_a}{2\pi} + \sum_b \int dv\, \frac{1}{2\pi i}\, \partial_v \log S_{ab}(u,v) \cdot \rho_b(v). \tag{8.11}$$

After a little algebra, we obtain

$$\delta\mathcal{F} = 0 \quad \Leftrightarrow \quad \log Y_b(u) = \underbrace{\frac{e_b(u)}{T}}_{\text{source term}} - \sum_a \int dv \log\left(1 + e^{i\gamma_a} Y_a(v)\right) K_{ab}(v,u) \tag{8.12}$$

which is the TBA equations. Our notation is

$$e^{i\gamma_a} Y_a(u) = \frac{\rho_a(v)}{\bar{\rho}_a(u)}\,, \qquad K_{ab}(v,u) = \frac{1}{2\pi i}\,\frac{\partial}{\partial v} \log S_{ab}(v,u). \tag{8.13}$$

The free energy at the extremum is

$$\mathcal{F} = -T \sum_a^{\infty} \int \frac{du}{2\pi} \log\left(1 + e^{i\gamma_a} Y_a(u)\right). \tag{8.14}$$

The sum over a can be finite for simple integrable models, but we have an infinite number of Y-functions in $\mathcal{N} = 4$ SYM. The TBA for $\mathcal{N} = 4$ SYM can be derived from the string hypothesis [58–60].

8.2 Y-system

It is known that TBA equations derived from the asymptotic spin chains is more or less equivalent to the Y-system equipped with discontinuity relations [61, 62] (see [63] for the Y-system in general integrable systems). The Y-system is a set of functional relations given by

$$Y^+_{a,s} Y^-_{a,s} = \frac{(1 + Y_{a,s-1})(1 + Y_{a,s+1})}{\left(1 + \frac{1}{Y_{a-1,s}}\right)\left(1 + \frac{1}{Y_{a+1,s}}\right)}, \qquad f^{\pm} = f\left(u \pm \frac{i}{g}\right). \tag{8.15}$$

If we introduce the T-functions by

$$Y_{a,s} = \frac{T_{a,s+1}\, T_{a,s-1}}{T_{a+1,s}\, T_{a-1,s}} \tag{8.16}$$

the Y-system can be rewritten as

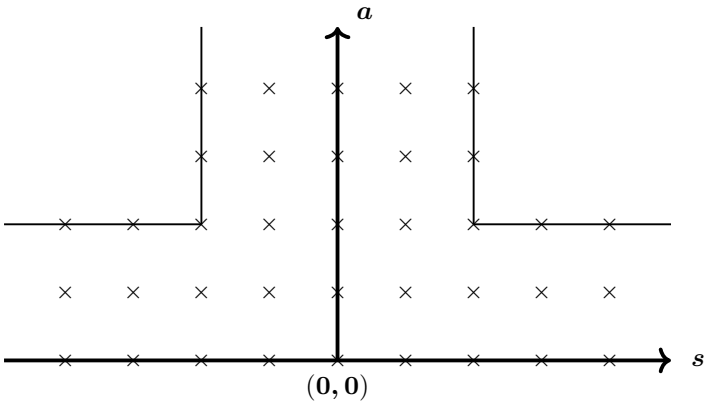
$$T^+_{a,s}\, T^-_{a,s} = T_{a,s-1}\, T_{a,s+1} + T_{a-1,s}\, T_{a+1,s}\ , \tag{8.17}$$

which is called the T-system (or Hirota equations). The T-system is invariant under the gauge transformation

$$T_{a,s} \quad \to \quad g_1^{[a+s]} g_2^{[a-s]} g_3^{[-a+s]} g_4^{[-a-s]}\, T_{a,s}\ , \tag{8.18}$$

where $f^{[n]} = f\left(u + \frac{in}{g}\right)$. The Y-functions $Y_{a,s}$ are gauge-invariant objects.

The T-functions in $\mathcal{N} = 4$ SYM are defined on the $\mathfrak{psu}(2,2|4)$ T-hook, which can be illustrated as



(8.19)

The T-functions $T_{a,s}$ with $a \in \mathbb{Z}_{\geqslant 0}\,, s \in \mathbb{Z}$ obey the following boundary conditions:

- $T_{a,s} = 0$ outside the T-hook (points without $\times$ in (8.19))
- $T_{0,s} = 1$ and $T_{2,\pm Q} = T_{Q,\pm 2}$ for $Q \geqslant 3$

The second condition comes from our choice of gauge fixing.

If $T_{a,s}(u)$ is a constant (i.e. independent of u), then $T^{\pm}_{a,s}$ becomes $T_{a,s}$, and we can identify the T-system as the Plücker relations for group characters.[30] T-functions are identified as

$$T_{a,s} = PSU(2,2|4) \text{ character for the } a \times s \text{ rectangular representation.} \tag{8.20}$$

8.3 $PSU(2,2|4)$ Characters

We follow [64, 65] and introduce $PSU(2,2|4)$ characters. See [66] for the definition of superalgebra $\mathfrak{psu}(2,2|4)$, and [67] for the unitary irreducible representations of supergroup $PSU(2,2|4)$.

The generating function for $GL(M|N)$ characters is known as

$$w_{M|N}(t,\xi) = \mathrm{Sdet}\,\frac{1}{1-\xi t} = \frac{\prod_{n=1}^{N}(1-y_n t)}{\prod_{m=1}^{M}(1-x_m t)} \equiv \sum_{s=1}^{\infty} t^s\, T^{(M|N)}_{1,s}(\xi), \tag{8.21}$$

where

$$\xi = \mathrm{diag}\,(x_1\,, x_2\,, \ldots\,, x_M \,|\, y_1\,, y_2\,, \ldots y_N) \in GL(M|N) \tag{8.22}$$

and $[1,s]$ in $T_{1,s}$ denotes the s-th totally symmetric representation of $GL(M|N)$. We can decompose (8.21) as

$$w_{4|4}(t, \xi_L \times \xi_R) = w_{(2|2)}(t,\xi_L) \times w_{(2|2)}(t,\xi_R) \tag{8.23}$$

[30] Some people call the T-system with the replacement $T^{\pm}_{a,s} \to T_{a,s}$ the Q-system.

giving us

$$T^{(4|4)}_{1,s}(\xi_L \times \xi_R) = \sum_{j=0}^{s} T^{(2|2)}_{1.s-j}(\xi_L)\, T^{(2|2)}_{1,j}(\xi_R). \tag{8.24}$$

We need $GL(2,2|4)$ characters and not $GL(4|4)$ characters. Let us try expanding $GL(2|2)_R$ in $1/t$ and ξ_R^{-1} as

$$T^{(2,2|4)}_{1,s}(\xi_L \times \xi_R) = \frac{y_3 y_4}{x_3 x_4} \sum_{j=\max(0,-s)}^{\infty} T^{(2|2)}_{1,s+j}(\xi_L) T^{(2|2)}_{1,j}(\xi_R^{-1}). \tag{8.25}$$

It turns out that (8.25) is the $GL(2,2|4)$ character for the $[1,s]$ representation. We parameterize $\xi \in PSU(2,2|4)$ as

$$\xi = \mathrm{diag}\Big(\underbrace{x_1\,, x_2\,, x_3\,, x_4}_{SU(2,2)} \,\Big|\, \underbrace{y_1\,, y_2\,, y_3\,, y_4}_{SU(4)_R}\Big), \qquad x_1\, x_2\, x_3\, x_4 = y_1\, y_2\, y_3\, y_4 = 1. \tag{8.26}$$

The last two equations reduce $U(2,2|4)$ to $PSU(2,2|4)$.

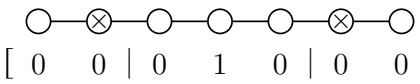
If we set

$$x_1 = x_2 = \frac{1}{x_3} = \frac{1}{x_4} = -e^{-\beta/2}\,, \qquad y_1 = y_2 = y_3 = y_4 = 1, \tag{8.27}$$

then we find

$$T_{1,0} = \zeta(x), \quad x = e^{-\beta}, \tag{8.28}$$

which is the single-letter function of $\mathcal{N}=4$ SYM, given by (3.17). The relation (8.28) comes from the fact that the irreducible representation $[1,0]$ can be identified as the singleton representation of $\mathfrak{psu}(2,2|4)$,



(8.29)

8.4 Solving TBA

There are two situations where TBA equations in $\mathcal{N}=4$ SYM have constant solutions given by the $PSU(2,2|4)$ characters. The first situation is the quasi-classical limit $g \equiv \frac{\sqrt{\lambda}}{2\pi} = \infty$. The T-functions $T^{\pm}_{a,s}$ become $T_{a,s}$, which can be identified as the trace of the monodromy of Lax connections on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ worldsheet [68],

$$T_{a,s} \simeq \mathrm{Str}_{a,s}\left[\mathrm{Pexp}\left(\oint d\sigma \mathcal{L}_\sigma\right)\right]. \tag{8.30}$$

The second situation is the zero coupling limit $g=0$ in Hagedorn TBA. We can neglect the source term in (8.12), and the constant Y-function becomes a solution of TBA.[31]

[31] Rigorously speaking, we should look at the simplified form of Hagedorn TBA to see the source term vanishes.

Let us look for the constant solution for the Hagedorn TBA around $g = 0$. The TBA equations consist of an infinite number of nonlinear integral equations of the form (8.12). One of them can be written as

$$\log Y_{1,1}Y_{2,2} = \sum_{Q=1}^{\infty} \log(1 + Y_{Q,0}) * K_{Q_y} \tag{8.31}$$

where

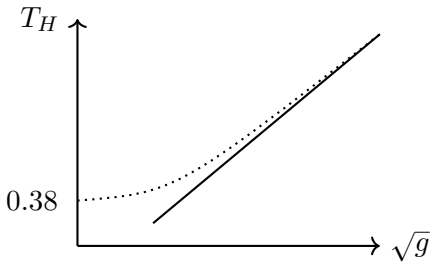
$$f * K(u) = \int_{-\infty}^{\infty} dv f(v)K(v, u). \tag{8.32}$$

Since all Y-functions are constant, the RHS of (8.31) becomes $\sim \int_{-\infty}^{\infty} dv K(v, u) =$ (constant) from (8.13). Thus

$$1 = Y_{1,1}Y_{2,2} = \frac{T_{1,0}}{T_{0,1}} \frac{T_{2,3}}{T_{3,2}} = T_{1,0} \qquad \text{(from our gauge choice in Section 8.2)} \tag{8.33}$$

Therefore, $T_{1,0} = 1$ solves TBA at $g = 0$, which is consistent with the condition that $\zeta(x_H) = 1$ through (8.28). However, the free energy vanishes for this constant solution, implying that the equation $\mathcal{F} = -1$ in (8.5) is not satisfied. To rescue the situation, we need to study the corrections at higher orders in g.

We can also solve TBA numerically, starting from the character solution at $g = 0$, and extending the solution towards $\lambda > 0$. For this type of computation, the Quantum Spectral Curve is particularly useful [69]. The numerical solution is given by (see [54] for the precise figure):



(8.34)

Here the dotted line represents the numerical solution, starting at the value $T_H(g = 0)$ corresponding to $x_H = 7 - 4\sqrt{3}$ in (3.28). The solid line represents the leading order approximation at the strong coupling, namely

$$T_H \sim \sqrt{\frac{g}{4\pi}} = \frac{R}{\sqrt{8\pi^2\alpha'}}. \tag{8.35}$$

One finds that $T_H(g \gg 1)$ approaches the Hagedorn temperature of the flat space superstring (4.19) measured in the unit of AdS radius R.

Here are miscellaneous remarks. First, we can include general chemical potentials in the identification (8.27) [70]. Second, Hagedorn TBA gives the Hagedorn temperature $T_H(\lambda)$, but not the value of the partition function $Z = \mathrm{tr}(e^{-\beta D})$ around $T = T_H$. Third, the

Hagedorn temperature at the next leading order of $1/g$ is computed in [71] by considering the winding mode on EAdS_5 at the radius where the ground state becomes massless,

$$T_H = \sqrt{\frac{g}{4\pi}} + \frac{1}{2\pi} + O(1/g). \tag{8.36}$$

9 Free Energy of $\mathcal{N} = 4$ SYM

We compute the free energy of $\mathcal{N} = 4$ SYM at zero coupling to examine if $F \sim O(N^2)$ or $\sim O(1)$. The derivation consists of five steps, and we omit some of the details. See [23, 72, 73] for technical aspects.

Step 1

As a warm-up, let us consider the partition function $\mathrm{tr}(e^{-\beta H})$ of a free boson $\phi(t)$ on a thermal circle. We heuristically evaluate the partition function by inserting the resolution of identity expanded by coherent states,

$$\mathrm{tr}(e^{-\beta H}) = \int \prod_k \frac{d\bar{f}_k\, df_k}{\pi} \left\langle f \right| e^{-\beta H} \left| f \right\rangle. \tag{9.1}$$

The free boson Hamiltonian is quantized as $H = \sum_k \omega_k a_k a_k^\dagger$, and the coherent state can be written as the product state $|f\rangle = \otimes_k |f_k\rangle$. Then

$$\begin{aligned} \left\langle f_k \right| e^{-\beta\omega_k a_k a_k^\dagger} \left| f_k \right\rangle &= \sum_n \langle f_k \,|\, n\rangle\, e^{-\beta\omega_k n}\, \langle n | f_k\rangle = \sum_n \frac{(\bar{f}_k\, f_k)^n}{n!}\, e^{-\beta\omega_k n - \bar{f}_k\, f_k} \\ &= \exp\Big(-\bar{f}_k\, f_k + \bar{f}_k\, f_k\, e^{-\beta\omega_k} \Big) \end{aligned} \tag{9.2}$$

where one can compute the overlap between the coherent state and the energy eigenstate in a textbook way [74]. Now the partition function (9.1) is a collection of Gaussian integrals. The partition function of a free fermion can be computed in a similar way.

Next, we count the number of G-singlets for a Lie group G. We introduce a Haar measure on G by

$$\int dg\, 1 = 1, \qquad \int dg\, (R_g)_{\alpha\beta} = \delta_{\alpha\beta} \tag{9.3}$$

and the resolution of identity by

$$\int \prod_\alpha \frac{d^2 f_\alpha}{\pi} \int dg_1 dg_2\, |R_{g_1}\, f\rangle\langle R_{g_2}\, f| = 1, \tag{9.4}$$

where R_g is the representation matrix of $g \in G$ in some representation. The partition function of the free boson system is written as

$$\mathrm{tr}(e^{-\beta H}) = \int \prod_{k,\alpha} \frac{d\bar{f}_{k,\alpha}\, df_{k,\alpha}}{\pi} \int dg_1 dg_2 \left\langle R_{g_2}\, f \right| e^{-\beta H} \left| R_{g_1}\, f \right\rangle. \tag{9.5}$$

After a little algebra, we arrive at

$$\mathrm{tr}(e^{-\beta H}) = \int dg \prod_k \frac{1}{\det(1 - e^{-\beta\omega_k} R_g)} \,. \tag{9.6}$$

The $\mathcal{N} = 4$ SYM at zero coupling consists of free bosons and free fermions in the adjoint representations of $SU(N)$. The partition function can be written as

$$\mathrm{tr}\left(e^{-\beta D_0}\right) = \int dg \prod_n \frac{\det\left(1 + e^{-\beta\omega_n} R_g\right)}{\det\left(1 - e^{-\beta\omega_n} R_g\right)} \,, \tag{9.7}$$

where R_g is in the adjoint representation. The numerator/denominator comes from the fermionic/bosonic oscillators, respectively. The frequency ω_n is related to the eigenvalue of D_0 .

Step 2

We rewrite the determinant in (9.7) by using

$$\det(1 \mp A)^{\mp 1} = \exp\Big(\mp \mathrm{tr}\log(1 \mp A)\Big) = \exp\left(\sum_{k=1} \frac{(-1)^{k+1}}{k} \mathrm{tr} A^k\right). \tag{9.8}$$

The structure of $PSU(2,2|4)$ symmetry is hidden in the index n in (9.7). Heuristically we can rewrite the product over n as

$$\mathrm{tr}\left(e^{-\beta D_0}\right) = \int dg \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\left[\zeta_B(x^n)\chi_{\mathrm{adj}}(g^n) + (-1)^{n+1}\zeta_F(x^n)\chi_{\mathrm{adj}}(g^n)\right]\right) \tag{9.9}$$

where ζ_B and ζ_F are the bosonic and fermionic parts of the single-letter function $\zeta(x)$ in (3.16). Since $N \otimes \bar{N} = \mathrm{adj} + 1$, we have[32]

$$\chi_{\mathrm{adj}}(g) = \chi_N(g)\chi_{\bar{N}}(g) - 1. \tag{9.10}$$

Suppose $g \in SU(N)$ is diagonal. We can parameterize g as

$$g = \mathrm{diag}\left(e^{i\alpha_1}, e^{i\alpha_2}, \ldots, e^{i\alpha_N}\right), \qquad \sum_{j=1}^{N} \alpha_j = 2\pi k \quad (k \in \mathbb{Z}) \tag{9.11}$$

which implies

$$\chi_{\mathrm{adj}}(g) = \sum_{j,k} e^{i(\alpha_j - \alpha_k)} - 1 = \sum_{j \leqslant k} 2\cos(\alpha_j - \alpha_k) - 1. \tag{9.12}$$

Step 3

We evaluate the Haar measure on $SU(N)$ [75].

[32] N and $\bar{N}$ is the fundamental and anti-fundamental representation of $SU(N)$.

An invariant measure on $U(N)$ is given by

$$dg = \frac{\prod_{i,j=1}^{N} dg_{ij}}{(\det g)^N} \tag{9.13}$$

and the invariance can be readily checked by substituting $\tilde{g} = Mg$,

$$\frac{\prod_{i,j=1}^{N} d\tilde{g}_{ij}}{(\det \tilde{g})^N} = \frac{(\det M)^N \prod_{i,j=1}^{N} dg_{ij}}{(\det Mg)^N} = \frac{\prod_{i,j=1}^{N} dg_{ij}}{(\det g)^N} \,. \tag{9.14}$$

For $g \in SU(N)$ we impose $\det g = 1$.

Since a unitary matrix is diagonalizable, we write

$$g = M\Lambda M^\dagger, \qquad \Lambda = \mathrm{diag}\left(e^{i\alpha_1}, e^{i\alpha_2}, \ldots, e^{i\alpha_N}\right). \tag{9.15}$$

We substitute this decomposition (9.15) into the Haar measure (9.13). Since our integrand (9.9) depends only on Λ, the integral over angular directions is trivial except for its contribution to the Jacobian. The result for the $SU(N)$ integral is

$$dg = \prod_{j=1}^{N} d\alpha_j \prod_{j<k} 4\sin^2\left(\frac{\alpha_j - \alpha_k}{2}\right) d\Omega \;\times\; \delta(\sum_i \alpha_i - 2\pi k), \qquad (k \in \mathbb{Z}) \tag{9.16}$$

where $d\Omega = M^\dagger dM$ is the collection of the integration measure for the angular directions.

In summary, we obtain

$$\begin{aligned}\mathrm{tr}\left(e^{-\beta D}\right) = \int \prod_j d\alpha_j \;\delta\left(\sum_j \alpha_j - 2\pi k\right) \exp\Bigg[\sum_{j\neq k} \log\left|2\sin\frac{\alpha_j - \alpha_k}{2}\right| + \\ \sum_{n=1}^{\infty} \frac{\zeta_B(x^n) - (-1)^n \zeta_F(x^n)}{n}\Big(\sum_{j\leqslant k} 2\cos\left(n\left(\alpha_j - \alpha_k\right)\right) - 1\Big)\Bigg].\end{aligned} \tag{9.17}$$

Step 4

We take the large N limit; namely we introduce the density of eigenvalues $\rho(\alpha)$ and look for the saddle point of the integrand. Operationally, in the large N limit we substitute

$$\sum_n f(\alpha_n) \to N \int d\alpha \;\rho(\alpha) f(\alpha) \tag{9.18}$$

for functions of a single variable, and

$$\sum_{m<n} f(\alpha_m, \alpha_n) \to \frac{N^2}{2} \;\mathrm{p.v.} \iint d\alpha d\beta \;\rho(\alpha)\rho(\beta) f(\alpha, \beta) \tag{9.19}$$

for $f(\alpha,\beta) = f(\beta,\alpha)$. Here p.v. denotes the principal value prescription coming from $\alpha_m \neq \alpha_n$ in the original sum. We also add a Lagrangian multiplier

$$\lambda\left(\int d\alpha\rho(\alpha) - 1\right) \tag{9.20}$$

for the normalization of $\rho(\alpha)$. The factors of N in (9.18), (9.19) should explain the scaling of the free energy $\mathcal{F} = O(N^2)$ unless the coefficient in front vanishes.

The partition function becomes

$$\mathrm{tr}\left(e^{-\beta D}\right) = \int D\rho\, d\lambda\, e^{-S[\rho]+\lambda(\int d\alpha\rho(\alpha)-1)}, \tag{9.21}$$

where[33]

$$S[\rho] = -N^2 \iint d\alpha d\beta\rho(\alpha)\rho(\beta)K(\alpha,\beta) + \sum_{n=1}^{\infty}\frac{\zeta_B\left(x^n\right) - (-1)^n\zeta_F\left(x^n\right)}{n}, \tag{9.22}$$

$$K = \log\left|2\sin\frac{\alpha-\beta}{2}\right| + \sum_{n=1}^{\infty}\frac{\zeta_B\left(x^n\right) - (-1)^n\zeta_F\left(x^n\right)}{n}\cos(n(\alpha-\beta)) \tag{9.23}$$

Since all eigenvalues show up in the difference form $(\alpha-\beta)$, $\rho =$ constant is a saddle point, which gives the free energy $S \sim O(1)$.

Step 5

Let us look for a non-constant solution. Again we expect that the $n = 1$ term gives the dominant contribution to the free energy. After focusing on $n = 1$, this problem becomes the Gross-Witten-Wadia matrix model [76–78].

Two types of solutions are known in the GWW model. In the first type of solution, $\rho(\alpha) \neq 0$ over the interval $-\pi \leqslant \alpha \leqslant \pi$. We can expand the density as

$$\rho(\alpha) \sim c_0 + c_1\cos\alpha, \tag{9.24}$$

which gives $S \sim O(1)$. In the second type, $\rho(\alpha) \neq 0$ over $-\alpha_c \leqslant \alpha \leqslant \alpha_c$ with $\alpha_c < \pi$. The density is given by

$$\rho(\alpha) = \frac{\cos\frac{\alpha}{2}}{\pi\sin^2\frac{\alpha_c}{2}}\sqrt{\sin^2\frac{\alpha_c}{2} - \sin^2\frac{\alpha}{2}}\,, \qquad \sin^2\frac{\alpha_c}{2} = 1 - \sqrt{1-\frac{1}{\zeta(x)}}\,, \tag{9.25}$$

which gives $S \sim O(N^2)$. It is complicated to derive these results analytically because we need to solve a Riemann-Hilbert problem with the periodicity condition on α.[34] According

[33] One should not forget the principal value prescription (9.19).

[34] General methods to solve Riemann-Hilbert problems can be found in e.g. [79, 80].

to [23], the free energy is given by

$$F_> \sim -\frac{N^2}{2}\left\{\zeta - 1 + \sqrt{\zeta^2-\zeta} - \log\left(\zeta + \sqrt{\zeta^2-\zeta}\right)\right\} + \zeta \tag{9.26}$$

$$F_< \sim \zeta \tag{9.27}$$

where $F_>$ ($F_<$) denotes the free energy after (before) the Hagedorn transition. Note that the first term in the RHS of (9.26) vanishes if $\zeta = 1$.

Acknowledgments

R.S thanks Troels Harmark, Yunfeng Jiang and Matthias Wilhelm for their comments about this manuscript.

A Basic Usage of Mathematica

Below we summarize basic methods in Mathematica. Mathematica is frequently used in many areas of physics, but it is usually not a topic covered by the standard curriculum in universities.

A.1 Why Mathematica?

Mathematica is suitable for symbolic or analytic computation more than other software. For example, you can

- integrate or series expand functions
- solve algebraic and differential equations
- guess the general formula from integer patterns

However, there are also troubles, such as

- license is expensive
- math kernel is a black box, sometimes has bugs

If you do not have a Mathematica license, try one of the followings,

- ask your University or Institute
- download the trial version
- use Raspberry Pi. It has Mathematica though it may not work as fast as usual computers.

If you prefer other software, you may try

- **Maple**; popular for engineers. Useful to double-check the numerical precision of Mathematica.[35]
- **SageMath**; free software. The cloud service (CoCalc) has a paid plan [81]

[35] For example, Maple also supports GNU Multiple Precision Arithmetic Library

- **Python**; free software, allows dynamic typing as in Mathematica.[36] Popular for deep learning, quantum computing, Google Colab, etc.

A.2 References

If you are puzzled by a concrete problem, you would probably find a solution at Mathematica StackExchange. At the same time, beginners are recommended to read through one of the textbooks such as [82, 83], because Mathematica has its own structure that you would not realize just by looking at help pages. If you want to learn how Mathematica is used in the actual problem of theoretical high-energy physics, take a look at [84].

A.3 Notation

We begin by explaining the notation. Sometimes you want to increase the readability of your code without changing its behavior. At other times you may want to save your energy.

A simple way to use a concise notation is to redefine built-in commands,

```
In[1]:= FS=FullSimplify;
        FF=FullForm;
```

A more sophisticated way is to use Notation package,

```
In[2]:= <<Notation`
```

We can redefine the human notation (LHS) as the machine notation (RHS),

$$\left\{\text{Notation}\left[\boxed{\delta_{k\_,l\_}} \Longleftrightarrow \boxed{\text{KroneckerDelta}[k\_,l\_]}\right]\right\};$$

Mathematica interpret the following two expressions in the same way,

```
In[3]:= KroneckerDelta[a,b]//FF
        δa,b//FF
```

```
Out[3]//FullForm=
        KroneckerDelta[a,b]
```

```
Out[4]//FullForm=
        KroneckerDelta[a,b]
```

Sometimes, the expressions are indistinguishable until you inspect `FullForm`. Define

$$\left\{\text{Notation}\left[\boxed{X^{a\_}{}_{i\_}^{j\_}} \Longleftrightarrow \boxed{X[a\_,i\_,j\_]}\right]\right\};$$

Here LHS is `Subsuperscript`, and not `Power`. For example,

```
In[5]:= Y^a_-j_-//FF
```

[36] It means that user-defined variables are initially all pointers, and Python determines the variable type during runtime.

```
Out[5]//FullForm=
        Subsuperscript[Superscript[Y,Pattern[a,Blank[]]],Pattern[i,Blank[]],
        Pattern[j,Blank[]]]
```

This is the usual notation.

```
In[6]:= A^2//FF

Out[6]//FullForm=
        Power[A,2]
```

A.4 Functional vs Procedural programming

Mathematica (or Lisp) is a functional programming language, whereas C or Fortran is a procedural programming language [85]. To illustrate what it actually means, let us compute a sum of random real numbers. We generate 10^4 random numbers in the interval $[0, 1]$,

```
In[7]:= repeat=10^4;
        array=RandomReal[1,repeat];
        sum=0;
```

and take the sum in three different methods. The first method is a `For` loop, the second method is replacing `List` by `Plus`, and the third method is applying a list operation `Total`.

```
In[8]:= AbsoluteTiming[For[i=1,i≤repeat,i++,sum+=array[[i]]];sum]
        AbsoluteTiming[array/.List→Plus]
        AbsoluteTiming[Total[array]]

Out[8]= {0.0105,5039.34}

Out[9]= {0.002681,5039.34}

Out[10]= {0.00009,5039.34}
```

The `For` loop is slow, and `Total` is fast.

A.5 Operators and Operations

Examples of Prefix, Postfix and Infix notations are explained below.

When we apply a function, we can use @ instead of square brackets [ ],

```
In[11]:= Table[i,{i,3}];
         {Total[%],Total@%}

Out[11]= {6,6}
```

We can apply a function after the expression by using //,

```
In[12]:= Table[a_i,{i,3}];
         %//Total
```

```
Out[12]= a1+a2+a3
```

If a function has two arguments, we can place the function in between the arguments by using ~,

```
In[13]:= x~f~y
         %/.f→Plus

Out[13]= f[x,y]

Out[14]= x+y
```

This operation may look strange, but the operator + behaves in this way. Finally, when joining text strings, use <>,

```
In[15]:= (* Join text strings *)
         a6=30;
         a<>6//FF
         ToExpression[%]

Out[16]//FullForm=
         a6

Out[17]= 30
```

Let us discuss advanced notation. Since Mathematica can quickly execute list operations, we should learn how to apply a function to a list, a list of lists, a list of lists of lists, and so on.

First, the symbol @@ rewrites the Head of the expressions

```
In[18]:= Apply[Times,Table[ai,{i,3}]]
         Times@@Table[ai,{i,3}]

Out[18]= a1 a2 a3

Out[19]= a1 a2 a3
```

We can understand why we obtained the product of a_i's by looking at `FullForm`,

```
In[20]:= %//FF

Out[20]//FullForm=
         Times[Subscript[a,1],Subscript[a,2],Subscript[a,3]]
```

Next, the symbol /@ distributes an operation over a list,

```
In[21]:= Map[Sqrt,Table[a_i,{i,3}]]
         Sqrt/@Table[a_i,{i,3}]
```

$$\text{Out[21]=}\ \{\sqrt{a_1},\sqrt{a_2},\sqrt{a_3}\}$$

$$\text{Out[22]=}\ \{\sqrt{a_1},\sqrt{a_2},\sqrt{a_3}\}$$

Finally, the symbol @@@ applies a function at one level deeper inside `Table`. The symbol `function@@@variable` is equivalent to `Apply[function, variable,{1}]`. For example, we generate a tensor by

```
In[23]:= mat=Table[a_i b_j c_k,{i,2},{j,3},{k, 4}];
         MatrixForm/@%
```

We obtain

$$\left\{\begin{pmatrix} a_1b_1c_1 & a_1b_1c_2 & a_1b_1c_3 & a_1b_1c_4 \\ a_1b_2c_1 & a_1b_2c_2 & a_1b_2c_3 & a_1b_2c_4 \\ a_1b_3c_1 & a_1b_3c_2 & a_1b_3c_3 & a_1b_3c_4 \end{pmatrix}, \begin{pmatrix} a_2b_1c_1 & a_2b_1c_2 & a_2b_1c_3 & a_2b_1c_4 \\ a_2b_2c_1 & a_2b_2c_2 & a_2b_2c_3 & a_2b_2c_4 \\ a_2b_3c_1 & a_2b_3c_2 & a_2b_3c_3 & a_2b_3c_4 \end{pmatrix}\right\}. \tag{A.1}$$

We apply `Times` on different levels, as

```
In[24]:= {Times@@mat,Times@@@mat,Apply[Times,mat,{2}]};
         MatrixForm/@%
```

The results are

$$\left\{\begin{pmatrix} a_1a_2b_1^2c_1^2 & a_1a_2b_1^2c_2^2 & a_1a_2b_1^2c_3^2 & a_1a_2b_1^2c_4^2 \\ a_1a_2b_2^2c_1^2 & a_1a_2b_2^2c_2^2 & a_1a_2b_2^2c_3^2 & a_1a_2b_2^2c_4^2 \\ a_1a_2b_3^2c_1^2 & a_1a_2b_3^2c_2^2 & a_1a_2b_3^2c_3^2 & a_1a_2b_3^2c_4^2 \end{pmatrix}, \begin{pmatrix} a_1^3b_1b_2b_3c_1^3 & a_1^3b_1b_2b_3c_2^3 & a_1^3b_1b_2b_3c_3^3 & a_1^3b_1b_2b_3c_4^3 \\ a_2^3b_1b_2b_3c_1^3 & a_2^3b_1b_2b_3c_2^3 & a_2^3b_1b_2b_3c_3^3 & a_2^3b_1b_2b_3c_4^3 \end{pmatrix},$$
$$\begin{pmatrix} a_1^4b_1^4c_1c_2c_3c_4 & a_1^4b_2^4c_1c_2c_3c_4 & a_1^4b_3^4c_1c_2c_3c_4 \\ a_2^4b_1^4c_1c_2c_3c_4 & a_2^4b_2^4c_1c_2c_3c_4 & a_2^4b_3^4c_1c_2c_3c_4 \end{pmatrix}\right\}.$$

Here we multiplied a's, b's or c's of (A.1) in different ways, depending on which level `Times` is applied.

A.6 Functions

There are several ways to define a function in Mathematica. Usually, we give a name to a function and specify the argument, like

```
In[25]:= f[x_]:=3+x;
         f[y]
```

$$\text{Out[26]=}\ 3+y$$

Alternatively, we may define a pure function, whose argument is specified by #. When we apply a pure function to a variable, we use @.

```
In[27]:= Function[3+#][x]
         (3+#)&@x

Out[27]= 3+x

Out[28]= 3+x
```

A pure function with two variables can be constructed as

```
In[29]:= (3+#1+#2)&@@{x,y}

Out[29]= 3+x+y
```

Pure functions are useful for manipulating differential equations. Consider the equations of motion of a harmonic oscillator,

```
In[30]:= eqϕ=-D[ϕ[x],{x,2}]-m^2ϕ[x]

Out[30]= -m^2 ϕ[x]-ϕ''[x]
```

This differential equation $\texttt{eq}\phi = 0$ has solutions $\phi(x) = e^{\pm imx}$. However, if we just replace $\phi(x)$ by $e^{i\omega x}$, Mathematica does not replace the derivative term in $\texttt{eq}\phi$. A better solution is to replace ϕ with a pure function,

```
In[31]:= eqϕ;
         %/.ϕ[x_]:→Exp[Iωx]
         %%/.ϕ:→(Exp[Iω#]&)//Factor

Out[32]= -e^(ixω) m^2-ϕ''[x]

Out[33]= e^(ixω)(-m+ω)(m+ω)
```

A.7 Tips for Writing Fast Mathematica Code

We choose two topics from [86] to demonstrate efficient ways of writing a Mathematica code.

The first topic is to remember the value that you will need in the future. Compare two ways of computing Fibonacci numbers,

```
In[34]:= fib[1]=1;
         fib[2]=2;
         fib[n_]:=fib[n]=fib[n-1]+fib[n-2];

         fib2[1]=1;
         fib2[2]=2;
         fib2[n_]:=fib2[n-1]+fib2[n-2];
```

We remember the functional value in the first method, while we do not remember in the second method.

```
In[40]:= AbsoluteTiming[fib[25]]
         AbsoluteTiming[fib2[25]]

Out[40]= {8.×10^-6,121393}

Out[41]= {0.077958,121393}
```

The first method is faster.

Incidentally, Mathematica can guess the functional form of a sequence of integers, ,

```
In[42]:= Table[fib[n],{n,30}]

Out[42]= {1,2,3,5,8,13,21,34,55,89,144,233,377,610,987,1597,2584,4181,6765,10946,

         17711,28657,46368,75025,121393,196418,317811,514229,832040,1346269}

In[43]:= FindSequenceFunction[%]

Out[43]= 1/2 (Fibonacci[#1]+LucasL[#1])&
```

Here the Lucas numbers L_n popped up because Mathematica's `Fibonacci` is defined by

$$\texttt{Fibonacci}[1] = \texttt{Fibonacci}[2] = 1, \quad \texttt{Fibonacci}[n+2] = \texttt{Fibonacci}[n+1]+\texttt{Fibonacci}[n].$$

The second topic is parallelization. `Parallelize` distributes the task to different CPUs. Since it requires some computational overhead, `Parallelize` may not speed up the computation if the task is too simple. Moreover, parallelization is not beneficial if the problem is not parallelizable.[37]

In order to parallelize, we launch multiple kernels,

```
In[44]:= LaunchKernels[]
```

Consider the prime factorization of a large integer,

```
In[45]:= fl[n_]:=Length[FactorInteger[(10^n-1)/9]];
```

We execute this job with and without parallelization,

```
In[46]:= ParallelMap[fl,Range[55,65]]//AbsoluteTiming
         Map[fl,Range[55,65]]//AbsoluteTiming

Out[46]= {0.833741,{8,12,6,8,2,20,7,5,13,15,7}}

Out[47]= {1.538477,{8,12,6,8,2,20,7,5,13,15,7}}
```

`ParallelMap` is faster in this parameter range.

[37] A problem is parallelizable if the computation at the m-th step does not require any of the results at the n-th steps with $n < m$.

B $XXX_{1/2}$ spin chain

We construct the spin-$\frac{1}{2}$ XXX spin chain and study its properties using Mathematica.

B.1 Definitions

```
In[48]:= (* notation *)
         FS=FullSimplify;
         MF=MatrixForm;

         (* commutation *)
         comm[a_,b_]:=a.b-b.a;

         (* 2x2 Pauli matrices *)
         id[n_]:=DiagonalMatrix[ConstantArray[1,n]];
         sp=Sqrt[2]{{0,1},{0,0}};
         sm=Transpose[sp];
         sx=PauliMatrix[1];
         sy=PauliMatrix[2];
         sz=PauliMatrix[3];
```

Let us check the notation,

```
In[49]:= MF/@{sx,sy,sz,sp,sm}
```

which gives

$$\left\{\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \begin{pmatrix}0&-i\\i&0\end{pmatrix},\ \begin{pmatrix}1&0\\0&-1\end{pmatrix},\ \begin{pmatrix}0&\sqrt{2}\\0&0\end{pmatrix},\ \begin{pmatrix}0&0\\\sqrt{2}&0\end{pmatrix}\right\}$$

We construct an operator L_{ij} acting on the i-th and j-th site, with $1 \leqslant i < j \leqslant L$,

$$L_{ij}(\{a_k\}) = a_1 + a_2\,\sigma_i^z\,\sigma_j^z + a_3\,\sigma_i^+\,\sigma_j^- + a_4\,\sigma_i^-\,\sigma_j^+\,. \tag{B.1}$$

This operator acts trivially on the remaining sites. We create L_{ij} by (1) preparing L identity matrices, (2) substituting with Pauli matrices at the sites i and j, and (3) replacing `List` with `KroneckerProduct`.

```
In[50]:= generateTensors[i_,j_,LL_]:=Module[
         {iTab=Table[id[2],{n,LL}],spm,smp,szz},
         spm=ReplacePart[iTab,{i→sp,j→sm}];
         smp=ReplacePart[iTab,{i→sm,j→sp}];
         szz=ReplacePart[iTab,{i→sz,j→sz}];
         Array[a#&,{4}].(KroneckerProduct@@@{iTab,szz,spm,smp})
         ];
```

At $L = 2$, the following matrix is generated,

In[51]:=
```
generateTensors[1,2,2]//MF
```

$$\begin{pmatrix} a_1+a_2 & 0 & 0 & 0 \\ 0 & a_1-a_2 & 2a_3 & 0 \\ 0 & 2a_4 & a_1-a_2 & 0 \\ 0 & 0 & 0 & a_1+a_2 \end{pmatrix}$$

B.2 Solving Yang-Baxter relations

We make the following ansatz for the R-matrix

$$R_{ij}(u_i-u_j) = L_{ij}(\{a_k = b_k(u_i-u_j)+c_k\}), \qquad (k=1,2,3,4) \tag{B.2}$$

In[52]:=
```
R[i_,j_]:=generateTensors[i,j,3]/.a_k_:→b_k(u_i-u_j)+c_k//FS;
```

and we solve the Yang-Baxter relations,

$$R_{12}(u_1-u_2)\,R_{13}(u_1-u_3)\,R_{23}(u_2-u_3) = R_{23}(u_2-u_3)\,R_{13}(u_1-u_3)\,R_{12}(u_1-u_2). \tag{B.3}$$

In[53]:=
```
R12=R[1,2];
R23=R[2,3];
R13=R[1,3];
R12.R13.R23-R23.R13.R12//FS//Flatten//Tally;
eqYB=%[[All,1]]
```

This equation should vanish for any value of u's,

In[54]:=
```
Fold[CoefficientList,eqYB,{u1,u2,u3}]//Flatten//FS//Tally;
eqYB2=%[[All,1]]
```

These equations can be solved in terms of $(b_k\,, c_k)$,

In[55]:=
```
Array[u b#+c#&,{4}]/.Solve[eqYB2==0,Variables[eqYB2]];
solYB=%//DeleteDuplicates
```

We want to pick up the solutions with $\prod_{k=1}^{4} a_k \neq 0$, and normalize them as $a_4 = 1$.

In[56]:=
```
Position[solYB,0];
solPos=Complement[Range[Length[solYB]],DeleteDuplicates[%[[All,1]]]];
Table[Table[solYB[[a,k]]/solYB[[a,4]],{k,4}],{a,solPos}]//DeleteDuplicates//FS
```

The results are

Out[58]=
```
{{1,1+u b2/c3,1,1},{1+u b1/c3,1,1,1},{1,1,1,1}}
```

The last solution does not depend on u's, so only two non-trivial solutions are found. We can remove b and c by rescaling the rapidity u. Thus the R-matrix is given by

```
In[59]:= I generateTensors[1,2,2]/.{a1→1/2-I u,a2→1/2,a3→1/2,a4→1/2}//FS;
         I generateTensors[1,2,2]/.{a1→1/2,a2→1/2-I u,a3→1/2,a4→1/2}//FS;
         MF/@{%%,%}
```

$$\left\{\begin{pmatrix} u+i & 0 & 0 & 0 \\ 0 & u & i & 0 \\ 0 & i & u & 0 \\ 0 & 0 & 0 & u+i \end{pmatrix}, \begin{pmatrix} u+i & 0 & 0 & 0 \\ 0 & -u & i & 0 \\ 0 & i & -u & 0 \\ 0 & 0 & 0 & u+i \end{pmatrix}\right\}.$$

The first solution agrees with [12].

B.3 Diagonalize the Hamiltonian numerically

Here we diagonalize the XXX Hamiltonian under periodic boundary conditions numerically for small L.[38]

$$H_{\rm XXX} = \sum_{i=1}^{L} H_{i,i+1}\,, \qquad H_{i,j} = I_{i,j} - \sigma_i^- \sigma_j^+ - \sigma_i^+ \sigma_j^- - \sigma_i^z \sigma_j^z. \tag{B.4}$$

We recycle the previous code to generate $H_{\rm XXX}$,

```
In[60]:= HXXX[L_]:=HXXX[L]=Sum[generateTensors[i,i+1,L],{i,L-1}]+
         generateTensors[L,1,L]/.a1→1/.ai_:→-1;
```

The Hamiltonian at $L = 2$ is

```
In[61]:= HXXX[2]//MF
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & -4 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and its eigenvalues are $\{8, 0, 0, 0\}$. At $L = 3$,

```
In[62]:= HXXX[3]//MF
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & -2 & 0 & -2 & 0 & 0 & 0 \\ 0 & -2 & 4 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & -2 & -2 & 0 \\ 0 & -2 & -2 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & -2 & 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and its eigenvalues are $\{6, 6, 6, 6, 0, 0, 0, 0\}$. At $L = 4$,

[38] We can derive this Hamiltonian from the transfer matrix, following the standard method in quantum integrable system.

In[63]:= `HXXX[4]//MF`

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & -2 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 4 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & -2 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 4 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 8 & -2 & 0 & 0 & -2 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 4 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & -2 & 0 & 0 & -2 \\
0 & -2 & 0 & 0 & -2 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 4 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 & -2 & 0 & 0 & -2 & 8 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 4 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & -2 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 4 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & -2 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

and its eigenvalues are

$$H_{\rm XXX}(L=4) \sim \mathrm{diag}\{12,8,8,8,4,4,4,4,4,4,4,0,0,0,0,0\}. \tag{B.5}$$

In general, eigenvalues are irrational numbers. For example, at $L=7$ we find roots of the polynomial

$$x^6 - 60\,x^5 + 1416\,x^4 - 16696\,x^3 + 102624\,x^2 - 306784\,x + 337856 = 0.$$

B.4 Solution of the Bethe Ansatz

We compare the energy spectrum of the XXX Hamiltonian with the solutions of Bethe Ansatz Equations,

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L + \prod_{j=1}^{M} \frac{u_k - u_j + i}{u_k - u_j - i} = 0, \qquad (k = 1, 2, \ldots, M). \tag{B.6}$$

We look for regular solutions, meaning that all u's are distinct. Some u's may be infinite, which corresponds to $\mathfrak{su}(2)$ descendants.[39] In $\mathcal{N}=4$ SYM we should impose the periodicity condition, $\sum_{k=1}^{L} p_k \in 2\pi\,\mathbb{Z}$ to guarantee the trace cyclicity in (6.20). The energy of an M-magnon state is given by

$$E = \sum_{j=1}^{M} \frac{8}{4\,u_j^2 + 1}\,. \tag{B.7}$$

In[64]:=
```
eqBAE[L_,M_]:=Table[(u[k]+I/2)/(u[k]-I/2)^L+Product[(u[k]-u[j]+I)/(u[k]-u[j]-I),{j,M}],{k,M}];
enBAE[M_]:=Sum[8/(4u[j]^2+1),{j,M}];
```

[39] The solution whose Bethe roots are all finite corresponds to a $\mathfrak{su}(2)$ highest weight state.

We take a closer look at the $L = 4$ states. For $M = 1$,

```
In[65]:= Solve[eqBAE[4,1]==0,Table[u[i],{i,1}]]//FS
         enBAE[1]/.%//FS
```

$$\text{Out[65]}= \{\{u[1] \to -\frac{1}{2}\},\{u[1] \to 0\},\{u[1] \to \frac{1}{2}\}\}$$
$$\{4,8,4\}$$

which is consistent with (B.5) up to degeneracy. For $M = 2$,

```
In[66]:= MM=2;
         Solve[eqBAE[4,MM]==0,Table[u[i],{i,MM}]]//FS;
         Table[u[i],{i,MM}]/.%
         enBAE[MM]/.%%//FS
```

$$\text{Out[66]}= \{\{-\frac{1}{2\sqrt{3}},\frac{1}{2\sqrt{3}}\},\{\frac{1}{2\sqrt{3}},-\frac{1}{2\sqrt{3}}\},\{-\frac{1}{2}-\frac{1}{\sqrt{2}},-\frac{1}{2}-\frac{1}{\sqrt{2}}\},\{\frac{1}{2}-\frac{1}{\sqrt{2}},\frac{1}{2}-\frac{1}{\sqrt{2}}\},$$
$$\{-\frac{1}{2}+\frac{1}{\sqrt{2}},-\frac{1}{2}+\frac{1}{\sqrt{2}}\},\{\frac{1}{2}+\frac{1}{\sqrt{2}},\frac{1}{2}+\frac{1}{\sqrt{2}}\}\}$$
$$\{12,12,8-4\sqrt{2},4\,(2+\sqrt{2}),4\,(2+\sqrt{2}),8-4\sqrt{2}\}$$

We find the solutions with irrational energies that are not found in (B.5). However, the irrational-energy solutions have coincident roots, and are irregular. The classification of the physical solutions in the $\mathrm{XXX}_{1/2}$ model has been studied in [41, 42].

B.5 Compute higher conserved charges

As an integrable spin chain, the XXX model has higher conserved charges besides the Hamiltonian $Q_2 = H_{\mathrm{XXX}}$. The next conserved charge is given by

$$Q_3 = i \sum_{j=1}^{L} [H_{j,j+1}\,, H_{j+1,j+2}] \tag{B.8}$$

which commutes with Hamiltonian. Equivalently, Q_3 is generated by applying the boost operator

$$Q_3 = -i[BQ_2\,, Q_2] + (\text{boundary term}), \qquad BQ_2 = \sum_{j=1}^{L} j\, H_{j,j+1}\,. \tag{B.9}$$

See [87–90] for further references about higher conserved charges.

We implement (B.8) explicitly,

```
In[67]:= Qsr[3,L_]:=Qsr[3,L]=
         I(Sum[comm[generateTensors[i,i+1,L],generateTensors[i+1,i+2,L]],{i,L-2}]+
         comm[generateTensors[L-1,L,L],generateTensors[L,1,L]+
         comm[generateTensors[L,1,L],generateTensors[1,2,L]])/.a_1→1/.a_i_:→-1;
```

We also implement the boost operator (B.9),

```
In[68]:= (* boost operator *)
         BXXX[L_]:=BXXX[L]=Sum[i generateTensors[i,i+1,L],{i,L-1}]+
         L generateTensors[L,1,L]/.a_1→1/.a_i_:→-1;

         (* for periodic spin chains, we need to subtract the boundary term *)
         Qbd[3,L_]:=Qbd[3,L]=
         comm[generateTensors[L,1,L],generateTensors[1,2,L]]/.a_1→1/.a_i_:→-1;
         QXXX[3,L_]:=QXXX[3,L]=-I(comm[BXXX[L],HXXX[L]]-L Qbd[3,L]);
```

We can check that the two results agree, and they commute with the XXX Hamiltonian,

```
In[69]:= Table[QXXX[3,L]==Qsr[3,L],{L,2,5}]
         Table[comm[Qsr[3,L],HXXX[L]]==0*IdentityMatrix[2^L],{L,2,5}]

Out[69]= {True,True,True,True}
         {True,True,True,True}
```

Let us evaluate Q_3 in the basis where $H_{\rm XXX}$ is diagonal.

```
In[70]:= Do[
         {eval[ll],evec[ll]}=FS@Eigensystem[HXXX[ll]];
         QsrDiag[3,ll]=Inverse[Transpose@evec[ll]].Qsr[3,ll].Transpose[evec[ll]]//FS;
         ,{ll,2,5}]
```

Actually this does not work, and Q_3 is not diagonal yet.

```
In[71]:= MF/@FS[{Inverse[Transpose@evec[4]].HXXX[4].Transpose[evec[4]],QsrDiag[3,4]}]
```

$$\left\{\begin{pmatrix}
12&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&8&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&8&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&8&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&4&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix},\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&16i&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-16i&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&8i&0&-8i&0&0&0&0&0&0\\
0&0&0&0&0&0&-16i&-8i&0&-8i&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-16i&0&0&0&0&0\\
0&0&0&0&0&0&16i&8i&0&8i&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&16i&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}\right\}$$

This is because we need to resolve the ambiguity among degenerate eigenvectors, particularly those corresponding to the eigenvalue 4; $Q_2\,\psi = 4\psi$,

```
In[72]:= Position[eval[4],4]//Flatten

Out[72]= {5,6,7,8,9,10,11}
```

Let us directly diagonalize Q_3 in the degenerate eigenspace,

```
In[73]:= {oval[4],ovec[4]}=Eigensystem[QsrDiag[3,4][[5;;11,5;;11]]];
         rot[4]=ArrayFlatten[{{IdentityMatrix[4],0,0},{0,ovec[4],0},
         {0,0,IdentityMatrix[5]}}];
```

Now Q_2 and Q_3 at $L = 4$ are both diagonal,

```
In[74]:= MF/@FS[{Inverse[Transpose@rot[4]].Inverse[Transpose@evec[4]].HXXX[4].
         Transpose[evec[4]].Transpose[rot[4]],
         Inverse[Transpose@rot[4]].Inverse[Transpose@evec[4]].Qsr[3,4].
         Transpose[evec[4]].Transpose[rot[4]]}]
```

$$\left\{\begin{pmatrix}
12&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&8&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&8&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&8&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&4&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix},\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-16&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-16&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-16&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&16&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&16&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&16&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}\right\}$$

References

[1] J.M. Maldacena, *The large n limit of superconformal field theories and supergravity*, arXiv:hep-th/9711200.

[2] G. 't Hooft, *A Planar Diagram Theory for Strong Interactions*, *Nucl. Phys.* **B72** (1974) 461.

[3] G. 't Hooft, *Dimensional Reduction in Quantum Gravity*, *Conf. Proc. C* **930308** (1993) 284 [gr-qc/9310026].

[4] B. Hoare, N. Levine and A.A. Tseytlin, *On the Massless Tree-Level S-Matrix in 2D Sigma Models*, *J. Phys. A* **52** (2019) 144005 [1812.02549].

[5] M.V. Berry and M. Tabor, *Level clustering in the regular spectrum*, *Proc. R. Soc. Lond. A* **356** (1977) 375.

[6] A.V. Kotikov and L.N. Lipatov, *DGLAP and BFKL equations in the $N = 4$ supersymmetric gauge theory*, *Nucl. Phys. B* **661** (2003) 19 [hep-ph/0208220].

[7] A.V. Kotikov, L.N. Lipatov, A.I. Onishchenko and V.N. Velizhanin, *Three loop universal anomalous dimension of the Wilson operators in $N = 4$ SUSY Yang-Mills model*, *Phys. Lett. B* **595** (2004) 521 [hep-th/0404092].

[8] E. Pomoni, R. Rabe and K. Zoubos, *Dynamical spin chains in 4D $\mathcal{N} = 2$ SCFTs*, *JHEP* **08** (2021) 127 [2106.08449].

[9] N. Beisert, C. Ahn, L.F. Alday, Z. Bajnok, J.M. Drummond, L. Freyhult et al., *Review of AdS/CFT Integrability: An Overview*, arXiv:1012.3982.

[10] N. Beisert, "*Introduction to Integrability*." https://people.phys.ethz.ch/~nbeisert/lectures/Int-16HS-Notes.pdf, 2016.

[11] D. Bombardelli, A. Cagnazzo, R. Frassek, F. Levkovich-Maslyuk, F. Loebbert, S. Negro et al., *An integrability primer for the gauge-gravity correspondence: An introduction*, *J. Phys. A* **49** (2016) 320301 [1606.02945].

[12] L.D. Faddeev, *How Algebraic Bethe Ansatz Works for Integrable Model*, in *Les Houches School of Physics: Astrophysical Sources of Gravitational Radiation*, pp. pp. 149–219, 5, 1996 [hep-th/9605187].

[13] F.H.L. Essler, H. Frahm, F. Göhmann, A. Klümper and V.E. Korepin, *The One-Dimensional Hubbard Model*, Cambridge University Press (2005), 10.1017/CBO9780511534843.

[14] London Integrability Journal Club. https://integrability-london.weebly.com/likbez.html.

[15] E. Witten, *Baryons and Branes in Anti-de Sitter Space*, *JHEP* **07** (1998) 006 [hep-th/9805112].

[16] Y. Nakayama, *Scale Invariance Vs Conformal Invariance*, *Phys. Rept.* **569** (2015) 1 [1302.0884].

[17] W. Nahm, *Supersymmetries and Their Representations*, *Nucl. Phys. B* **135** (1978) 149.

[18] M. Günaydin and D. Minic, *Singletons, Doubletons and M Theory*, *Nucl. Phys. B* **523** (1998) 145 [hep-th/9802047].

[19] E. Witten, *Conformal Field Theory in Four and Six Dimensions*, in *Symposium on Topology, Geometry and Quantum Field Theory (Segalfest)*, pp. 405–419, 12, 2007 [0712.0157].

[20] M.F. Sohnius, *Introducing Supersymmetry*, *Phys. Rept.* **128** (1985) 39.

[21] E. D'Hoker and D.Z. Freedman, *Supersymmetric Gauge Theories and the AdS/CFT Correspondence*, arXiv:hep-th/0201253.

[22] J.A. Minahan, *Review of AdS/CFT Integrability, Chapter I.1: Spin Chains in N=4 Super Yang-Mills*, arXiv:1012.3983.

[23] B. Sundborg, *The Hagedorn Transition, Deconfinement and N=4 SYM Theory*, arXiv:hep-th/9908001.

[24] A.M. Polyakov, *Gauge fields and space-time*, arXiv:hep-th/0110196.

[25] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas and M.V. Raamsdonk, *The hagedorn/deconfinement phase transition in weakly coupled large n gauge theories*, arXiv:hep-th/0310285.

[26] B. Feng, A. Hanany and Y.-H. He, *Counting Gauge Invariants: the Plethystic Program*, *JHEP* **03** (2007) 090 [hep-th/0701063].

[27] F. Harary and E. Palmer, *The power group enumeration theorem*, *Journal of Combinatorial Theory* **1** (1966) 157.

[28] M. Spradlin and A. Volovich, *The one-loop partition function of super-Yang–Mills theory on*, *Nuclear Physics B* **711** (2005) 199.

[29] M. Bianchi, *Higher Spins and Stringy AdS5 x S5*, arXiv:hep-th/0409304.

[30] D. Yamada and L.G. Yaffe, *Phase Diagram of N=4 Super-Yang-Mills Theory with R-Symmetry Chemical Potentials*, arXiv:hep-th/0602074.

[31] M. Bianchi, F.A. Dolan, P.J. Heslop and H. Osborn, *N=4 superconformal characters and partition functions*, arXiv:hep-th/0609179.

[32] B. Sundborg, *Thermodynamics of Superstrings at High-energy Densities*, *Nucl. Phys. B* **254** (1985) 583.

[33] M. Laine and A. Vuorinen, *Basics of thermal field theory - a tutorial on perturbative computations*, arXiv:1701.01554.

[34] J.J. Atick and E. Witten, *The Hagedorn Transition and the Number of Degrees of Freedom of String Theory*, *Nucl. Phys. B* **310** (1988) 291.

[35] M. Kruczenski and A. Lawrence, *Random Walks and the Hagedorn Transition*, *JHEP* **07** (2006) 031 [hep-th/0508148].

[36] S.W. Hawking and D.N. Page, *Thermodynamics of Black Holes in anti-De Sitter Space*, *Commun. Math. Phys.* **87** (1983) 577.

[37] E. Witten, *Anti-de Sitter Space, Thermal Phase Transition, And Confinement In Gauge Theories*, arXiv:hep-th/9803131.

[38] M. Natsuume, *AdS/CFT Duality User Guide*, vol. 903 (2015), 10.1007/978-4-431-55441-7, [1409.3575].

[39] T. Hartman, "*Lectures on Quantum Gravity and Black Holes*." http://www.hartmanhep.net/topics2015/gravity-lectures.pdf, 2015.

[40] A. Doikou, S. Evangelisti, G. Feverati and N. Karaiskos, *Introduction to quantum integrability*, arXiv:0912.3350.

[41] W. Hao, R.I. Nepomechie and A.J. Sommese, *Completeness of Solutions of Bethe's Equations*, *Phys. Rev. E* **88** (2013) 052113 [1308.4645].

[42] C. Marboe and D. Volin, *Fast Analytic Solver of Rational Bethe Equations*, *J. Phys. A* **50** (2017) 204002 [1608.06504].

[43] N. Beisert, *The su(2|2) Dynamic S-Matrix*, arXiv:hep-th/0511082.

[44] N. Beisert, *The Analytic Bethe Ansatz for a Chain with Centrally Extended su(2|2) Symmetry*, arXiv:nlin/0610017.

[45] S. Komatsu and Y. Wang, *Non-perturbative Defect One-Point Functions in Planar $\mathcal{N} = 4$ Super-Yang-Mills*, arXiv:2004.09514.

[46] N. Gromov and G. Sizov, *Exact Slope and Interpolating Functions in ABJM Theory*, arXiv:1403.1894.

[47] N. Beisert, *The S-Matrix of AdS/CFT and Yangian Symmetry*, arXiv:0704.0400.

[48] G. Arutyunov and S. Frolov, *On String S-matrix, Bound States and TBA*, arXiv:0710.1568.

[49] R.A. Janik, *The* $\mathrm{AdS}_5 \times S^5$ *Superstring Worldsheet S-Matrix and Crossing Symmetry*, *Phys. Rev. D* **73** (2006) 086006 [hep-th/0603038].

[50] L. Castillejo, R.H. Dalitz and F.J. Dyson, *Low's Scattering Equation for the Charged and Neutral Scalar Theories*, *Phys. Rev.* **101** (1956) 453.

[51] N. Beisert, B. Eden and M. Staudacher, *Transcendentality and Crossing*, arXiv:hep-th/0610251.

[52] N. Gromov, V. Kazakov, S. Leurent and D. Volin, *Quantum Spectral Curve for Planar $\mathcal{N} = 4$ Super-Yang-Mills Theory*, *Phys. Rev. Lett.* **112** (2014) 011602 [1305.1939].

[53] T. Harmark and M. Wilhelm, *The Hagedorn temperature of AdS5/CFT4 via integrability*, arXiv:1706.03074.

[54] T. Harmark and M. Wilhelm, *The Hagedorn temperature of AdS5/CFT4 at finite coupling via the Quantum Spectral Curve*, arXiv:1803.04416.

[55] R. Suzuki, *Refined Counting of Necklaces in One-loop $\mathcal{N} = 4$ SYM*, *JHEP* **06** (2017) 055 [1703.05798].

[56] A.B. Zamolodchikov, *Thermodynamic Bethe Ansatz in Relativistic Models. Scaling Three State Potts and Lee-yang Models*, *Nucl. Phys. B* **342** (1990) 695.

[57] Y. Jiang, S. Komatsu and E. Vescovi, *Structure constants in $\mathcal{N} = 4$ SYM at finite coupling as worldsheet g-function*, *JHEP* **07** (2020) 037 [1906.07733].

[58] M. Takahashi, *One-Dimensional Hubbard Model at Finite Temperature: *, *Progress of Theoretical Physics* **47** (1972) 69 [https://academic.oup.com/ptp/article-pdf/47/1/69/5305353/47-1-69.pdf].

[59] G. Arutyunov and S. Frolov, *String Hypothesis for the $\mathrm{AdS}_5 \times S^5$ Mirror*, *JHEP* **03** (2009) 152 [0901.1417].

[60] D. Volin, *String Hypothesis for Gl(N|M) Spin Chains: a Particle/Hole Democracy*, *Lett. Math. Phys.* **102** (2012) 1 [1012.3454].

[61] N. Gromov, V. Kazakov and P. Vieira, *Exact Spectrum of Anomalous Dimensions of Planar $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory*, *Phys. Rev. Lett.* **103** (2009) 131601 [0901.3753].

[62] A. Cavaglia, D. Fioravanti and R. Tateo, *Extended Y-system for the AdS_5/CFT_4 correspondence*, *Nucl. Phys. B* **843** (2011) 302 [1005.3016].

[63] A. Kuniba, T. Nakanishi and J. Suzuki, *T-Systems and Y-Systems in Integrable Systems*, *J. Phys. A* **44** (2011) 103001 [1010.1344].

[64] N. Gromov, V. Kazakov and P. Vieira, *Integrability for the Full Spectrum of Planar AdS/CFT*, arXiv:0901.3753.

[65] N. Gromov, V. Kazakov and Z. Tsuboi, *PSU(2,2|4) Character of Quasiclassical AdS/CFT*, arXiv:1002.3981.

[66] N. Beisert, *Review of AdS/CFT Integrability, Chapter Vi.1: Superconformal Symmetry*, *Lett. Math. Phys.* **99** (2012) 529 [1012.4004].

[67] Bars, I. and Günaydin, M., *Unitary Representations of Noncompact Supergroups*, *Commun. Math. Phys.* **91** (1983) 31.

[68] N. Gromov, V. Kazakov, S. Leurent and Z. Tsuboi, *Wronskian Solution for AdS/CFT Y-System*, *JHEP* **01** (2011) 155 [1010.2720].

[69] N. Gromov, V. Kazakov, S. Leurent and D. Volin, *Quantum spectral curve for AdS_5/CFT_4*, arXiv:1305.1939.

[70] T. Harmark and M. Wilhelm, *Solving the Hagedorn Temperature of AdS5/CFT4 via the Quantum Spectral Curve: Chemical Potentials and Deformations*, 2109.09761.

[71] E.Y. Urbach, *String Stars in Anti De Sitter Space*, 2202.06966.

[72] B.S. Skagerstam, *On the Large N_c Limit of the SU(N_c) Color Quark - Gluon Partition Function*, *Z. Phys. C* **24** (1984) 97.

[73] A.T. Kristensson and M. Wilhelm, *From hagedorn to lee-yang: Partition functions of $\mathcal{N} = 4$ sym theory at finite n*, arXiv:2005.06480.

[74] R.A. Bertlmann, "*Theoretical Physics, Quantum Mechanics*." https://homepage.univie.ac.at/reinhold.bertlmann/lectures/.

[75] M.L. Mehta, *Random Matrices*, Elsevier, 3rd ed. (2004).

[76] D.J. Gross and E. Witten, *Possible Third Order Phase Transition in the Large N Lattice Gauge Theory*, *Phys. Rev. D* **21** (1980) 446.

[77] S.R. Wadia, *N = Infinity Phase Transition in a Class of Exactly Soluble Model Lattice Gauge Theories*, *Phys. Lett. B* **93** (1980) 403.

[78] J.G. Russo and M. Tierz, *Multiple phases in a generalized Gross-Witten-Wadia matrix model*, arXiv:2007.08515.

[79] N.I. Muskhelishvili, *Singular Integral Equations: Boundary Problems of Function Theory and Their Application to Mathematical Physics*, Springer, Dordrecht (1958).

[80] N. Mann and J. Polchinski, *Bethe Ansatz for a Quantum Supercoset Sigma Model*, arXiv:hep-th/0508232.

[81] https://www.sagemath.org/index.html.

[82] S. Wolfram, *The Mathematica Book*, Wolfram Media, Inc., 5th ed. (2003).

[83] S. Mangano, *Mathematica Cookbook: Building Blocks for Science, Engineering, Finance, Music, and More*, O'Reilly Media, 1st ed. (2010).

[84] Mathematica Summer School on Theoretical Physics. http://msstp.org/.

[85] "Comparisons between procedural and functional programming." https://community.wolfram.com/groups/-/m/t/1681993.

[86] J. McLoone, "10 tips for writing fast mathematica code." https://blog.wolfram.com/2011/12/07/10-tips-for-writing-fast-mathematica-code/, 2011.

[87] M.P. Grabowski and P. Mathieu, *Integrability Test for Spin Chains*, *J. Phys. A* **28** (1995) 4777 [hep-th/9412039].

[88] N. Beisert, L. Fiévet, M. de Leeuw and F. Loebbert, *Integrable Deformations of the XXZ Spin Chain*, *J. Stat. Mech.* **1309** (2013) P09028 [1308.1584].

[89] Y. Jiang, I. Kostov, F. Loebbert and D. Serban, *Fixing the Quantum Three-Point Function*, *JHEP* **04** (2014) 019 [1401.0384].

[90] M. De Leeuw, A. Pribytok and P. Ryan, *Classifying Two-Dimensional Integrable Spin Chains*, *J. Phys. A* **52** (2019) 505201 [1904.12005].