The Spectrum of Classical String Theory

and

Integrability in the AdS/CFT Correspondence

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Plan of Presentation

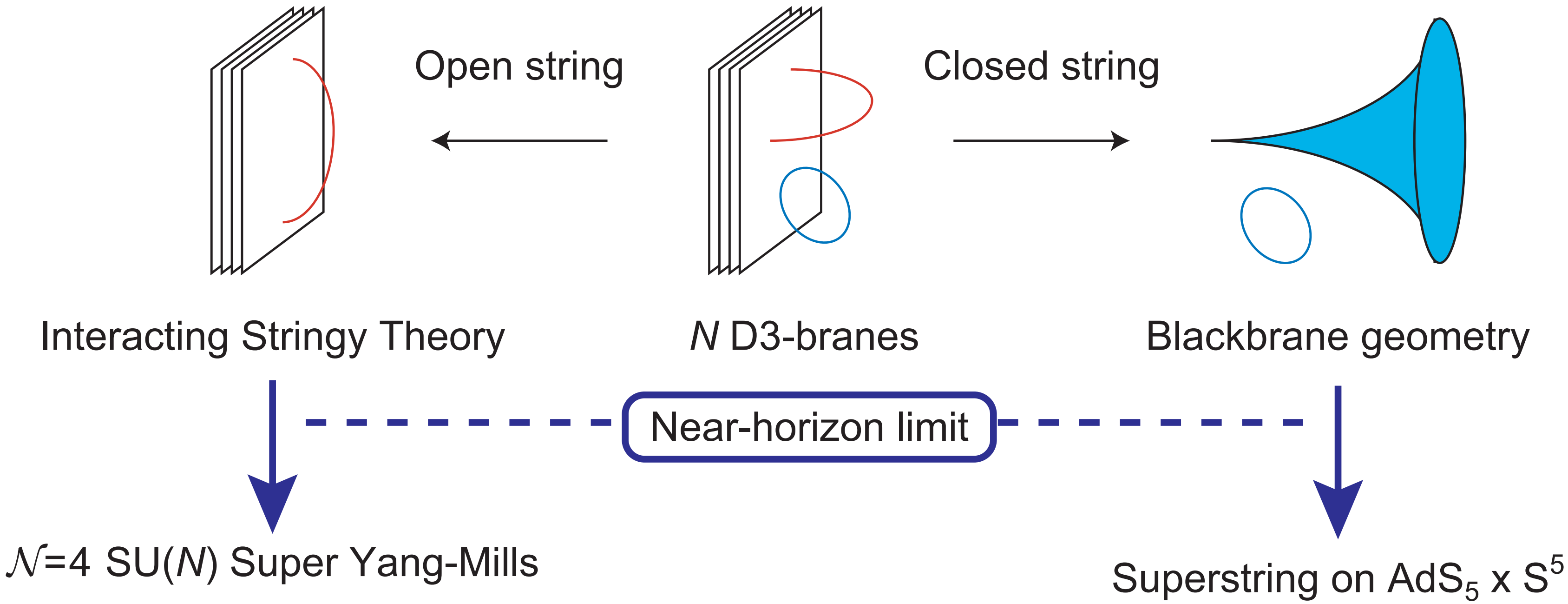
- Introduction
- Review of AdS/CFT with Integrability
- Sine-Gordon and Classical Strings
- Finite-Size Effects
- Summary and Outlook

Section 1

Introduction

Maldacena Conjecture

Maldacena, *Adv.Theor.Math.Phys.* **2** (1998)



Coupling constants: if $g_{\mathrm{YM}}^2 = g_{\mathrm{str}}$,

$$\lambda = N g_{\mathrm{YM}}^2 \,(\ll 1) \quad \text{and} \quad \lambda = N g_{\mathrm{str}} = R^4/\alpha'^2 \,(\gg 1)$$

Basic Questions

Is Maldacena conjecture

(called AdS/CFT correspondence)

really correct?

If gauge theory and string theory

can describe the same physics,

then how both are related,

under the strong/weak duality?

Matching Global Symmetry

$\mathcal{N} = 4$ SYM v.s. Superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$

$$psu(2,2|4) \underset{\text{bosonic}}{\supset} so(2,4) \times so(6)$$

$$R \text{ symmetry} \quad \longleftrightarrow \quad \text{Isometry of } \mathrm{S}^5$$

$$so(6)_R \text{ Cartan} : (J_1, J_2, J_3)$$

$$\text{Conformal symmetry} \quad \longleftrightarrow \quad \text{Isometry of } \mathrm{AdS}^5$$

$$so(2,4) \text{ Cartan} : (\Delta \text{ or } E, S_1, S_2)$$

The Spectrum of Both Theories

Gauge Theory (CFT)

$\equiv$ Eigenstates of Dilatation operator Δ

$$\langle \mathcal{O}_i^\dagger(x)\mathcal{O}_i(y)\rangle \sim 1/\left|x-y\right|^{2\Delta_i}$$

Quantum effects mix different operators

$\Rightarrow$ Dilatation operator become matrix Δ_{ij}

String Theory (AdS)

$\equiv$ (Classical) string states on $\mathrm{AdS}_5 \times \mathrm{S}^5$

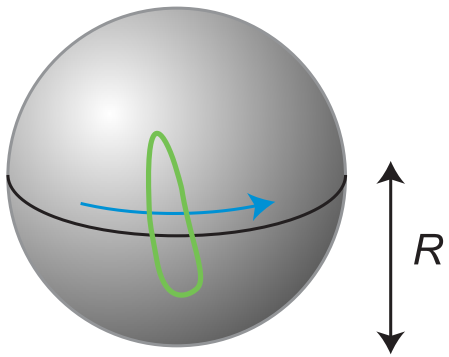
Correspondence of the Spectrum

Gauge theory | **String theory**

$$\mathcal{O} = \mathrm{tr}\left[\Phi^{i_1}\Phi^{i_2}\right] + \cdots$$

$$\mathcal{O}' = \mathrm{tr}\left[\Phi^{i_1}\Phi^{i_2}\Phi^{i_3}\Phi^{i_4}\right] + \cdots \quad \leftrightarrow$$

$$\mathcal{O}'' = \mathrm{tr}\left[\Phi^{i_1}\Phi^{i_2}\ldots\Phi^{i_L}\right] + \cdots$$



S^5

- Prediction of the Maldacena conjecture
- Provide strong evidences by themselves

Correspondence of the Spectrum

Gauge theory | String theory

$$\Delta(\lambda; J_i, \{x^{\alpha}_{\text{gauge}}\}) \quad \leftrightarrow \quad E(\lambda; J_i, \{x^{\alpha}_{\text{string}}\})$$

- Need to know which corresponds to which
- Should consider a family of operators/strings

To know how to identify extra parameters:

$$x^{\alpha}_{\text{gauge}} = f^{\alpha}\left(\{x^{\beta}_{\text{string}}\}\right)$$

Section 2

Review of AdS/CFT

with Integrability

SYM Operator as Spin Chain

Complex scalars of $\mathcal{N} = 4$ SYM :

$$(Z, W, X, \overline{Z}, \overline{W}, \overline{X})$$

Consider the $su(2)$ sector, $(L = J_1 + J_2)$,

$$\mathcal{O} \sim \mathrm{tr}\,[ZZ \ldots WZ \ldots WZ] + \cdots \qquad \Delta \cdot \mathcal{O} = \Delta_{\mathcal{O}}\,\mathcal{O}$$

$$|\mathcal{O}\rangle \sim \mathrm{tr}\,[\uparrow\uparrow \ldots \downarrow\uparrow \ldots \downarrow\uparrow] + \cdots \qquad H\,|\mathcal{O}\rangle = E_{\mathcal{O}}\,|\mathcal{O}\rangle$$

Dilatation operator $\Delta(\lambda)$ (at 1-loop) $\leftrightarrow$

Hamiltonian of ($\mathrm{XXX}_{1/2}$) integrable spin chain

Minahan, Zarembo, *JHEP* **0303** (2003)

Diagonalize Δ using Integrability

1. Ansatz for the eigenstates of Δ

$$\mathcal{O} \sim \sum_{x_1 < x_2} \left\{ e^{ip_1 x_1 + ip_2 x_2} + S(p_2, p_1) e^{ip_2 x_1 + ip_1 x_2} \right\} |\ldots ZWZ \ldots WZ \ldots\rangle$$

2. Periodicity condition $=$ Bethe Ansatz

$$\Delta = \sum_{j=1}^{J_2} \frac{\lambda}{2\pi^2} \frac{1}{u_j^2 + \frac{1}{4}}, \quad e^{ip_j L} = \prod_{k \neq j}^{J_2} \frac{u_j - u_k + i}{u_j - u_k - i}, \quad u_j \equiv \frac{1}{2} \cot\left(\frac{p_j}{2}\right)$$

3. Thermodynamic limit ($\sim$ Integral equation)

$\Rightarrow$ Solution as an algebraic curve

Classical Integrability of Strings

1. Rewrite e.o.m. on $\mathbb{R}_t \times S^3$ in Lax-pair form

$$E.o.M. \quad \Leftarrow \quad [\partial_\sigma - L(x), \partial_\tau - M(x)] = 0 \quad \forall \boldsymbol{x} \in \mathbb{CP}^{\mathbf{1}}$$

2. Monodromy matrix defines spectral curve

$$\Omega(x) \equiv \bar{P} \exp\left(\oint d\sigma \, L(x;\tau,\sigma) \right), \quad \det \, (y \, \mathbf{1}_2 - \Omega(x)) = 0$$

3. Constraints on $\mathrm{p}(x)$, $\quad \Omega(x) \sim \mathrm{diag}\left(e^{i\mathrm{p}}, e^{-i\mathrm{p}}\right)$

Comparison of Integrability

Gauge theory

Diagonalize Δ
by Bethe Ansatz

Rapidity $\tilde{x}L \equiv u = \frac{1}{2}\cot\left(\frac{p}{2}\right)$

String theory

Rewrite e.o.m.
in Lax-pair form

Quasi-momentum $\mathrm{p}(x)$

Introduce the density $\tilde{\rho}(\tilde{x})$ and $\rho(x)$, (& rescale x)

$$\Delta - L = \frac{\lambda}{8\pi^2 L}\oint_{\mathcal{C}} d\tilde{x}\,\frac{\tilde{\rho}(\tilde{x})}{\tilde{x}^2} \quad \leftrightarrow \quad \frac{\lambda}{8\pi^2 J}\oint_{\mathcal{C}} dx\,\frac{\rho(x)}{x^2} = E - J$$

Formal agreement at one-loop in $\tilde{\lambda} \equiv \lambda/J^2$

Kazakov, Marshakov, Minahan, Zarembo, *JHEP* **0405** (2004)

Correspondence at $L(\text{or } J) = \infty$

Beisert, hep-th/0511082

Bethe Ansatz conjectured to all orders in λ

- Dispersion for an elementary magnon

$$\varepsilon_1(p) = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p}{2}\right)}$$

$$Q\text{-magnon boundstate}: \ \varepsilon_Q(p) = \sqrt{Q^2 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p}{2}\right)}$$

- The $su(2|2)^2$ invariant two-body S-matrix

$$\hat{S}(x^{\pm}, y^{\pm}) = S_0 \left[\hat{S}_{su(2|2)_L} \otimes \hat{S}_{su(2|2)_R}\right]$$

Input in $su(2)$ sector & Symmetry $\rightarrow$ The $su(2|2)^2$ S-matrix

Section 3

Sine-Gordon

and Classical Strings

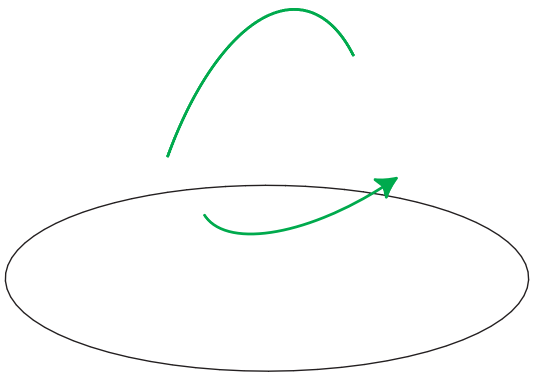
[Okamura, R.S.] *Phys. Rev.* **D75** (2007) 046001

On Classical String Solutions

Different ways of comparison in different limits:

Folded spinning string

$\leftrightarrow$ Symmetric 2-cut sol.

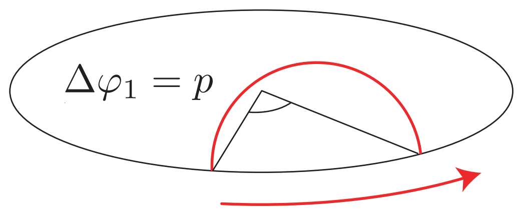


Gubser, Klebanov, Polyakov
Nucl. Phys. **B636** (2002)

Frolov, Tseytlin, *Phys. Lett.* **B570** (2003)

(Dyonic) giant magnon

$E - J_1 = \varepsilon_1(p)$ and $E, J_1 = \infty$



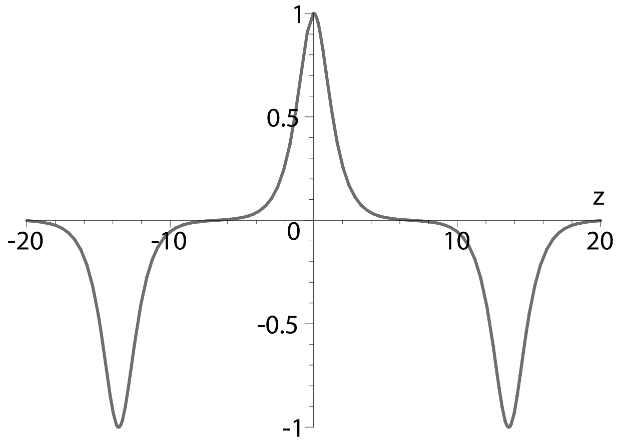
Hofman, Maldacena, *J. Phys.* **A39** (2006)

Chen, Dorey, Okamura, *JHEP* **0609** (2006)

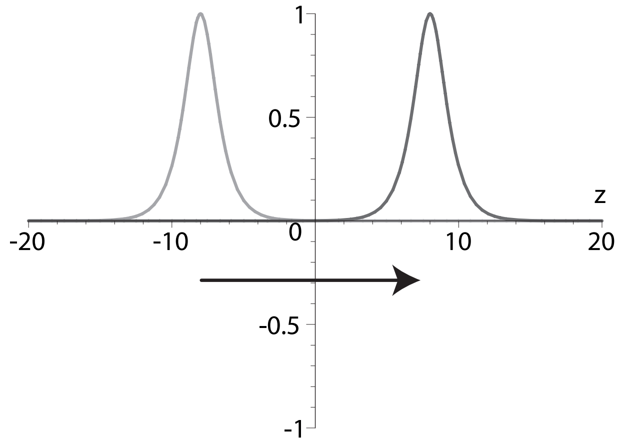
Perspective from Sine-Gordon

Classical string on $\mathbb{R}_t \times S^2 \longrightarrow$ sine-Gordon solution

Helical-wave at rest



Moving one-soliton



$\ell = 4\mathbf{K}(k)$	Periodicity	$\ell = \mathbf{K}(1) = \infty$
$v = 0$	Velocity	$v = \cos(p/2)$

Pohlmeyer-Lund-Regge Reduction

String e.o.m. on S^3, with $(\xi_1, \xi_2) \equiv (X_{1+i2}, X_{3+i4})$

$$\partial_a \partial^a \vec{\xi} + \left(\boldsymbol{\partial_a \vec{\xi}^* \cdot \partial^a \vec{\xi}} \right) \vec{\xi} = 0$$

Define $\psi \equiv \cos(\frac{\alpha}{2}) e^{i\frac{\beta}{2}}$, with $K_i \equiv \epsilon_{ijkl} X^j \partial_+ X^k \partial_- X^l$

$$\cos \boldsymbol{\alpha} \equiv -\boldsymbol{\partial_+ \vec{X} \cdot \partial_- \vec{X}}, \quad \partial_\pm \beta \sin^2 \left(\frac{\alpha}{2} \right) \equiv \pm \frac{1}{2} \partial_\pm^2 \vec{X} \cdot \vec{K}$$

$\vec{\xi}(\tau, \sigma)$: any classical string solution on $\mathbb{R}_t \times S^3 \Rightarrow$

$\psi(\tau, \sigma)$ solves Complex sine-Gordon equations

The Solution Connecting Them

Complex sine-Gordon (CsG) Model:

$$\mathcal{L} = \frac{-\partial_\tau \psi^* \, \partial_\tau \psi + \partial_\sigma \psi^* \, \partial_\sigma \psi}{1 - \psi^* \psi} + \psi^* \psi$$

$\Rightarrow$ Solutions with general k and v exist

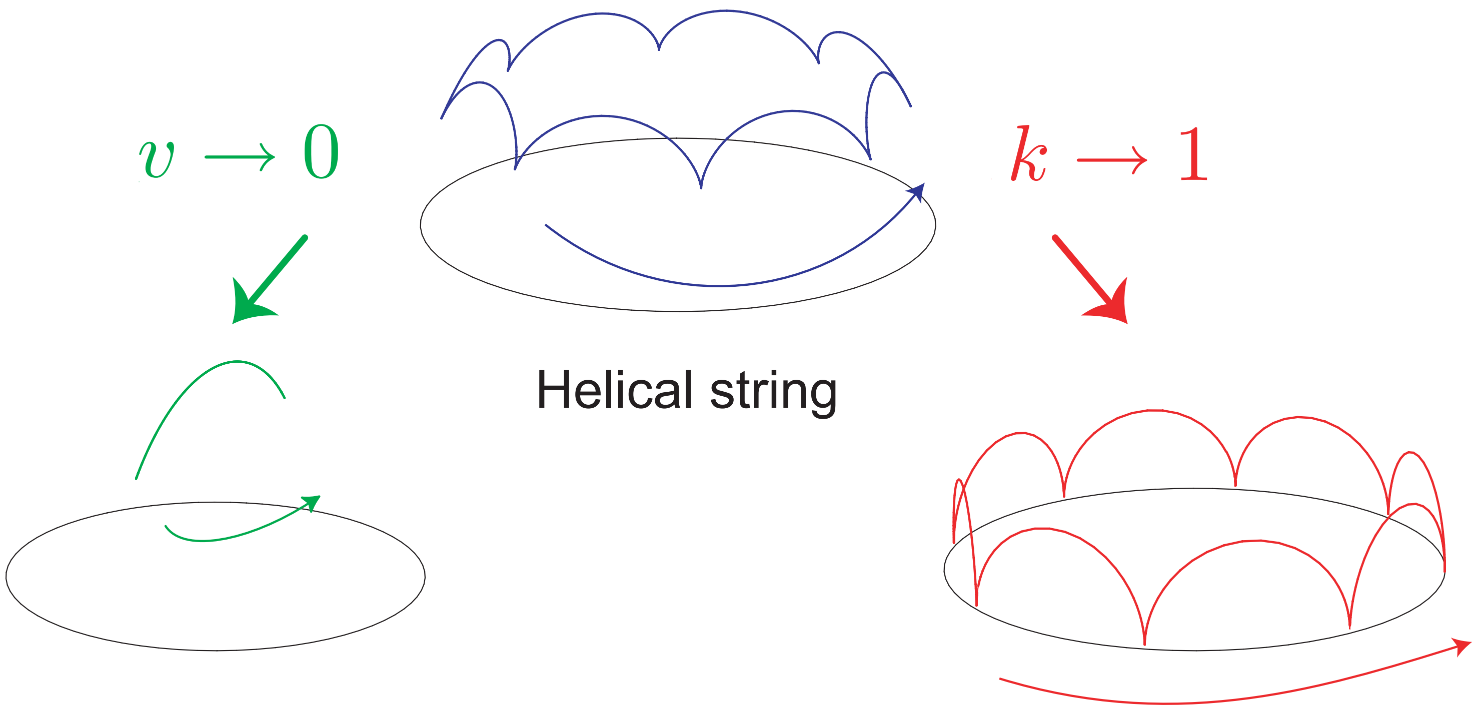
Can construct corresponding classical strings?

$\rightarrow$ Consider the inverse of PLR reduction

$\psi_{\rm cn}$: helical wave sol. $\rightsquigarrow$ $\vec{\xi}$: “Helical string"

Helical Spinning Strings

Spacetime profile of type (i) solution:



Folded string

with $J_1\,, J_2 \neq 0$

Dyonic Giant Magnons

with $J_1 \to \infty, J_2 \neq 0$

Charges and Winding Numbers

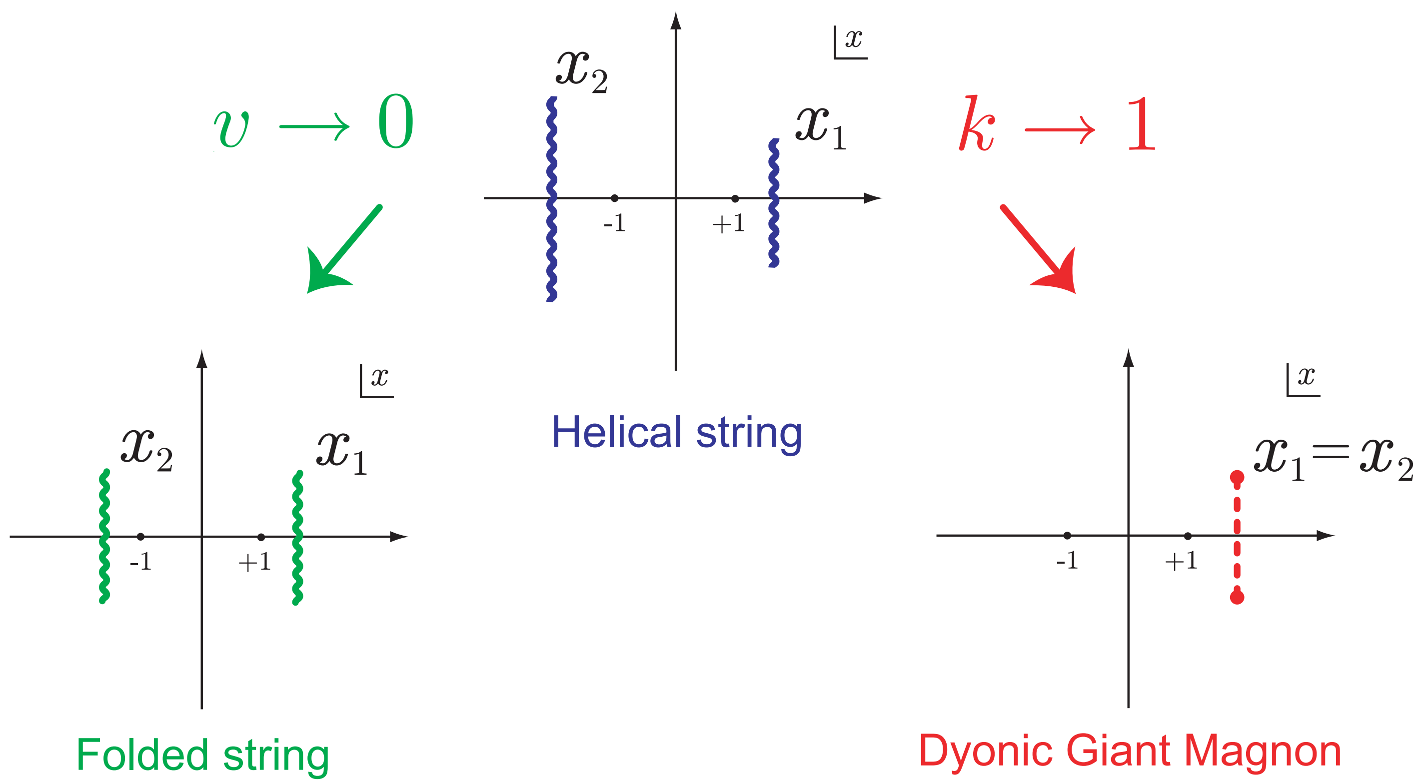
Parameters: $(k\,, v\,, u_1\,, u_2) \leftrightarrow (J_1\,, J_2\,, N_1\,, N_2)$

$$\begin{aligned}
\mathcal{E} &= n\,a\left(1 - v^2\right)\mathbf{K} \\
\mathcal{J}_1 &= \frac{n\,C^2\,u_1}{k^2}\left[-\mathbf{E} + \left(\mathrm{dn}^2(i\omega_1) + \frac{v\,k^2}{u_1}\,i\,\mathrm{sn}(i\omega_1)\,\mathrm{cn}(i\omega_1)\,\mathrm{dn}(i\omega_1)\right)\mathbf{K}\right] \\
\mathcal{J}_2 &= \frac{n\,C^2\,u_2}{k^2}\left[\mathbf{E} + (1-k^2)\left(\frac{\mathrm{sn}^2(i\omega_2)}{\mathrm{cn}^2(i\omega_2)} - \frac{v}{u_2}\,\frac{i\,\mathrm{sn}(i\omega_2)\,\mathrm{dn}(i\omega_2)}{\mathrm{cn}^3(i\omega_2)}\right)\mathbf{K}\right] \\
\frac{2\pi N_1}{n} &= 2\mathbf{K}\left(-iZ_0(i\omega_1) - v\,u_1\right) + (2n'_1 + 1)\pi \\
\frac{2\pi N_2}{n} &= 2\mathbf{K}\left(-iZ_2(i\omega_2) - v\,u_2\right) + 2n'_2\,\pi
\end{aligned}$$

Finite-Gap interpretation

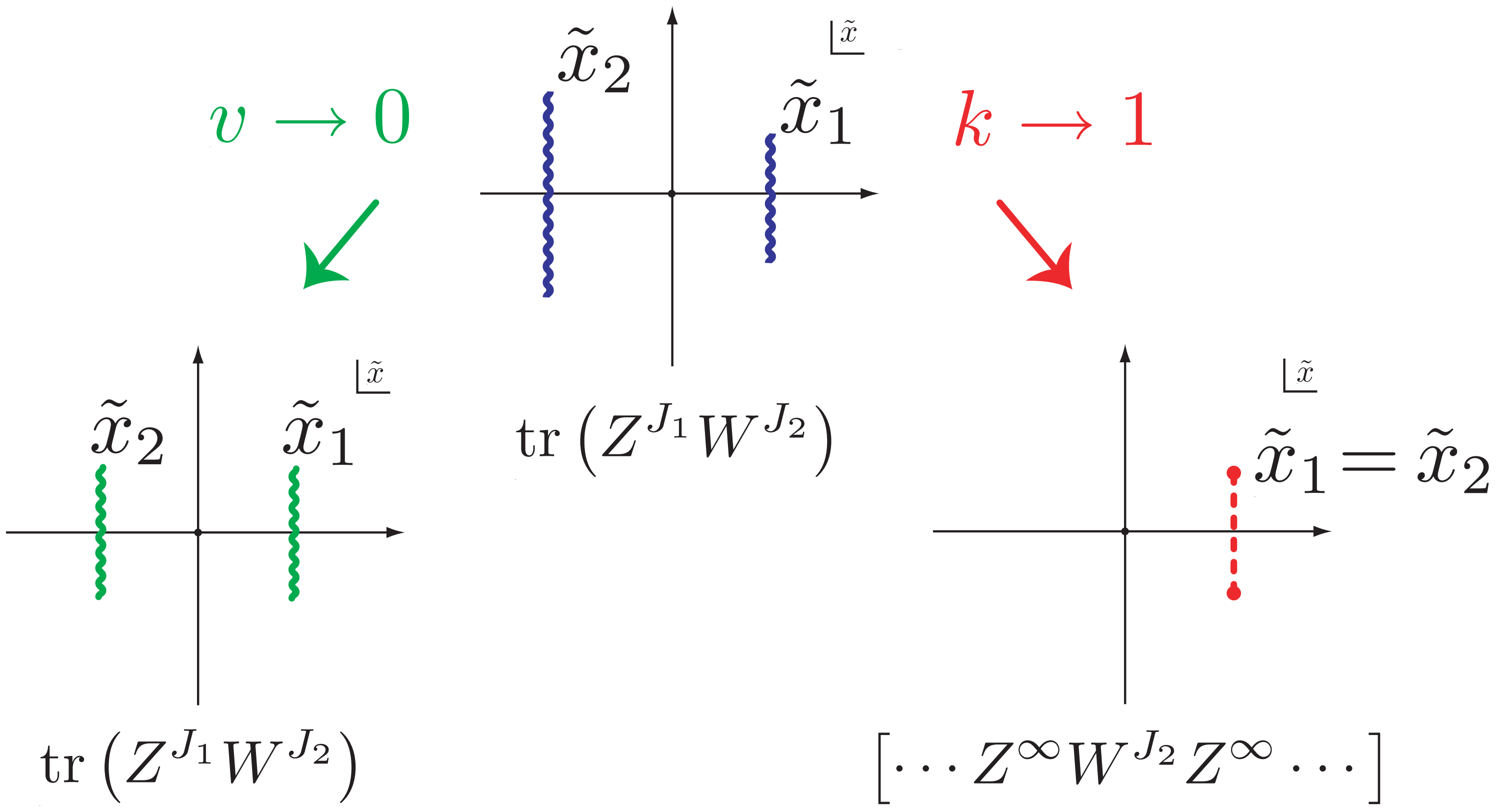
Vicedo, `arXiv:hep-th/0703180`

Helical strings = General 2-cut FG solutions



On Gauge Theory Dual

FG sol. $\leftrightarrow$ Algebraic curve from Bethe Ansatz



Section 4

Finite-Size Effects

[Hatsuda, R.S.] hep-th/0801.0747

The All-loop Bethe Ansatz

Correspondence at infinite L:

(Classical) AdS (Perturbative)CFT

$\lambda \gg 1$ $\nwarrow$ $\nearrow$ $\lambda \ll 1$

Conjectured Bethe Ansatz

(Dispersion and S-matrix)

$\cdots$ breaks down at finite L, because

- Wrapping interaction starting at $\mathcal{O}\left(\lambda^L\right)$
- Exponential correction (1-loop in $\lambda^{-1/2}$) $\sim e^{-cJ}$

Beyond the All-loop Bethe Ansatz

How to check AdS/CFT at finite L?

(Classical) AdS (Perturbative)CFT

$\lambda \gg 1$ ⤡ ⤢ $\lambda \ll 1$

Effective Field Theory

(Generalized Luscher formula)

Use information of the infinite-L theory (exact in λ)

to predict the leading $L < \infty$ correction

On Exponential Corrections

Finite-J correction to giant magnon is $\sim e^{-cJ}$

[Arutyunov, Frolov, Zamaklar], [Astolfi, Forini, Grignani, Semenoff]

Can Evaluate in two ways Janik, Łukowski, *Phys. Rev.* **D76** (2007)

- Asymptotics of classical strings
- The generalized Lüscher formula

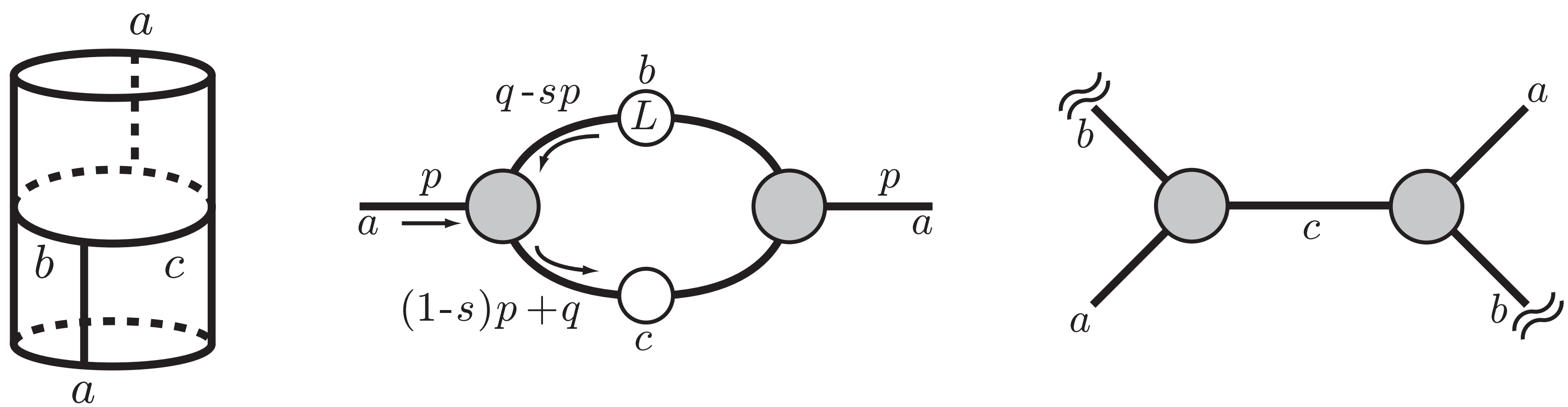
The leading correction to dyonic giant magnon:

$$\delta\left[E - J_1\right] = \left\{\boldsymbol{\alpha_0} + \frac{1}{\sqrt{\lambda}}\alpha_1 + \cdots\right\} \boldsymbol{e^{-cJ_1}} + \mathcal{O}\left(e^{-c'J_1}\right)$$

The generalized Lüscher formula

$$\delta\varepsilon_a(p) \;\leftrightarrow\; \text{finite-size self-energy } \Sigma_L(p)$$

$$\leftrightarrow\; [a+b \to a+b] \text{ scattering process}$$



b, c on-shell $\Rightarrow$ Poles of the $su(2|2)^2$ S-matrix

$$\delta\varepsilon_a^{\mu} = \pm \left| \left(1 - \frac{\varepsilon'_Q(p)}{\varepsilon'_1(\tilde{q}_*)}\right) e^{-i(\tilde{q}_*+sp)L} \operatorname*{Res}_{q=\tilde{q}} \sum_b S_{ba}^{ba}(q_*, p) \right|$$

Relevant Poles and the Residues

Criteria for the relevance of a pole:

- Gives the smallest $|\mathrm{Im}\, p_b|$ with $\mathrm{Im}\, p_b < 0$
- Comes from the s- or t-type diagram

  $\rightarrow Y^- = X^+$ (s-channel) and $Y^+ = X^+$ (t-channel)

Consistency of the Landau-Cutkosky diagram

$\rightarrow$ t-channel contribution is a half of s-channel

$$\frac{\pi}{\sqrt{\lambda}}\delta\varepsilon_a^\mu = \pm\frac{4\sin^3(\frac{p}{2})}{\cosh(\frac{\theta}{2})}\exp\left[-\frac{2\sin^2(\frac{p}{2})\cosh^2(\frac{\theta}{2})}{\sin^2(\frac{p}{2})+\sinh^2(\frac{\theta}{2})}\left(\frac{\mathcal{L}-\mathcal{Q}}{\sin(\frac{p}{2})\cosh(\frac{\theta}{2})}+1\right)\right]$$

Comparison with helical string

Evaluate charges at $k \sim 1$, $\Delta\varphi_1 \equiv p_1$, $\sinh\left(\frac{\theta}{2}\right) \equiv \frac{\mathcal{J}_2}{\sin\left(\frac{p_1}{2}\right)}$

$$\mathcal{E} - \mathcal{J}_1 \approx \sqrt{\mathcal{J}_2^2 + \sin^2\left(\frac{p_1}{2}\right)}$$

$$\mp 4\,\frac{\sin^3\left(\frac{p_1}{2}\right)}{\cosh\left(\frac{\theta}{2}\right)}\,\exp\left[-\frac{2\sin^2\left(\frac{p_1}{2}\right)\cosh^2\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{p_1}{2}\right)+\sinh^2\left(\frac{\theta}{2}\right)}\left(\frac{\mathcal{J}_1}{\sin\left(\frac{p_1}{2}\right)\cosh\left(\frac{\theta}{2}\right)}+1\right)\right]$$

Both sides agree if (in spin chain frame)

$$J_1 + J_2 \leftrightarrow L, \quad J_2 \leftrightarrow Q, \quad \Delta\varphi_1 \equiv p_1 \leftrightarrow p$$

The limit $Q \to 0$ $(\theta \to 0)$ coincides with Janik & Łukowski

Section 5

Summary and Outlook

Summary

- Reviewed AdS/CFT correspondence
  from integrability-based approach
- Constructed helical spinning strings
  ($=$ general 2-cut finite-gap solution)
- Computed finite-size correction
  to dyonic giant magnon

Outlook

- Towards finite-size effects exact in L

  c.f. Thermodynamic Bethe Ansatz

  Zamolodchikov, *Nucl. Phys.* **B342** (1990)

  Arutyunov, Frolov, `arXiv:0710.1568 [hep-th]`

  $\leftrightarrow$ Wrapping effects $(\sim \lambda^L)$ at weak coupling?

- Quantum superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$

  Lüscher formula agrees with known 1-loop results?

  Is there quantum integrability?

...Many questions worth investigation!