Gauge theory correlators, moduli space and quiver calculus

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based on aXiv:1810.09478 + work in progress

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Motivation

4D $\mathcal{N}$=4 Super Yang-Mills ($\mathcal{N}$=4 SYM) with U(*Nc*) or SU(*Nc*) gauge group in the planar limit is (expected to be) integrable

$$\lambda = N_c \, g_{\mathrm{YM}}^2 \; (N_c \to \infty)$$

Integrability helps compute the (connected, planar) n-point functions

Single-trace operator = Bethe Ansatz eigenstate

$$\mathcal{O}(x) = \mathrm{tr}\,(Z X Z \ldots) + \mathrm{tr}\,(Z Z X \ldots) + \ldots$$

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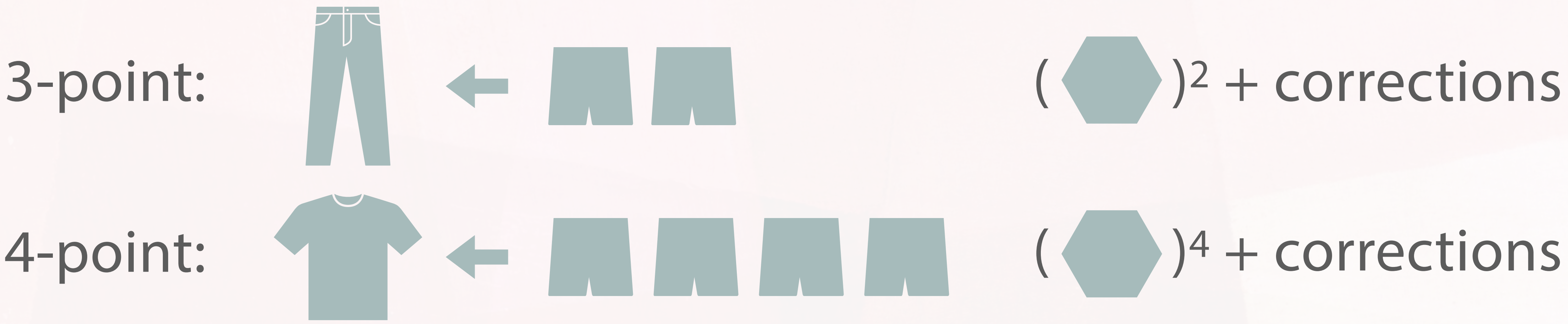
$$\mathcal{O}(x) = \mathrm{tr}\,(ZXZ\ldots) + \mathrm{tr}\,(ZZX\ldots) + \ldots$$

2-point: $\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = |x|^{-2\Delta_{\mathcal{O}}}$ = Bethe Ansatz + corrections

n-point: = Hexagon bootstrap

[Basso, Komatsu, Vieira] (2015) [Fleury, Komatsu] (2016) [Bargheer, Caetano, Fleury, Komatsu, Vieira] (2017)

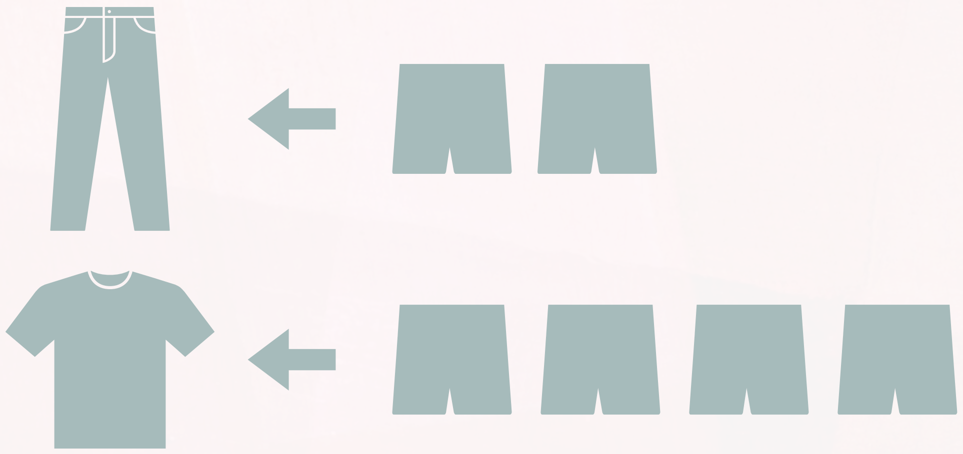
Motivation (cont'd)



3-point: ← (⬢)^2 + corrections

4-point: ← (⬢)^4 + corrections

Motivation (cont'd)



3-point: (⬢)^2 + corrections

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$$\textbf{Wrapping corrections} \;\sim \boldsymbol{\lambda}^{2\ell_{ij}} \; (\boldsymbol{\lambda} \ll 1)$$

$$\ell_{ij} = \#\ \textbf{of Wick contractions between}\ \mathcal{O}_i, \mathcal{O}_j$$

Corrections may be singular for extremal n-point $(\ell_{ij} = 0)$

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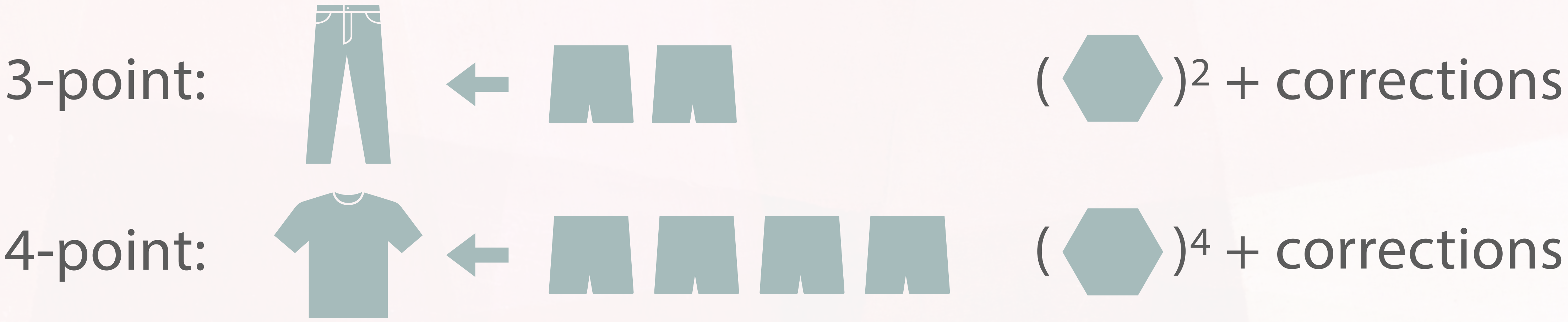

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Why?

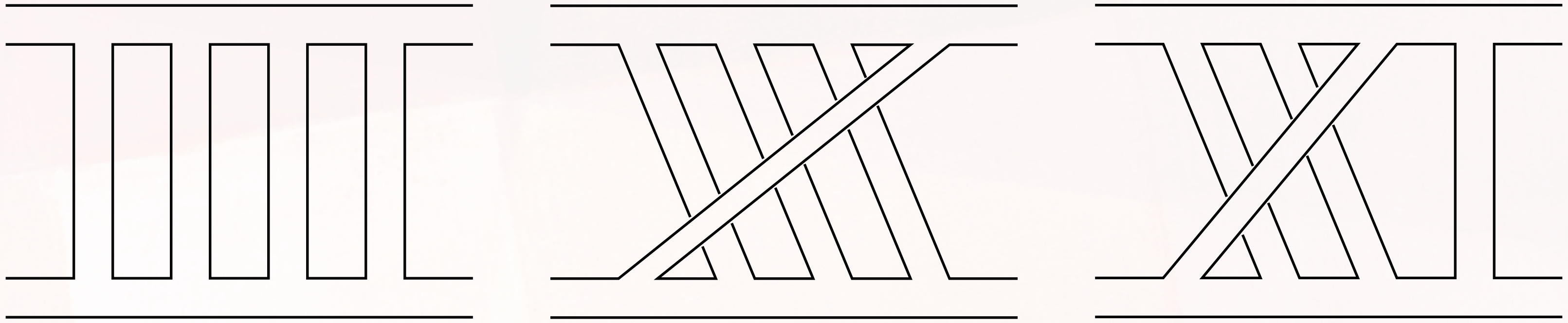
- Mixing between single- and multi-trace states
- Connected n-pt ~ $O\,(1/Nc^{\,n\text{-}2})$ ~ knows non-planar effects

To study non-planar effects, needs two-parameter (g,n) space

e.g. $\Sigma_{g=1,n-2}$ degenerates to $\Sigma_{g=0,n}$ in the moduli space $\mathcal{M}_{g,n}$

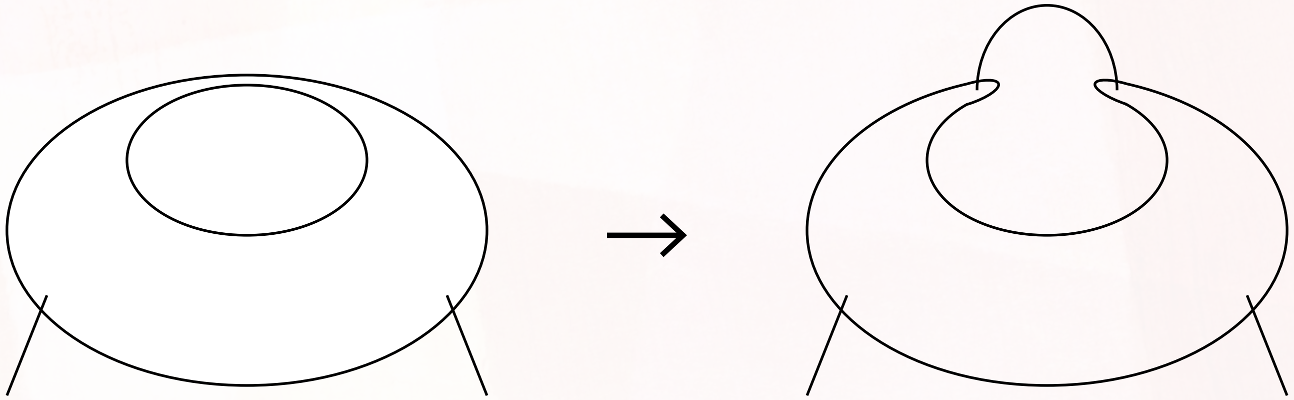
Study $\Sigma_{g,n}$

$1/N_c$ expansion ⇒ $g > 0$



Study $\mathcal{M}_{g,n}$

Boundary of moduli space ⇒ $n > 2$



Main Questions

Find formulae of the non-planar **multi-trace** n-point

Clarify the relation between the single-trace n-point and the **moduli space** of Riemann surfaces from gauge theory (AdS/CFT)

Old ideas

Relate Permutations and Riemann surfaces!

[BPS correlators]

Permutations ~ $c = 1$ Matrix Models

(→ 2d Yang-Mills →TQFT) →Riemann surfaces

Gopakumar, Pius (1212.1236)

[Belyi map]

$$(\boldsymbol{\sigma_0}\,, \boldsymbol{\sigma_1}\,, \boldsymbol{\sigma_\infty}) \in \boldsymbol{S_L}\ (\boldsymbol{L \to \infty}) \quad \to \quad \mathbb{C}\boldsymbol{P_1}\ \textbf{with 3 punctures}$$

Around each puncture, σ permutes the *L* sheets of complex plane

Relevant in arithmetic geometry,
but unclear if this idea is useful for physics

de Mello Koch, Ramgoolam (1002.1634)
From Matrix Models and Quantum Fields to Hurwitz Space and the absolute Galois Group

Tree-level Multi-trace Correlators

Notation

Define **permutation basis** for general scalar multi-trace operators of $\mathcal{N}$=4 SYM (or any gauge theory with adjoint matters)

$$\textbf{Take } \alpha \in S_L = \textbf{Symmetric group of order } L$$

$$\mathcal{O}_\alpha^{A_1 A_2 \ldots A_L} = \mathrm{tr}_L(\alpha\, \Phi^{A_1} \Phi^{A_2} \ldots \Phi^{A_L})$$

$$\equiv \sum_{a_1, a_2 \ldots, a_L = 1}^{N_c} (\Phi^{A_1})^{a_1}_{a_{\alpha(1)}} (\Phi^{A_2})^{a_2}_{a_{\alpha(2)}} \ldots (\Phi^{A_L})^{a_L}_{a_{\alpha(L)}}$$

Multi-trace structure of $\mathcal{O}$ = Cycle type of permutation α

$$\text{e.g. Double-trace} \quad \leftrightarrow \quad \alpha = (1, 2, \ldots, J)(J+1, J+2, \ldots, L) \in \mathbb{Z}_J \times \mathbb{Z}_{L-J}$$

Sum over Wick contractions = Permutation

Tree-level 2-point in permutation basis:

$$\langle \mathcal{O}^{\vec{A}}_{\alpha}(x)\mathcal{O}^{\vec{B}}_{\beta}(0)\rangle = \sum_{\sigma\in S_L}\left(\prod_{p=1}^{L}\frac{g^{A_{\sigma^{-1}(p)}B_p}}{|x|^2}\right)N_c^{C_L(\alpha\sigma^{-1}\beta\sigma)}$$

$$C_L(\lambda) = \#\text{ of cycles in }\lambda\in S_L,\ C_L(\text{id}) = C_L\Big((1)(2)(3)\ldots(L)\Big) = L$$

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Main Problem

$$\text{Sum over }\left\{\ell_{ij}\in\mathbb{Z}_{\geq 0}\middle|\ \sum_{j\neq i}\ell_{ij} = L_i\right\}\ \ (\text{for } n\geq 4)$$

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Edge-based formula

Several formulae are derived in different ways. Here is one:

1) Define the extended operator $(L_i \to L)$

$$\hat{\mathcal{O}}_i \equiv \mathcal{O}_{\alpha_i} \times \mathrm{tr}\,(1)^{L-L_i} \equiv \prod_{p=1}^{L} (\Phi^{\hat{A}_p^{(i)}})^{a_p}_{a_{\hat{\alpha}_i(p)}} , \quad (\hat{\alpha}_i \in S_L)$$

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2) Introduce n-tuple Wick contraction

$$(\Phi^{\hat{A}_1})^{b_1}_{a_1} \, (\Phi^{\hat{A}_2})^{b_2}_{a_2} \, (\Phi^{\hat{A}_3})^{b_3}_{a_3} \, \dots \, (\Phi^{\hat{A}_n})^{b_n}_{a_n} = \underline{h^{\hat{A}_1 \hat{A}_2 \dots \hat{A}_n}} \; \delta^{b_2}_{a_1} \, \delta^{b_3}_{a_2} \, \dots \, \delta^{b_n}_{a_1}$$

Equal to $g^{A_i A_j} |x_i - x_j|^{-2}$ if $\hat{A}_k = 1 \ (k \neq i, j)$

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Equal to $g^{A_i A_j} |x_i - x_j|^{-2}$ if $\hat{A}_k = 1 \ (k \neq i, j)$

3) Take all n-tuple Wick contractions, individually specified by

$$(W_{12}\,, W_{23}\,, \dots \,, W_{n1}) \in S_L^{\otimes n}, \quad W_{12} \, W_{23} \, \dots \, W_{n1} = 1$$

Edge-based formula (cont'd)

Tree-level n-point in permutation basis:

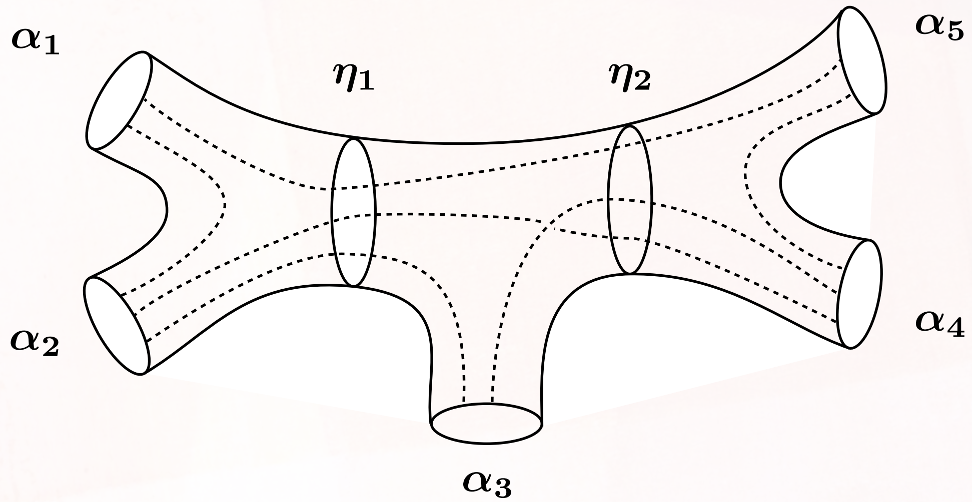
$$\left\langle \prod_{i=1}^{n} \mathcal{O}_i(x_i) \right\rangle = \frac{1}{L!\prod_{i=1}^{n}(L-L_i)!} \sum_{\{U_i\}\in S_L^{\otimes n}} \underline{\mathfrak{H}_n}\, N_c^{C_L(\breve{\alpha}_1\, \breve{\alpha}_2 \dots \breve{\alpha}_n)}$$

Flavor factor:

$$\mathfrak{H}_n = \prod_{p=1}^{L} h^{\breve{A}_p^{(1)}\, \breve{A}_p^{(2)} \dots \breve{A}_p^{(n)}}$$

Notation: $W_{ij} = U_i\, U_j^{-1}, \quad \breve{\alpha}_k = U_k^{-1}\, \hat{\alpha}_k\, U_k\,, \quad \breve{A}_p^{(k)} = \hat{A}^{(k)}_{U_k(p)}$

Pants decomposition:



Correlators and Moduli space

Decorated Moduli Space

Difficult to construct & integrate over $\mathcal{M}_{g,n}$ of Riemann surface $\Sigma_{g,n}$

$$\mathcal{M}_{g,n} \overset{\text{locally}}{\sim} \mathbb{C}^{n-3+3g}, \quad \mathcal{M}_{g,n} \simeq \begin{cases} \{z_4, \ldots, z_n\} & (g=0) \\ \{z_2, \ldots, z_n\} \cup \{\tau\} & (g=1) \\ \{z_1, \ldots, z_n\} \cup \{\mu_1, \ldots, \mu_{3g-3}\} & (g \geq 2) \end{cases}$$

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- Consider the decorated moduli space (extra n parameters)
- We can specify the complex structure of $\Sigma_{g,n}$ uniquely by the Jenkins-Strebel quadratic differential

$$\varphi = -\frac{L_i^2}{4\pi^2}\frac{dz^2}{(z-z_i)^2} + O(|z-z_i|^{-1}), \quad (i = 1, 2, \dots, n)$$

(~ classical stress-energy tensor on worldsheet)

[Jenkins, Strebel, Harer, Mumford, Thurston, …]

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$$\int \sqrt{\varphi} \leftrightarrow \textbf{Dual length between punctures}$$

[Jenkins, Strebel, Harer, Mumford, Thurston, …]

Moduli space from Gauge theory

Idea

$$\textbf{Sum over } \left\{ \ell_{ij} \in \mathbb{Z}_{\geq 0} \middle| \sum_{j \neq i} \ell_{ij} = L_i \right\} \ \ (\textbf{for } n \geq 4)$$

= Discretized moduli space of Riemann surfaces $\mathcal{M}_{g,n}$

[Gopakumar, Aharony, Komargodski, Razamat, David, Charbonnier, Eynard, …]

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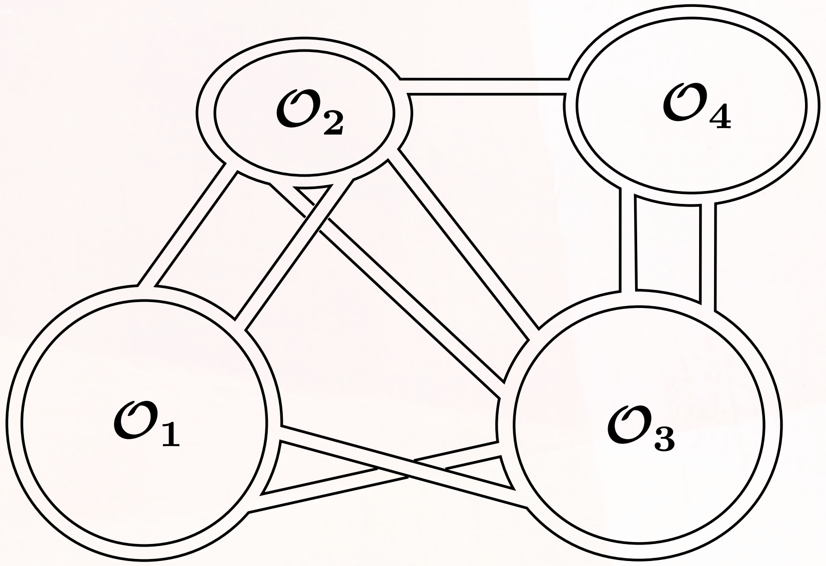
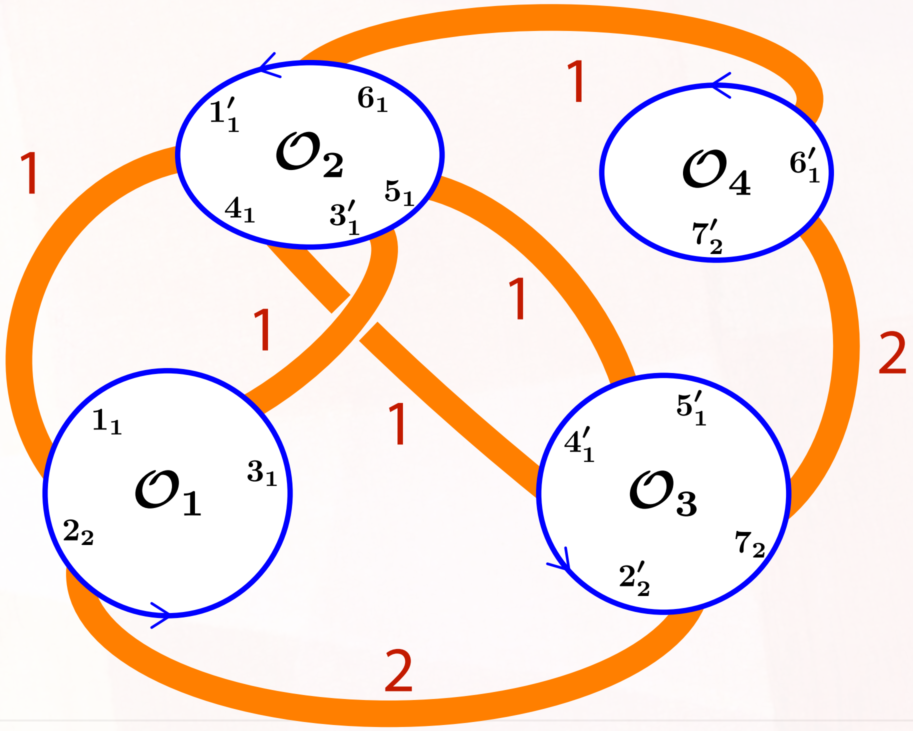
Apply a skeleton reduction to Feynman graphs (not unique), and count the number of consecutive Wick contractions

→ (connected) metric ribbon graph





Correlator as Moduli space

- JS length ↔ Decorated moduli space $\Sigma_{g,n}$
- Sum over the number of Wick contractions $\{\ell_{ij}\}$

  ↭ Decorated moduli space in gauge theory via $1/N_c$ expansion

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➤ The dimensions of both moduli spaces agree:

$$\mathcal{M}_{g,n} \sim \mathbb{C}^{n-3+3g}, \quad \mathcal{M}_{g,n}^{\text{gauge}}(\{L_i\}) \sim \mathbb{Z}^{n-3+3g} \ (\ell_{ij} \gg 1)$$

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Self-contractions are forbidden in gauge theory, unlike matrix models

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Flavor data decouple from color data in the correlators

AdS/CFT: Differential = Stress-energy tensor?

Quiver Calculus

Multi-trace from Permutation

Non-planar BPS states are not single-traces, but a sum of multi-traces

$$\underset{\text{representation basis}}{\underline{\mathcal{O}^R(Z)}} = \sum_{\underline{\alpha \in S_L}} \underline{\chi^R(\alpha)}\, \underset{\text{permutation basis}}{\underline{\mathrm{tr}_L(\alpha\, Z^{\otimes L})}}$$

S_L : symmetric group

χ^R : character in the representation R

[Corley, Jevicki, Ramgoolam (2001)]

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General scalar multi-trace operators of $\mathcal{N}$=4 SYM in permutation basis

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- Why the representation basis is good?
- How to define general scalar scalar multi-trace operators in the representation basis?

Tree-level 2-pt revisited

$$\mathcal{O}_\alpha \in \ \mathbf{Hilb} \ \Rightarrow \ \alpha \in S_L$$

Element of abstract group (quantum)

⇩

δ function

Character decomposition

$$\delta_L(\alpha) = \begin{cases} 1 & \text{if } \alpha = \mathrm{Id} \\ 0 & \text{otherwise} \end{cases}$$

⇩

$D^R_{ij}(\alpha)$ Matrix elements of an irreducible representation *R* (classical)

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$D^R_{ij}(\alpha)$ Matrix elements of an irreducible representation *R* (classical)

⇒ Evaluate two-point functions for any *Nc*

$$\langle \mathcal{O}^R(Z)\mathcal{O}^S(\overline{Z})\rangle \approx N_c^L \delta^{RS}$$

Quiver calculus

[de Mello Koch, Smolic, Smolic] (hep-th/0701066) [Paskounis, Ramgoolam] (1301.1980), [Mattioli, Ramgoolam] (1601.06086)

General operators ⇒ General "restricted" characters

Permutation basis for "simplest" non-BPS operators

$$\mathcal{O}_{\alpha}^{m,n} = \mathrm{tr}_{L}(\alpha\, W^{\otimes m} Z^{\otimes n})$$

$$\alpha \in S_{m+n} \ \ \text{mod conjugacy class of } \gamma \in S_m \times S_n$$

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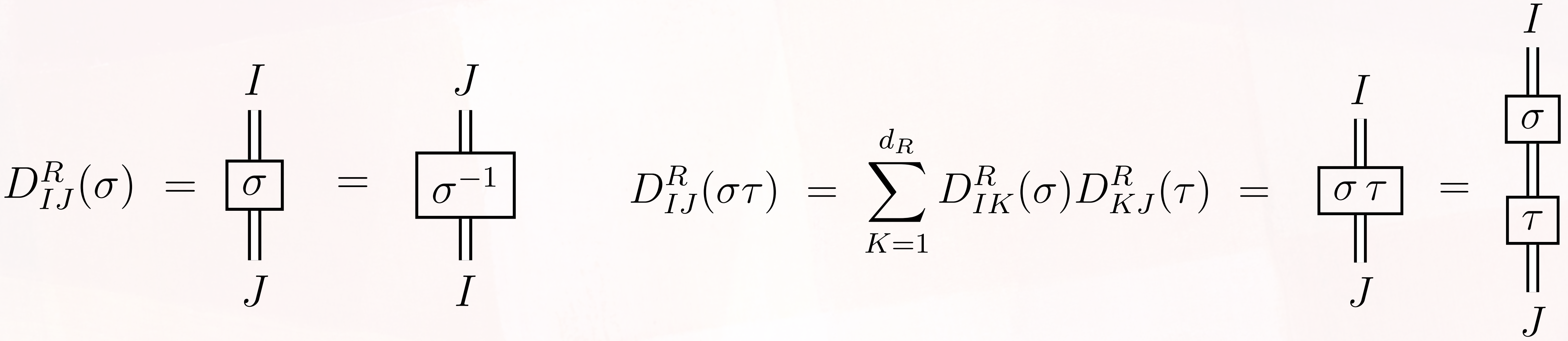
Computation using quivers

Introduce “quiver diagrams” for the representation matrices



$$D^R_{IJ}(\sigma) = \qquad = \qquad$$

$$D^R_{IJ}(\sigma\tau) = \sum_{K=1}^{d_R} D^R_{IK}(\sigma) D^R_{KJ}(\tau) = \qquad = \qquad$$

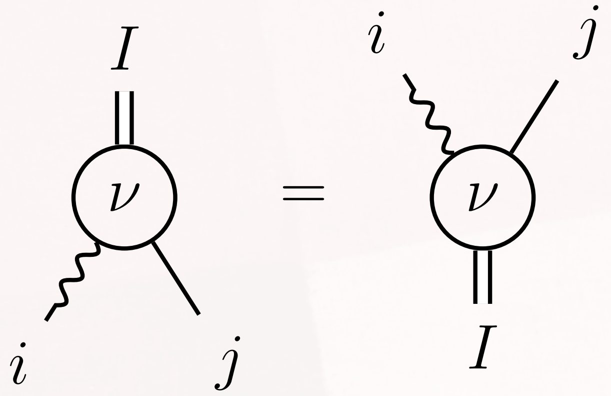
Quiver calculus

[de Mello Koch, Smolic, Smolic] (hep-th/0701066) [Paskounis, Ramgoolam] (1301.1980), [Mattioli, Ramgoolam] (1601.06086)

Introduce branching coefficients to the irreps of $S_m \times S_n$

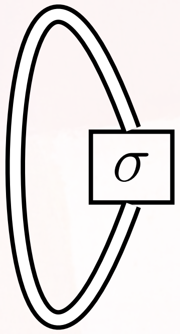
$$\boldsymbol{R} = \bigoplus_{\boldsymbol{\nu}} (\boldsymbol{r_1} \otimes \boldsymbol{r_2})_{\boldsymbol{\nu}}$$

$$B^{R\to(r_1,r_2)\nu}_{I\to(i,j)} =$$

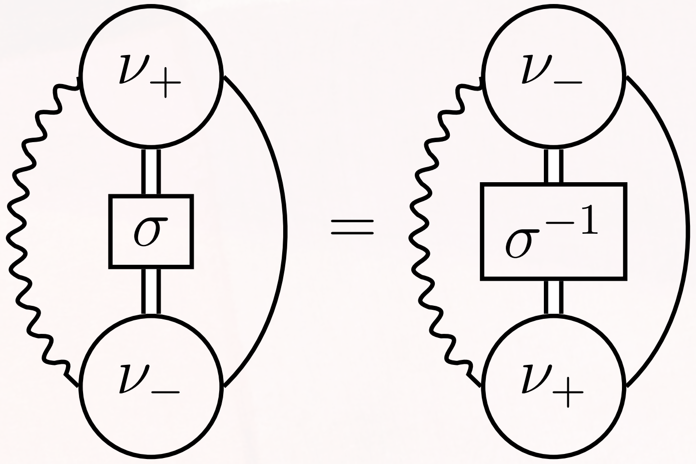


Characters and restricted characters

$$\chi^R(\sigma) = \chi^R(\sigma^{-1}) =$$



$$\chi^{R(r_1,r_2)(\nu_+,\nu_-)}(\sigma) =$$



After various identities, tree-level 2-pt is diagonalized

Use of restricted characters

1. Representation basis solves finite *Nc* constraints

$$F_{[i_1 i_2 \ldots i_{N+1}]} = 0 \quad \text{if} \quad 1 \leq i_k \leq N$$

Cut-off for the height of irreps (not more than *Nc*)

2. Quiver calculus generalizable to non-extremal *n*-pt

(work in progress)

3. Non-diagonalizable observables in complex multi-matrix model

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle \sim \int [DZ D\overline{Z} DW D\overline{W}] \, e^{-\mathrm{tr}\,(Z\overline{Z}+W\overline{W})} \, \underline{\mathrm{tr}\,(ZWZW)} \, \underline{\mathrm{tr}\,(\overline{Z}\overline{Z}\overline{W}\overline{W})}$$

On the diagonal basis of *Z*, *W* has generally off-diagonal elements

Conclusion
and
Outlook

Conclusion and Outlook

- 🍏 Studied n-point functions of gauge theory

- 🍓 Obtained the edge-based formula (like closed string)
- 🍐 Obtained the discretized moduli space $\mathcal{M}_{g,n}$ (like open string)

🍓 = 🍐 : **open-closed duality**

- 🍌 Application to AdS/CFT?
  - 1/*Nc* or finite *Nc* effects of n-point
  - Simplest four-point correlator at tree-level (Octagon frame)

Coronado (1811.00467)

Thank you

Tree-level 2-pt revisited

Tree-level 2-point in permutation basis:

$$\langle \mathcal{O}_{\alpha}^{\vec{A}}(x)\mathcal{O}_{\beta}^{\vec{B}}(0)\rangle = \sum_{\sigma\in S_L}\left(\prod_{p=1}^{L}\frac{g^{A_{\sigma^{-1}(p)}B_p}}{|x|^2}\right)N_c^{C_L(\alpha\sigma^{-1}\beta\sigma)}$$

Choose half-BPS ops $\mathcal{O}_{\alpha}^{\vec{A}} = \mathrm{tr}\,_L(\alpha Z^{\otimes L}),\quad \mathcal{O}_{\beta}^{\vec{B}} = \mathrm{tr}\,_L(\beta \overline{Z}^{\otimes L})$

Tree-level 2-pt revisited

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Choose half-BPS ops $\mathcal{O}_\alpha^{\vec{A}} = \mathrm{tr}_L(\alpha Z^{\otimes L}),\quad \mathcal{O}_\beta^{\vec{B}} = \mathrm{tr}_L(\beta \overline{Z}^{\otimes L})$

In the large *Nc* limit,

$$\langle \mathcal{O}_\alpha\mathcal{O}_\beta\rangle \approx \sum_{\sigma\in S_L} N_c^L\,\delta_L(\beta\sigma^{-1}\alpha\sigma),\quad \delta_L(\sigma)=1\ \text{ iff } \sigma=1$$

Any class function is a linear combination of irreducible characters

$$\delta_L(\sigma) = \sum_{R\vdash L}\frac{d_R}{L!}\chi^R(\sigma),\quad d_R = \dim(S_L\ \mathrm{irrep}\ R)$$

Thus

$$\langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle \approx \sum_{\sigma \in S_L} N_c^L \sum_{R \vdash L} \frac{d_R}{L!} \chi^R(\beta \sigma^{-1} \alpha \sigma)$$

Apply the grand orthogonality theorem

$$\sum_{\sigma \in S_L} D^R_{ij}(\sigma) D^S_{kl}(\sigma^{-1}) = \frac{L!}{d_R} \delta_{il} \delta_{jk} \delta^{RS}$$

$$D^R_{ij}(\sigma) \ : \ \text{matrix elements } (i,j) \text{ of } \sigma \text{ in irrep } R$$

Thus
$$\langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle \approx \sum_{\sigma \in S_L} N_c^L \sum_{R \vdash L} \frac{d_R}{L!} \chi^R(\beta \sigma^{-1} \alpha \sigma)$$

Apply the grand orthogonality theorem

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$$D^R_{ij}(\sigma) \ : \ \text{matrix elements } (i,j) \text{ of } \sigma \text{ in irrep } R$$

to get
$$\langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle \approx \sum_{R \vdash L} N_c^L \, \chi^R(\alpha) \, \chi^R(\beta)$$

The character orthogonality says that 2-pt in rep. basis is orthogonal

$$\langle \mathcal{O}^R(Z) \mathcal{O}^S(\overline{Z}) \rangle \approx N_c^L \delta^{RS}$$